Solution of the mid term exam 2020

Exercice 1 : Invariance under dilatations

1.1.1

$$S_{kin}[\phi'] = \int d^4x' \, \partial'_\mu \phi'(x') \, \partial'^\mu \phi'(x')$$

(x' is a dummy variable ⇒ we could have called it x)

$$= \int d^4x' \, d^{-2D_\phi} \, \partial'_\mu \phi(d^{-1}x') \partial'^\mu \phi(d^{-1}x')$$

$(x'^\mu = d x^\mu)$

$$= \int d^4x \, d^4 \, d^{-2D_\phi} \, d^{-2} \, \partial_\mu \phi(x) \partial^\mu \phi(x)$$

$$S_{kin}[\phi'] = S_{kin}[\phi] \quad \Rightarrow \quad \underline{D_\phi = 1}$$

Remark: it is important here to consider S and not $\mathcal{L}$ since the jacobian is not vanishing: $d^4x' = d^4 d^4x$

1.1.2 For the mass and potential term:

$$S_{pot}[\phi'] = \int d^4x' \left\{ -\frac{1}{2} m^2 \phi'^2(x') - g \, \phi'^p(x') \right\}$$

$$= \int d^4x' \left\{ -\frac{1}{2} m^2 d^{-2} \phi^2(d^{-1}x') - g \, d^{-p} \phi^p(d^{-1}x') \right\}$$

$$= \int d^4x \, d^4 \left\{ -\frac{1}{2} m^2 d^{-2} \phi^2(x) - g \, d^{-p} \phi^p(x) \right\}$$

$$S_{pot}[\phi'] = S_{pot}[\phi] \quad \Rightarrow \quad \begin{cases} m = 0 \\ p = 4 \end{cases}$$

1.2.3 $*$ $\phi'(x) = d^{-1} \phi(d^{-1}x)$ $\qquad d = 1 + \varepsilon \qquad \varepsilon \ll 1$

$$= (1-\varepsilon) \, \phi\big((1-\varepsilon)x^\mu\big)$$

$$= (1-\varepsilon)\big(\phi(x) - \varepsilon x^\mu \partial_\mu \phi(x)\big) + O(\varepsilon^2)$$

$$= \phi(x) - \varepsilon\phi(x) - \varepsilon x^\mu \partial_\mu \phi(x) + O(\varepsilon^2)$$

$$\Big(\phi((1-\varepsilon)x^\mu) = \phi(x^0 - \varepsilon x^0, x^1 - \varepsilon x^1, \dots) = \phi(x^\mu) - \varepsilon x^0 \frac{\partial}{\partial x^0}\phi(x) \dots = \phi(x) - \varepsilon x^\mu \partial_\mu \phi(x) \Big)$$

$*$ $x'^\mu = d \, x^\mu$

$$= (1+\varepsilon) x^\mu$$

$$= x^\mu + \varepsilon x^\mu$$

<u>1.2.4 & 1.2.5</u> We can repeat the different steps of the derivation of the Noether current seen during the lectures.

We call $\mathcal{L}(x') = \mathcal{L}(\phi(x'), \partial'_\mu \phi(x'))$

and $\mathcal{L} = \mathcal{L}(x) = \mathcal{L}(\phi(x), \partial_\mu \phi(x))$

(1) $$\int_{V'} d^4x' \, \mathcal{L}(x') = \int_V d^4x \, \mathcal{L}(x) + \int_V d^4x \, \partial_\mu (\mathcal{L} \delta x^\mu)$$

which becomes here:

$$\int_{V'} d^4x' \, \mathcal{L}(x') = \int_V d^4x \, (1+4\epsilon)(\mathcal{L} + \epsilon x^\mu \partial_\mu \mathcal{L})$$
$$= \int_V d^4x \, (\mathcal{L} + 4\epsilon\mathcal{L} + \epsilon x^\mu \partial_\mu \mathcal{L})$$

