# Mid-term exam NPAC
November 2019

## Exercice 1

1. $$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = U \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$\begin{aligned}
\mathcal{L}_0(\varphi_1', \partial_\mu \varphi_1', \varphi_2', \partial_\mu \varphi_2') &= \frac{1}{2}\Big( \partial_\mu(\alpha\varphi_1 + \beta\varphi_2)\partial^\mu(\alpha\varphi_1 + \beta\varphi_2) \\
&\qquad + \partial_\mu(\gamma\varphi_1 + \delta\varphi_2)\partial^\mu(\gamma\varphi_1 + \delta\varphi_2)\Big) \\
&= \frac{1}{2}(\alpha^2+\gamma^2)(\partial\varphi_1)^2 + \frac{1}{2}(\beta^2+\delta^2)(\partial\varphi_2)^2 \\
&\quad + (\alpha\beta + \gamma\delta)\,\partial_\mu\varphi_1\partial^\mu\varphi_2 \\
&= \mathcal{L}_0(\varphi_1, \partial_\mu\varphi_1, \varphi_2, \partial_\mu\varphi_2)
\end{aligned}$$

$$\Rightarrow \left\{ \begin{array}{l} \alpha\beta + \gamma\delta = 0 \\ \alpha^2 + \gamma^2 = 1 \\ \beta^2 + \delta^2 = 1 \end{array} \right\} \Rightarrow \left| \begin{array}{l} \alpha = \cos\theta \\ \gamma = \sin\theta \end{array} \right. \text{ and } \left| \begin{array}{l} \beta = \cos\varphi \\ \delta = \sin\varphi \end{array} \right.$$

$$\Rightarrow \cos\theta\cos\varphi + \sin\theta\sin\varphi = 0 \Rightarrow \cos(\theta - \varphi) = 0$$

$$\Rightarrow \varphi = \theta + (2n+1)\frac{\pi}{2}$$

① $\varphi = \theta + \frac{\pi}{2} \Rightarrow U = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \det U = 1$

② $\varphi = \theta + \frac{3\pi}{2} \Rightarrow U' = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix} \Rightarrow \det U' = -1$

The matrices $U'$ are the product of a mirror symmetry and a rotation: $U' = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = S \cdot U(-\theta)$

The set of matrices $U$ when $\theta$ varies between 0 and $2\pi$ is $SO(2)$ and the set of matrices $U'$ is obtained by multiplying them by mirror symmetries that are such that $S^2 = \mathbb{1}$. The set $\{U(\theta), U'(\theta)\} = SO(2) \otimes \mathbb{Z}_2 = O(2)$.

2. $\vec{\varphi}(x) = \begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix}$

$$\mathcal{L}_0 = \frac{1}{2}\, \partial_\mu \vec{\varphi}(x) \cdot \partial^\mu \vec{\varphi}(x) = \frac{1}{2}\, \partial_\mu \varphi_i(x)\, \partial^\mu \varphi_i(x)$$

$$\vec{\varphi}'(x) = U\, \vec{\varphi}(x)$$

$$\mathcal{L}_0 = \frac{1}{2}\, \partial_\mu \vec{\varphi}\ {}^tU\, U\, \partial^\mu \vec{\varphi}(x) \quad \Rightarrow \quad {}^tU \cdot U = \mathbb{1} \quad \Rightarrow \quad U \in O(2)$$

3. $\varphi(x) = \dfrac{\varphi_1(x) + i\,\varphi_2(x)}{\sqrt{2}} \quad \Rightarrow \quad \varphi^*(x) = \dfrac{\varphi_1(x) - i\,\varphi_2(x)}{\sqrt{2}}$

$$\Rightarrow \mathcal{L}_0 = \partial_\mu \varphi^*(x)\, \partial^\mu \varphi(x)$$

$\Rightarrow \varphi'(x) = e^{i\theta}\varphi(x)$ and $\varphi'(x) = e^{i\theta}\varphi^*(x)$ keep $\mathcal{L}_0$ invariant

$$\varphi'(x) = e^{i\theta}\varphi(x) \quad \Rightarrow \quad \begin{cases} \varphi_1' = \cos\theta\, \varphi_1 - \sin\theta\, \varphi_2 \\ \varphi_2' = \sin\theta\, \varphi_1 + \cos\theta\, \varphi_2 \end{cases} \Leftrightarrow SO(2) \text{ or } U \text{ transfo.}$$

$$\varphi'(x) = e^{i\theta}\varphi^*(x) \quad \Rightarrow \quad \begin{cases} \varphi_1' = \cos\theta\, \varphi_1 + \sin\theta\, \varphi_2 \\ \varphi_2' = \sin\theta\, \varphi_1 - \cos\theta\, \varphi_2 \end{cases} \Leftrightarrow U' \text{ transfo}$$

The set of numbers $\{e^{i\theta}\}$ when $\theta$ varies between 0 and $2\pi$ is $U(1)$. $U(1)$ is "identical" to $SO(2)$ (homeomorphic).

