Mid-term exam TQC 2019-2020

Preliminaries

We say that a monomial is of degree n in the fields $\varphi_i(x)$ if it is proportional to the product of n fields: $\varphi_{i_1}(x)\cdots\varphi_{i_n}(x)$.

We consider in the first and second exercices below, transformations that are independent of space and time, that is, that are such that $x'_\mu = x_\mu$. They therefore act the same way on $\varphi(x)$ and $\varphi(x')$ where $\varphi(x)$ is any field. We call them "internal" transformations: they commute with all transformations of the Poincaré group.

Exercice 1

We consider two real scalar fields $\varphi_1(x)$ and $\varphi_2(x)$. The lagrangian $\mathcal{L}_0$ reads:

$$\mathcal{L}_0 = \frac{1}{2}\Big(\partial_\mu\varphi_1(x)\partial^\mu\varphi_1(x) + \partial_\mu\varphi_2(x)\partial^\mu\varphi_2(x)\Big) \tag{1}$$

1. The most general linear internal transformation on the fields $\varphi_1(x)$ and $\varphi_2(x)$ is such that the transformed fields $\varphi'_1(x)$ and $\varphi'_2(x)$ are linear functions of $\varphi_1(x)$ and $\varphi_2(x)$: the transformation can be realized as a matrix acting on $\varphi_1(x)$ and $\varphi_2(x)$. Determine the complete set of these transformations that leave $\mathcal{L}_0$ invariant. Give a name to this group of transformations.

2. To retrieve the result of question 1, it is convenient to gather $\varphi_1(x)$ and $\varphi_2(x)$ into a single "vector" of components: $\varphi_i(x)$ with $i = 1, 2$. Rewrite $\mathcal{L}_0$ in terms of this vector and retrieve the result of question 1 in a trivial way.

3. A third possibility to retrieve the result of question 1 is to gather $\varphi_1(x)$ and $\varphi_2(x)$ into a complex scalar field: $\varphi(x) = \frac{\varphi_1(x)+i\varphi_2(x)}{\sqrt{2}}$. Rewrite $\mathcal{L}_0$ in terms of $\varphi(x)$. Find all the linear transformations acting on $\varphi(x)$ that leave $\mathcal{L}_0$ invariant. How would you call this group of transformations? Are the two groups found in questions 1 and 2 identical (give a precise meaning to the word "identical")?

4. We now add a mass term to $\mathcal{L}_0$ and we call $\mathcal{L}'_0$ the resulting lagrangian. It reads:

$$\mathcal{L}'_0 = \mathcal{L}_0 - \frac{1}{2}m_1^2\varphi_1^2(x) - \frac{1}{2}m_2^2\varphi_2^2(x).$$

Find under which condition is $\mathcal{L}'_0$ invariant under the transformations found in question 1? Rewrite $\mathcal{L}'_0$ using the "vector" introduced in question 2 and also the complex field $\varphi(x)$ introduced in question 3.

4. Find the term of lowest degree in the fields $\varphi_i(x)$ which is larger than two and which is invariant under the group found in question 1. Rewrite it in terms the "vector" introduced in question 2 and also the complex field $\varphi(x)$ introduced in question 3.

5. We call "invariant" a tensor whose components are unchanged under any transformation of the group. We define the tensor ϵ_{ij} with i, j running on the two values 1 and 2 and such that in a given basis $\epsilon_{12} = 1$ and $\epsilon_{ij} = -\epsilon_{ji}$ for all i and j. Show taht it is an invariant tensor for the subgroup of the group found in question 1 which is connected to the identity. Find how ϵ_{ij} transforms under the other part (the one which is not connected to the identity) of the group found in question 1.

6. We consider two vectors of components $\varphi_i(x)$ and $\psi_i(x)$ with $i = 1, 2$ similar to the one introduced in question 2, that is, they both transform in the same way as the vector $\varphi_i(x)$ introduced in question 2. How does $\epsilon_{ij}\varphi_i(x)\psi_j(x)$ transform under the group found in question 1?

7. Compute the Noether current associated with the lagrangian $\mathcal{L}'_0$ when the condition found in question 3 is fulfilled. What is the tensor nature of this current for the group found in question 1. Show that it is conserved for physical fields, that is, fields satisfying the Euler-Lagrange equations.

Exercice 2: This exercice is independent of exercice 1. However, it is recommended to solve it after exercice 1.

1. We consider N real scalar fields $\varphi_1(x), \cdots, \varphi_N(x)$. The lagrangian $\mathcal{L}_0$ reads:

$$\mathcal{L}_0 = \frac{1}{2}\Big(\partial_\mu\varphi_1(x)\partial^\mu\varphi_1(x) + \cdots + \partial_\mu\varphi_N(x)\partial^\mu\varphi_N(x)\Big) - \frac{1}{2}m^2\Big(\varphi_1^2(x) + \cdots + \varphi_N^2(x)\Big) \tag{2}$$

m is called the mass of the fields. We define $\vec{\varphi}(x) = (\varphi_1(x), \cdots, \varphi_N(x))$.