(2) $$\int_V d^4x \, \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) = \int_V d^4x \, \mathcal{L} + \int_V d^4x \left(\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$$
$$= S + \int_V d^4x \, \delta\phi \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)}_{\text{Euler-Lagrange}} + \int_V d^4x \, \partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$$

We can now compute the variation of the action and impose that it vanishes

$$S' = \int_{V'} d^4x' \, \mathcal{L}\left(\phi'(x'), \partial'_\mu \phi'(x')\right)$$

(Notice that x' is a dummy variable: we could have called it x here also)

$$= \int_V d^4x \left(\mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) + \partial_\mu \left(\mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) \, \delta x^\mu \right) \right)$$

can be replaced by $\phi(x)$ up to $O(\epsilon^2)$ terms

$$= S + \int_V d^4x \, \delta\phi \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) + \int_V d^4x \, \partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \mathcal{L} \delta x^\mu \right)$$

for the physical fields $\Leftrightarrow$ solutions of the Euler-Lagrange equations, we thus obtain (since $S' = S \ \forall V$):

$$\partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \mathcal{L} \delta x^\mu \right) = 0$$

$$\Rightarrow j^\mu = x_\mu \mathcal{L} - (\phi + x^\nu \partial_\nu \phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$$

We can directly check that it is conserved when

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - g\phi^4$$

$$j_\mu = \frac{1}{2}x_\mu(\partial\phi)^2 - g\,x_\mu\phi^4 - (\phi + x^\nu\partial_\nu\phi)\,\partial_\mu\phi$$

$$\Rightarrow \partial^\mu j_\mu = \cancel{2(\partial\phi)^2} + \cancel{x^\mu\partial_\mu\partial_\nu\phi\,\partial^\nu\phi} - 4g\phi^4 - 4g\,x^\mu\partial_\mu\phi\,\phi^3$$
$$- (\phi + x^\nu\partial_\nu\phi)\,\Box\phi - (\cancel{2\partial_\mu\phi} + \cancel{x^\nu\partial_\nu\partial_\mu\phi})\,\partial^\mu\phi$$

we now use the equations of motion $\Box\phi + 4g\phi^3 = 0$

$$= -4g\phi^4 - 4g\,x^\mu\partial_\mu\phi\,\phi^3 + (\phi + x^\nu\partial_\nu\phi)\,4g\phi^3$$

$= 0$ as expected

1.2.6 $T_{\mu\nu} = -\mathcal{L}\eta_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi$

$$= -\left(\frac{1}{2}(\partial\phi)^2 - g\phi^4\right)\eta_{\mu\nu} + \partial_\mu\phi\,\partial_\nu\phi$$

We observe that with this Lagrangian $T_{\mu\nu}$ is symmetric: $T_{\mu\nu} = T_{\nu\mu}$

$$x^\mu T_{\mu\nu} = -\frac{1}{2}x_\nu(\partial\phi)^2 + g\,x_\nu\phi^4 + x^\mu\partial_\mu\phi\,\partial_\nu\phi$$

$$\Rightarrow \quad x^\mu T_{\mu\nu} = -j_\nu - \phi\,\partial_\nu\phi$$

$$\underline{j_\mu = -x^\nu T_{\mu\nu} - \frac{1}{2}\partial_\mu\phi^2}$$

Much could be said about this relation...

Exercise 2

1. We must compute $\langle\Phi|\hat{\phi}(x)|\Phi\rangle \overset{\text{def}}{=} \phi(x)$ with $|\Phi\rangle$ satisfying $a_{\vec{q}}|\Phi\rangle = \alpha(\vec{q})|\Phi\rangle$ and show that

$$(\Box + m^2)\,\phi(x) = 0$$

We first notice that $a_{\vec{q}}|\Phi\rangle = \alpha(\vec{q})|\Phi\rangle \Rightarrow \langle\Phi|a^\dagger_{\vec{q}} = \alpha^*(\vec{q})\langle\Phi|$

$$\phi(x) = \int\frac{d^3q}{(2\pi)^3\,2\omega_{\vec{q}}}\left\{\langle\Phi|a_{\vec{q}}|\Phi\rangle e^{-iqx} + \langle\Phi|a^\dagger_{\vec{q}}|\Phi\rangle e^{iqx}\right\}$$

$$= \quad '' \quad \left\{\alpha(\vec{q})e^{-iqx} + \alpha^*(\vec{q})e^{iqx}\right\}\langle\Phi|\Phi\rangle$$