4. One element of $O(2)$ is $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$: it changes $\varphi_1 \rightleftarrows \varphi_2$:

$\begin{cases} \varphi_1' = \varphi_2 \\ \varphi_2' = \varphi_1 \end{cases}$. The mass term is invariant under this transformation if $m_1 = m_2$. In this case

$$\mathcal{L}_0' = \mathcal{L}_0 - \frac{1}{2} m^2 (\varphi_1^2 + \varphi_2^2) = \mathcal{L}_0 - \frac{1}{2} m^2 \vec{\varphi}^2 = \mathcal{L}_0 - m^2 \varphi^* \varphi.$$

This shows that reciprocally when $m_1^2 = m_2^2 = m^2$, $\mathcal{L}_0'$ is $O(2)$-invariant.

5. Under $O(2)$, only $\varphi_1^2(x) + \varphi_2^2(x)$ is invariant. This "implies" that only $\left(\varphi_1^2(x) + \varphi_2^2(x)\right)^2$ is invariant at order 4. This term can also be written as $\left(\vec{\varphi}^2(x)\right)^2$ or $\left(\varphi^*(x)\varphi(x)\right)^2$.

6. $SO(2)$ is the subgroup of $O(2)$ connected to $\mathbb{1}_2$.
Under $SO(2)$, the tensor $\varepsilon_{ij}$ transforms as

$$\varepsilon'_{ij} = U_{ik} U_{jl} \varepsilon_{kl} \qquad \text{with } U \in SO(2)$$
$$= U_{ik} \varepsilon_{kl} ({}^tU)_{lj}$$
$$= (U \varepsilon {}^tU)_{ij}$$

$$U \varepsilon {}^tU = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$
$$= \quad '' \quad \cdot \begin{pmatrix} -\sin\theta & \cos\theta \\ -\cos\theta & -\sin\theta \end{pmatrix}$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \varepsilon$$

Thus in any frame $\varepsilon$ has the same coordinates
$\Rightarrow$ it is an invariant tensor of $SO(2)$.

For the other part of the group, it is sufficient to consider one element of $\mathbb{Z}_2$ since all the matrices $U'$ can be written $U' = S \cdot U$ and

$$\varepsilon' = U' \varepsilon {}^tU'$$
$$= S U \varepsilon {}^tU {}^tS$$
$$= S \varepsilon {}^tS \qquad \text{because } U \varepsilon {}^tU = \varepsilon \quad \forall U \in SO(2)$$

We choose $S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$$\Rightarrow S \varepsilon {}^tS = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} = -\varepsilon$$

Thus, under the $\mathbb{Z}_2$ transformations $\varepsilon'_{ij} = -\varepsilon_{ij}$

7. Under $SO(2)$:

$$(\varepsilon_{ij}\varphi_i\psi_j)' = \varepsilon_{ij}\varphi_i'\psi_j'$$
$$= \varepsilon_{ij} U_{ik} U_{je}\varphi_k\psi_e$$
$$= ({}^tU)_{ki}\,\varepsilon_{ij}\,U_{je}\,\varphi_k\psi_e$$
$$= ({}^tU\,\varepsilon\,U)_{ke}\,\varphi_k\psi_e$$
$$= \varepsilon_{ke}\,\varphi_k\psi_e \qquad \text{car } U\in SO(2) \Rightarrow {}^tU = U^{-1}\in SO(2)$$
$$\text{et } U^{-1}\,\varepsilon\,(U^{-1})^{-1} = \varepsilon$$

$\Rightarrow$ $\varepsilon_{ij}\varphi_i\psi_k$ = scalar for $SO(2)$

This is clear when we write it as

$$\varepsilon_{ij}\varphi_i\psi_k = \varphi_1\psi_2 - \psi_2\varphi_1 = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}\wedge\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \vec{\varphi}\wedge\vec{\psi}$$

which is obviously invariant under the rotations in the plane $(\vec{\varphi},\vec{\psi})$. It is also a pseudo vector and thus we expect it to be transformed in its opposite under "parity", that is mirror symmetry:

$$(\varphi_1\psi_2 - \varphi_2\psi_1)' = \varphi_2\psi_1 - \varphi_1\psi_2 = -(\varphi_1\psi_2 - \varphi_2\psi_1)$$

for the transformation $S = \begin{pmatrix} & 1\\ 1 & \end{pmatrix}$.