1. Rewrite $\mathcal{L}_0$ in terms of $\vec{\varphi}(x)$ and find its invariance group. We call $\mathcal{L}_0\big(\vec{\varphi}(x), \partial_\mu\vec{\varphi}(x)\big)$ this lagrangian.

2. We now consider two real scalar N-component fields $\vec{\varphi}_1(x)$ and $\vec{\varphi}_2(x)$ having the same mass. We choose for lagrangian of this model: $\mathcal{L}'_0 = \mathcal{L}_0\big(\vec{\varphi}_1(x), \partial_\mu\vec{\varphi}_1(x)\big) + \mathcal{L}_0\big(\vec{\varphi}_2(x), \partial_\mu\vec{\varphi}_2(x)\big)$. Write explicitly $\mathcal{L}'_0$ in terms of the

components of both $\vec{\varphi}_1(x)$ and $\vec{\varphi}_2(x)$. What is the invariance group of $\mathcal{L}'_0$, that is, the largest group of transformations of the fields leaving $\mathcal{L}'_0$ invariant?

3. Find a term of degree 4 in the fields which is invariant under the group found in question 2. We call this term $U_1(\vec{\varphi}_1(x), \vec{\varphi}_2(x))$. Are there other terms of degree 4 in the fields that are invariant under this group (no general proof needed)?

4. We now consider the term of degree 4 that reads:

$$U_2(\vec{\varphi}_1(x), \vec{\varphi}_2(x)) = \vec{\varphi}_1^2(x)\,\vec{\varphi}_2^2(x) - (\vec{\varphi}_1(x) \cdot \vec{\varphi}_2(x))^2. \tag{3}$$

Gather the components of the two vectors $\vec{\varphi}_1(x)$ and $\vec{\varphi}_2(x)$ into a $N \times 2$ rectangular matrix $\Phi(x)$ (N rows and 2 columns) and show

(a) that the lagrangian $\mathcal{L}'_0$ can be written in terms of $\Phi(x)$,

(b) that $U_1(\vec{\varphi}_1(x), \vec{\varphi}_2(x))$ and $U_2(\vec{\varphi}_1(x), \vec{\varphi}_2(x))$ can be written in terms of a trace and a determinant of matrices built with $\Phi(x)$.

5. We define the lagrangian:

$$\mathcal{L} = \mathcal{L}'_0 - v_1 U_1(\vec{\varphi}_1(x), \vec{\varphi}_2(x)) - v_2 U_2(\vec{\varphi}_1(x), \vec{\varphi}_2(x)) \tag{4}$$

where v_1 and v_2 are real numbers. We consider the transformation of the fields induced by the transformation of $\Phi(x)$:

$$\Phi'(x) = R\,\Phi(x)\,U \tag{5}$$

where R and U are matrices. Find the groups of matrices R and U that leave $\mathcal{L}$ invariant. Give a name (and justify it) to the invariance group of $\mathcal{L}$.

Exercice 3:

We are interested in this exercice in the $(\frac{1}{2}, \frac{1}{2})$ representation of the Lorentz group. This representation is obtained by making the tensor product of the representations: $(\frac{1}{2}, 0) \otimes (0, \frac{1}{2})$.

1. The left spin 1/2 representation in $(\frac{1}{2}, \frac{1}{2})$ acts in an Hilbert space H_L spanned by two vectors that we call $\{|+\rangle_L, |-\rangle_L\}$. Similarly, the right spin 1/2 representation acts in an Hilbert space H_R spanned by two vectors that we call $\{|+\rangle_R, |-\rangle_R\}$. In which Hilbert space does the representation $\frac{1}{2} \otimes \frac{1}{2}$ act? What are the basis vectors of this space? Give them a convenient name.

2. Compute the matrix elements of Q_i and N_i in this basis.

3. Find the generators J_i of the rotations and the generators K_i of the Lorentz boosts in the basis found in question 1.

4. Check that $[J_1, J_2]$, $[J_1, K_2]$, $[K_1, K_2]$ are what they should be.

5. Consider the Lie algebra of a group G with hermitic generators $\{t_a\}$ that satisfy $[t_a, t_b] = i f_{abc} t_c$ and consider a representation of this Lie algebra where the t_a are $N \times N$ matrices. Show that if U is a unitary matrix of dimension N, then the matrices $t'_a = U t_a U^{-1}$ are also an hermitic representation of the same Lie algebra.

6. We now assume that the elements of the group G can all be obtained by "exponentiating the Lie algebra". Then, show that the exponentiations of the $\{t_a\}$ and of the $\{t'_a\}$ lead to two equivalent representations of G. Explain what the matrix U corresponds to in the representation space, that is, in the vector space in which the group elements act.

7. We define the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \end{pmatrix}. \tag{6}$$

Compute the new generators of the rotations and of the boosts $J'_i = U J_i U^{-1}$ and $K'_i = U K_i U^{-1}$. Conclude about the representation $(\frac{1}{2}, \frac{1}{2})$. In practice, with the Weyl or Dirac spinors, have you already encountered objects such that their product spans a representation of spin higher than 1/2?