Thus:

$$(\Box + m^2)\phi(x) = \int \frac{d^3q}{(2\pi)^3 2\omega_{\vec q}} \left\{ \alpha(\vec q)\,(\Box+m^2)\,e^{-iqx} + \alpha^*(\vec q)\,(\Box+m^2)\,e^{iqx} \right\} \langle \Phi | \Phi \rangle$$

By definition of $q^\mu = \begin{pmatrix} \omega_{\vec q} = \sqrt{\vec q^{\,2}+m^2} \\ q^i \end{pmatrix}$: $(\Box+m^2)\,e^{\pm iqx} = 0$.

$\Rightarrow$ $\underline{(\Box+m^2)\,\phi(x) = 0}$ (which is non trivial only if $|\Phi\rangle$ is non vanishing, of course!)

2. Assume that $|\Phi\rangle = |n\rangle = a^\dagger_{\vec k_1} \cdots a^\dagger_{\vec k_n} |0\rangle = |1_{\vec k_1}; 1_{\vec k_2}; \ldots; 1_{\vec k_n}\rangle$
(we disregard here the normalizations of the states that play no role in this argument).

Thus $a_{\vec q}|\Phi\rangle = a_{\vec q}\,|1_{\vec k_1}; \ldots; 1_{\vec k_n}\rangle$

Since $a_{\vec q}|0\rangle = 0$, the only possibility to get something non vanishing is that $\vec q$ is one of the $\vec k_i$'s. But in this case $a_{\vec q}|\Phi\rangle$ is a state describing a system of $n-1$ particles and can therefore not be proportional to $|\Phi\rangle$.
This result can be generalized to a state which is a linear combination of states having at most n particles:
$|\Phi\rangle = a_0|0\rangle + a_1|1\rangle + \cdots + a_n|n\rangle$ cannot be an eigenstate of $a_{\vec q}$. Thus it can only be an infinite series of such states.

The argument above can be retrieved in a rigorous way:

* $a_{\vec q}|0\rangle = 0$
* $a_{\vec q}\, a^\dagger_{\vec k_1}|0\rangle = [a_{\vec q}, a^\dagger_{\vec k_1}]|0\rangle = (2\pi)^3 2\omega_{\vec q}\, \delta(\vec q - \vec k_1)|0\rangle$
* $$\begin{aligned} a_{\vec q}\, a^\dagger_{\vec k_1} a^\dagger_{\vec k_2}|0\rangle &= \left(a^\dagger_{\vec k_1} a_{\vec q} + [a_{\vec q}, a^\dagger_{\vec k_1}]\right) a^\dagger_{\vec k_2}|0\rangle \\ &= \left(a^\dagger_{\vec k_1} a_{\vec q} a^\dagger_{\vec k_2} + (2\pi)^3 2\omega_{\vec q}\,\delta(\vec q-\vec k_1)\, a^\dagger_{\vec k_2}\right)|0\rangle \\ &= \left(\cancel{a^\dagger_{\vec k_1} a^\dagger_{\vec k_2} a_{\vec q}} + (2\pi)^3 2\omega_{\vec q}\left[\delta(\vec q-\vec k_1)\,a^\dagger_{\vec k_2} + \delta(\vec q-\vec k_2)\,a^\dagger_{\vec k_1}\right]\right)|0\rangle \\ &= (2\pi)^3 2\omega_{\vec q}\left(\delta(\vec q-\vec k_1)\,a^\dagger_{\vec k_2} + \delta(\vec q-\vec k_2)\,a^\dagger_{\vec k_1}\right)|0\rangle \end{aligned}$$