8. $$\mathcal{L}_0' = \frac{1}{2}(\partial_\mu\vec{\varphi}).(\partial^\mu\vec{\varphi}) - \frac{1}{2}m^2\vec{\varphi}^2 = \frac{1}{2}\partial_\mu\varphi_i\,\partial^\mu\varphi_i - \frac{1}{2}m^2\varphi_i\varphi_i$$

$$j^\mu(x) = \frac{\partial\mathcal{L}_0'}{\partial(\partial_\mu\varphi(x))}\,\delta\varphi(x) + \mathcal{L}_0'\,\delta x^\mu$$

- $\delta x^\mu = 0$
- $$\begin{pmatrix}\varphi_1'\\ \varphi_2'\end{pmatrix} = \begin{pmatrix}1 & -\delta\theta\\ +\delta\theta & 1\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} + \delta\theta\begin{pmatrix} & -1\\ +1 & \end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

$$\Rightarrow \delta\varphi_i(x) = -\delta\theta\,\varepsilon_{ij}\,\varphi_j(x)$$

$$\Rightarrow \quad j^\mu(x) = \partial^\mu\varphi_i\,(-\varepsilon_{ij}\varphi_j(x)) \qquad (\text{discarding } \delta\theta)$$
$$= \vec{\varphi}\wedge\partial^\mu\vec{\varphi}$$

$\Rightarrow$ $j^\mu(x)$ is invariant under $SO(2)$ and is changed into its opposite under "parity" $\Rightarrow$ it is a pseudo-scalar for $O(2)$.

$$\partial_\mu j^\mu = \underbrace{\partial_\mu \vec{\varphi} \wedge \partial^\mu \vec{\varphi}}_{=0} + \underbrace{\vec{\varphi} \wedge \underbrace{\Box \vec{\varphi}}_{\hookrightarrow m^2 \vec{\varphi}}}_{=0} \text{ by Euler-Lagrange}$$

because it is antisymmetric in the exchange $\partial_\mu \vec{\varphi} \leftrightarrow \partial^\mu \vec{\varphi}$:

$$= \varepsilon_{ij} \eta^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_j$$

$$= \varepsilon_{ij} \eta^{\nu\mu} \partial_\nu \varphi_i \partial_\mu \varphi_j$$

$$= \varepsilon_{ji} \eta^{\nu\mu} \partial_\nu \varphi_j \partial_\mu \varphi_i$$

$$= -\varepsilon_{ij} \eta^{\mu\nu} \partial_\mu \varphi_i \partial_\nu \varphi_j$$

$\Rightarrow \partial_\mu j^\mu = 0$ as it should for the physical fields.

## Exercice 2

1. $\mathcal{L}_0 = \frac{1}{2} \partial_\mu \vec{\varphi}(x) \cdot \partial^\mu \vec{\varphi}(x) - \frac{1}{2} m^2 \vec{\varphi}^2(x)$ It is $O(N)$-invariant: $\vec{\varphi}' = R\vec{\varphi}$ $R \in O(N)$

2. 
$$\mathcal{L}_0' = \frac{1}{2}\Big(\partial_\mu(\varphi_1)_1 \partial^\mu(\varphi_1)_1 + \dots + \partial_\mu(\varphi_1)_N \partial^\mu(\varphi_1)_N + \partial_\mu(\varphi_2)_1 \partial^\mu(\varphi_2)_1 + \dots + \partial_\mu(\varphi_2)_N \partial^\mu(\varphi_2)_N\Big)$$
$$- \frac{1}{2} m^2 \Big((\varphi_1)_1^2 + \dots + (\varphi_1)_N^2 + (\varphi_2)_1^2 + \dots + (\varphi_2)_N^2\Big)$$

If we gather the components of $\vec{\varphi}_1(x)$ and $\vec{\varphi}_2(x)$ in a $2N$-component vector $\vec{\varphi}(x) = ((\varphi_1)_1 \dots (\varphi_2)_N)$, then $\mathcal{L}_0'$ can be written:

$$\mathcal{L}_0' = \frac{1}{2} \partial_\mu \vec{\varphi} \cdot \partial^\mu \vec{\varphi} - \frac{1}{2} m^2 \vec{\varphi}^2$$

$\Rightarrow \mathcal{L}_0'$ is $O(2N)$-invariant

3. Same argument as in exercice 1: $\left(\vec{\varphi}^2(x)\right)^2$
There is no other term of order 4 which is $O(2N)$-invariant.