It is trivial to iterate this relation:

* $$\begin{aligned} a_{\vec q}\, a^\dagger_{\vec k_1} \cdots a^\dagger_{\vec k_n}|0\rangle = (2\pi)^3 2\omega_{\vec q}\Big(&\delta(\vec q-\vec k_1)\, a^\dagger_{\vec k_2} \cdots a^\dagger_{\vec k_n} \\ &+ \delta(\vec q-\vec k_i)\, a^\dagger_{\vec k_1} \cdots a^\dagger_{\vec k_{i-1}} a^\dagger_{\vec k_{i+1}} \cdots a^\dagger_{\vec k_n} \\ &+ \cdots + \delta(\vec q-\vec k_n)\, a^\dagger_{\vec k_1} \cdots a^\dagger_{\vec k_{n-1}} \Big)|0\rangle \end{aligned}$$

which is not proportional to $a^\dagger_{\vec k_1} \cdots a^\dagger_{\vec k_n}|0\rangle$ even after integration on momenta.

3. $|\Phi\rangle$ is taken as a general series of n-particle states:

$$|\Phi\rangle = N \sum_{n=0}^{\infty} C_n \int \frac{d^3k_1}{(2\pi)^3} \cdots \frac{d^3k_n}{(2\pi)^3} \, z(\vec{k}_1)\cdots z(\vec{k}_n)\, a^\dagger_{\vec{k}_1}\cdots a^\dagger_{\vec{k}_n} |0\rangle$$

$$a_{\vec{q}}|\Phi\rangle = N \sum_{n=0}^{\infty} C_n \int \prod_{i=1}^{n}\left(\frac{d^3k_i}{(2\pi)^3} z(\vec{k}_i)\right) a_{\vec{q}}\, a^\dagger_{\vec{k}_1}\cdots a^\dagger_{\vec{k}_n}|0\rangle$$

$$= \quad // \qquad (2\pi)^3\, 2\omega_{\vec{q}} \left\{ \delta(\vec{q}-\vec{k}_1)\, a^\dagger_{\vec{k}_2}\cdots a^\dagger_{\vec{k}_n} + \cdots + \delta(\vec{q}-\vec{k}_n)\, a^\dagger_{\vec{k}_1}\cdots a^\dagger_{\vec{k}_{n-1}} \right\}|0\rangle$$

$$= N \sum_{n=1}^{\infty} n\, C_n \int \prod_{i=1}^{n-1}\left(\frac{d^3k_i}{(2\pi)^3} z(\vec{k}_i)\right) a^\dagger_{\vec{k}_1}\cdots a^\dagger_{\vec{k}_{n-1}}\, 2\omega_{\vec{q}}\, z(\vec{q})$$

$$= N\, 2\omega_{\vec{q}}\, z_{\vec{q}} \sum_{n=0}^{\infty} (n+1)\, C_{n+1} \int \frac{d^3k_1}{(2\pi)^3}\cdots\frac{d^3k_n}{(2\pi)^3}\, z(\vec{k}_1)\cdots z(\vec{k}_n)\, a^\dagger_{\vec{k}_1}\cdots a^\dagger_{\vec{k}_n}|0\rangle$$

$$= \alpha(\vec{q})\, N \sum_{n=0}^{\infty} \quad C_n \qquad //$$

$$\Rightarrow \begin{cases} \alpha(\vec{q}) = 2\omega_{\vec{q}}\, z(\vec{q}) \\ C_{n+1} = \dfrac{C_n}{n+1} \end{cases} \Rightarrow \quad C_n = \frac{C_0}{n!}$$

4.

$$|\Phi\rangle = N C_0\, e^{\int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \alpha(\vec{k})\, a^\dagger_{\vec{k}}} |0\rangle$$

5.

$$\langle\Phi|\Phi\rangle = |NC_0|^2 \langle 0|\, e^{\underbrace{\int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \alpha^*(\vec{k})\, a_{\vec{k}}}_{A}}\; e^{\underbrace{\int \frac{d^3k'}{(2\pi)^3 2\omega_{\vec{k}'}} \alpha(\vec{k}')\, a^\dagger_{\vec{k}'}}_{B}} |0\rangle$$

$$[A,B] = \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \frac{d^3k'}{(2\pi)^3 2\omega_{\vec{k}'}} \alpha^*(\vec{k})\,\alpha(\vec{k}')\, [a_{\vec{k}}, a^\dagger_{\vec{k}'}]$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} |\alpha(\vec{k})|^2 \; \mathbb{1}$$

since $[A,B]$ is proportional to the identity, it commutes with both A and B and the Campbell-Hausdorf formula boils down to $\quad e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$ (no nested commutator)

Thus:

$$\langle\Phi|\Phi\rangle = |NC_0|^2 \langle 0|\, e^{\int \frac{d^3k}{(2\pi)^3 2\omega_{\vec{k}}} \left(\alpha_{\vec{k}} a^\dagger_{\vec{k}} + \alpha^*_{\vec{k}} a_{\vec{k}}\right) + \frac{|\alpha_{\vec{k}}|^2}{2}} |0\rangle$$

But it is not so easy to compute $\langle\Phi|\Phi\rangle$ with this expression because $\quad \langle 0|(\alpha_{\vec{k}_1} a^\dagger_{\vec{k}_1} + \alpha^*_{\vec{k}_1} a_{\vec{k}_1})\cdots(\alpha_{\vec{k}_n} a^\dagger_{\vec{k}_n} + \alpha^*_{\vec{k}_n} a_{\vec{k}_n})|0\rangle$ is not simple. What is simple to compute is $A|0\rangle$ and $\langle 0|B$. The idea is therefore to commute A and B.

$$e^B e^A = e^{A+B+\frac{1}{2}[B,A]}$$
$$= e^{A+B+\frac{1}{2}[A,B]-[A,B]}$$
$$= e^{A+B+\frac{1}{2}[A,B]} e^{-[A,B]} \qquad \text{because } [A,B] \propto 1\!\!1$$
$$= e^A e^B e^{-[A,B]}$$

$$\Rightarrow \quad \langle 0| e^A e^B |0\rangle = \langle 0| e^B e^A e^{[A,B]} |0\rangle = e^{[A,B]} \langle 0| e^B e^A |0\rangle$$

$$e^A|0\rangle = |0\rangle \quad \text{and} \quad \langle 0| e^B = \langle 0|$$

$$\Rightarrow \quad \langle \Phi|\Phi\rangle = |NC_0|^2 \, e^{\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k} |\alpha(\vec{k})|^2}$$

we can choose $NC_0 \in \mathbb{R}$ $\Rightarrow$ $\underline{NC_0 = e^{-\frac{1}{2}\int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} |\alpha(\vec{k})|^2}}$

<u>6.</u> We want to show that there is no state satisfying

$$\langle \Phi | \hat{\psi}_\alpha(x) | \Phi \rangle = \psi_\alpha(x) \quad \text{with} \quad (i\not{\partial} - m)\psi(x) = 0$$

or that there is no eigenstate of $b_{s,\vec{q}}$ and/or $c_{s,\vec{q}}$ where

$$\hat{\psi}(x) = \sum_{s=1}^{2} \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_{\vec{k}}} \left\{ b_{s,\vec{k}} u_{s,\vec{k}} e^{-ik\cdot x} + c^\dagger_{s,\vec{k}} v_{s,\vec{k}} e^{ik\cdot x} \right\}$$