4. $U_2 = \vec{\varphi}_1^{\,2}\, \vec{\varphi}_2^{\,2} - (\vec{\varphi}_1 . \vec{\varphi}_2)^2$

$$\Phi(x) = \begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \vdots & \vdots \\ \varphi_{1N} & \varphi_{2N} \end{pmatrix} \Rightarrow \Phi_{i\alpha}(x) = (\varphi_\alpha)_i\,(x)$$

with $\alpha = 1,2$ and $i = 1, \dots, N$

a. $$\mathcal{L}_0' = \frac{1}{2}\left((\partial_\mu \vec{\varphi}_1)^2 + (\partial_\mu \vec{\varphi}_2)^2\right) - \frac{1}{2} m^2 (\vec{\varphi}_1^{\,2} + \vec{\varphi}_2^{\,2})$$

$$= \frac{1}{2}\left(\partial_\mu(\varphi_\alpha)_i\right)\left(\partial^\mu(\varphi_\alpha)_i\right) - \frac{1}{2} m^2 (\varphi_\alpha)_i (\varphi_\alpha)_i$$

$$= \frac{1}{2}\partial_\mu \Phi_{i\alpha}\, \partial^\mu \Phi_{i\alpha} - \frac{1}{2} m^2 \Phi_{i\alpha}\Phi_{i\alpha}$$

$$= \frac{1}{2}\mathrm{Tr}\left[\partial_\mu {}^t\Phi . \partial^\mu \Phi\right] - \frac{1}{2} m^2\, \mathrm{Tr}({}^t\Phi . \Phi)$$

b. * $(\vec{\varphi}^{\,2})^2 = (\vec{\varphi}_1^{\,2} + \vec{\varphi}_2^{\,2})^2 = (\mathrm{Tr}\,{}^t\Phi\Phi)^2$

* $${}^t\Phi . \Phi = \begin{pmatrix} \varphi_{11} & \dots & \varphi_{1N} \\ \varphi_{21} & \dots & \varphi_{2N}\end{pmatrix} \begin{pmatrix} \varphi_{11} & \varphi_{21} \\ \vdots & \vdots \\ \varphi_{1N} & \varphi_{2N} \end{pmatrix}$$

$$= \begin{pmatrix} \vec{\varphi}_1^{\,2} & \vec{\varphi}_1 . \vec{\varphi}_2 \\ \vec{\varphi}_1 . \vec{\varphi}_2 & \vec{\varphi}_2^{\,2} \end{pmatrix}$$

$$\det({}^t\Phi\Phi) = \vec{\varphi}_1^{\,2}\vec{\varphi}_2^{\,2} - (\vec{\varphi}_1 . \vec{\varphi}_2)^2$$

$$= U_2$$

5. $\Phi' = R\,\Phi\,U \Rightarrow$ $R$ is a $N \times N$ matrix
$U$ " " $2\times 2$ "

$${}^t\Phi'\,\Phi' = {}^tU\,{}^t\Phi\,{}^tR R\,\Phi\,U$$

* $\Rightarrow \mathrm{Tr}({}^t\Phi'\Phi') = \mathrm{Tr}({}^tU\,{}^t\Phi\,{}^tRR\,\Phi\,U)$
$= \mathrm{Tr}({}^t\Phi\Phi) \quad \forall \Phi$ $\Bigg\} \Rightarrow$ ${}^tRR = \mathbb{1}_N \Rightarrow R \in O(N)$
${}^tU U = \mathbb{1}_2 \Rightarrow U \in O(2)$

* $\Rightarrow \det({}^t\Phi'\Phi') = \det({}^tU\,{}^t\Phi\,{}^tRR\,\Phi\,U)$
$= \det({}^t\Phi\Phi) \quad \forall \Phi$ $\Bigg\} \Rightarrow$ same constraints on $R$ and $U$

The kinetic term $Tr[\partial_\mu \Phi^t \partial^\mu \Phi]$ is for the same reasons invariant for $R \in O(N)$ and $U \in O(2)$.

The $O(N)$ and $O(2)$ transformations commute $\Rightarrow$ the invariance group of $\mathcal{L}$ is $O(N) \otimes O(2)$

## Exercice 3

1. Both $H_L$ and $H_R$ are of dimension 2.

$(\frac{1}{2}, \frac{1}{2})$ acts and $H_L \otimes H_R$

A convenient basis in $H_L$ : $\{|+\rangle_L, |-\rangle_L\}$

" " " $H_R$ : $\{|+\rangle_R, |-\rangle_R\}$

" " " $H_L \otimes H_R$ : $\{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$

with $|+\rangle_L$ and $|-\rangle_L$ the eigenvectors of $\sigma_3$ with eigenvalues $+1$ and $-1$ (same thing for $|+\rangle_R, |-\rangle_R$) and $|++\rangle = |+\rangle_L \otimes |+\rangle_R$ and so on.

$H_L \otimes H_R$ is a 4 dimensional space.

2. $Q_i = \frac{\sigma_i}{2} \otimes \mathbb{1}$ , $N_i = \mathbb{1} \otimes \frac{\sigma_i}{2}$

and, for instance, $Q_i |++\rangle = \frac{\sigma_i}{2} |+\rangle_L \otimes \mathbb{1} |+\rangle_R$ while

$N_i |++\rangle = \mathbb{1} |+\rangle_L \otimes \frac{\sigma_i}{2} |+\rangle_R$

We have

$$\sigma_1 = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} & -i \\ i & \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$|+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\sigma_1 |+\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |-\rangle$ $\quad$ $\sigma_2 |+\rangle = \begin{pmatrix} 0 \\ i \end{pmatrix} = i|-\rangle$ $\quad$ $\sigma_3 |+\rangle = |+\rangle$

$\sigma_1 |-\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |+\rangle$ $\quad$ $\sigma_2 |-\rangle = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i|+\rangle$ $\quad$ $\sigma_3 |-\rangle = -|-\rangle$

We therefore find

$$\begin{cases} N_3|++\rangle = \frac{1}{2}|++\rangle \\ N_3|+-\rangle = -\frac{1}{2}|+-\rangle \\ N_3|-+\rangle = \frac{1}{2}|-+\rangle \\ N_3|--\rangle = -\frac{1}{2}|--\rangle \end{cases}, \quad \begin{cases} Q_3|++\rangle = \frac{1}{2}|++\rangle \\ Q_3|+-\rangle = +\frac{1}{2}|+-\rangle \\ Q_3|-+\rangle = -\frac{1}{2}|-+\rangle \\ Q_3|--\rangle = -\frac{1}{2}|--\rangle \end{cases}$$

Therefore

$$N_3 = \begin{array}{c} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} \\ \begin{pmatrix} \frac{1}{2} & & & \\ & -\frac{1}{2} & & \\ & & \frac{1}{2} & \\ & & & -\frac{1}{2} \end{pmatrix} \end{array} \qquad Q_3 = \begin{array}{c} \begin{array}{cccc} |++\rangle & |+-\rangle & |-+\rangle & |--\rangle \end{array} \\ \begin{pmatrix} \frac{1}{2} & & & \\ & \frac{1}{2} & & \\ & & -\frac{1}{2} & \\ & & & -\frac{1}{2} \end{pmatrix} \end{array}$$

Similarly we find:

$$N_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix} \qquad Q_1 = \begin{pmatrix} 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

and

$$N_2 = \begin{pmatrix} 0 & -i/2 & 0 & 0 \\ \frac{i}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & -i/2 \\ 0 & 0 & \frac{i}{2} & 0 \end{pmatrix} \qquad Q_2 = \begin{pmatrix} 0 & 0 & -\frac{i}{2} & 0 \\ 0 & 0 & 0 & -i/2 \\ \frac{i}{2} & 0 & 0 & 0 \\ 0 & \frac{i}{2} & 0 & 0 \end{pmatrix}$$

These matrices are of course hermitic.

3. We have $J_i = N_i + Q_i$ and $K_i = N_i - Q_i$

We therefore find

$$J_1 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -\frac{i}{2} & -\frac{i}{2} & 0 \\ \frac{i}{2} & 0 & 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 & 0 & -\frac{i}{2} \\ 0 & \frac{i}{2} & \frac{i}{2} & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$$

$$K_1 = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & -\frac{i}{2} & \frac{i}{2} & 0 \\ \frac{i}{2} & 0 & 0 & i/2 \\ -\frac{i}{2} & 0 & 0 & -\frac{i}{2} \\ 0 & -\frac{i}{2} & \frac{i}{2} & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & & & \\ & -1 & & \\ & & 1 & \\ & & & 0 \end{pmatrix}$$

4. These matrices should satisfy the Lie algebra of the Lorentz group

$$J_1 J_2 = \begin{pmatrix} i/2 & 0 & 0 & -i/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i/2 & 0 & 0 & -i/2 \end{pmatrix} \quad \text{and} \quad J_2 J_1 = \begin{pmatrix} -i/2 & 0 & 0 & -i/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i/2 & 0 & 0 & i/2 \end{pmatrix}$$

Thus

$[J_1, J_2] = i\, J_3$ as it should.

All the other commutators can be checked in the same way.

5.
$$\begin{aligned} [t'_a, t'_b] &= t'_a t'_b - t'_b t'_a \\ &= U t_a U^{-1} U t_b U^{-1} - U t_b U^{-1} U t_a U^{-1} \\ &= U [t_a, t_b] U^{-1} \\ &= i f_{abc} U t_c U^{-1} = i f_{abc} t'_c \end{aligned}$$

$$t_a'^{\dagger} = (U\, t_a\, U^{-1})^{\dagger} = U^{-1\dagger}\, t_a^{\dagger}\, U^{\dagger} = U\, t_a\, U^{-1} = t_a'$$

because $U$ is unitary.

We conclude that the $t_a'$'s are hermitic and satisfy the same Lie algebra as the $t_a$

$\Rightarrow$ they are an hermitic representation of Lie$(G)$

6. $g \in G \Rightarrow g = e^{i\alpha_a t_a}$ with $\alpha_a \in \mathbb{R}$

Since the $t_a'$'s are also a representation of Lie $G$ $\Rightarrow g' = e^{i\alpha_a t_a'}$ with $\alpha_a' \in \mathbb{R}$ is also an element of $G$

$$g' = \sum_{m=0}^{\infty} \frac{i^m}{m!}(\alpha_a t_a')^m$$

$$= \sum_{m=0}^{\infty} \frac{i^m}{m!}\left(\alpha_a U t_a U^{-1}\right)^m$$

$$= \sum_{m=0}^{\infty} \frac{i^m}{m!} \sum_{a_1 \dots a_m} \alpha_{a_1}\alpha_{a_2}\cdots\alpha_{a_m} U t_{a_1} U^{-1} \cdots U t_{a_m} U^{-1}$$

$$= U \sum_{m=0}^{\infty} \frac{i^m}{m!} (\alpha_a t_a)^m\, U^{-1}$$

$$= U\, e^{i\alpha_a t_a}\, U^{-1}$$

Therefore the sets of matrices $\{e^{i\alpha_a t_a}\}$ and $\{U e^{i\alpha_a t_a} U^{-1} = e^{i\alpha_a t_a'}\}$ are equivalent representations of $G \Rightarrow e^{i\alpha_a t_a}$ and $e^{i\alpha_a t_a'}$

represent the same linear operator in two different basis and $U$ is the transformation matrix between the two basis.

7.
$$U J_3 U^{-1} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 0 & -1 & i & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} & & \text{"} & & \end{pmatrix}\begin{pmatrix} 0 & -1 & i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & -i & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is the "usual" generator of the rotations in the $(x,y)$ plane.

Notice that $U$ is unitary since

$$U U^{\dagger} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 & i & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \end{pmatrix} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \mathbb{1}_4$$

The same kind of calculation for all $J_i$ and $K_i$ shows that we retrieve all the usual generators in the basis $(t,x,y,z) \Rightarrow$ this set of matrices $J'_i, K'_i$ constitute the spin 1 representation of the Lorentz group: they transform the 4-vectors under Lorentz transformations.

This has been already encountered with Dirac bi-spinors for instance: $\bar{\Psi} \gamma^\mu \Psi$ is a 4-vector $\Rightarrow$ it spans the $(J'_i, K'_i)$ representation of the Lorentz group while $\Psi$ and $\bar{\Psi}$ span the spin $\frac{1}{2}$ representation.