The argument follows the one of the scalar field with the difference that now the b and c operators satisfy anti commutation relations that prevent two particles to be in the same state. Put it differently, $|\Phi\rangle$ is a general linear combination of states:

$$|b, \vec{k}_1, s_1; \dots; b, \vec{k}_n, s_n; c, \vec{k}'_1, s'_1; \dots c\, \vec{k}'_p, s'_p\rangle$$

describing a state involving n particles of type b with momenta and spin states $\vec{k}_i, s_i$ and p particles of type c with momenta and spins $\vec{k}'_i, s'_i$.
Acting with either $b_{\vec{k},s}$ or $c_{\vec{k},s}$ on such a state gives something non vanishing only if (respectively) $(\vec{k}, s)$ is one of the $(\vec{k}_i, s_i)$ (resp. $(\vec{k}, s)$ is one of the $(\vec{k}'_i, s'_i)$).
In this case, the action of either $b_{\vec{k},s}$ or $c_{\vec{k},s}$ yields a state with one less particle that can therefore not be proportional to $|\Phi\rangle$ $\Rightarrow$ an infinite series is again needed.

We therefore consider an infinite series of states proportional to $|b\vec{k}_1 s_1; \dots; c\vec{k}'_p s'_p\rangle$.

Let us first consider the series of states involving only type b particles:

$$|\Phi\rangle = C_0|0\rangle + C_1\sum_{s_1=1}^{2}\int_{\vec{k}_1} z_1 b_1^+|0\rangle + C_2\sum_{s_1,s_2}\int_{\vec{k}_1,\vec{k}_2} z_1 z_2 b_1^+ b_2^+|0\rangle + \cdots$$

whe we have defined

$$\int_{\vec{k}_1} = \int\frac{d^3\vec{k}_1}{(2\pi)^3\, 2\omega_{\vec{k}_1}}\quad,\quad z_1 = z(\vec{k}_1, s_1)\quad,\quad b_1^+ = b^+_{\vec{k}_1,s_1}$$

(here we change the normalization of the $z(q)$ as compared to Eq. (10))

$$b_{\vec{q},s}|\Phi\rangle = C_1\sum_{s_1}\int_{\vec{k}_1} z_1 b_{\vec{q}s} b_1^+|0\rangle + C_2\sum_{s_1,s_2}\int_{\vec{k}_1,\vec{k}_2} z_1 z_2 b_{\vec{q},s} b_1^+ b_2^+|0\rangle + \cdots$$

$$= C_1\sum_{s_1}\int_{\vec{k}_1} z_1(2\pi)^3 2\omega_{\vec{k}_1}\delta(\vec{k}_1-\vec{q})\delta_{ss_1}|0\rangle$$

$$\cancel{+ C_2\sum_{s_1 s_2}\int_{\vec{k}_1,\vec{k}_2} z_1 z_2 (2\pi)^3 2\omega_{\vec{q}}\left(\delta(\vec{k}_1-\vec{q})\delta_{ss_1} b_2^+|0\rangle - \delta(\vec{k}_2-\vec{q})\delta_{ss_2} b_1^+|0\rangle\right)}$$

$$+C_3\sum_{s_1 s_2 s_3}\int_{\vec{k}_1,\vec{k}_2,\vec{k}_3} z_1 z_2 z_3\, 2\omega_{\vec{q}}\left(\delta(\vec{k}_1-\vec{q})\delta_{ss_1} b_2^+ b_3^+ \cancel{- \delta(\vec{k}_2-\vec{q})\delta_{ss_2} b_1^+ b_3^+} + \delta(\vec{k}_3-\vec{q})\delta_{ss_3} b_1^+ b_2^+\right)|0\rangle$$

$$+ \cdots$$

$$= C_1 z_{\vec{q},s}|0\rangle + C_3\sum_{s_1,s_2}\int_{\vec{k}_1,\vec{k}_2} z_{\vec{q},s} z_1 z_2 b_1^+ b_2^+|0\rangle + C_5\cdots$$

If $|\Phi\rangle$ is an eigenstate of $b_{\vec{q}s}$: $b_{\vec{q},s}|\Phi\rangle = \beta_{\vec{q},s}|\Phi\rangle$ then:

(notice that the c particles are spectators in this argument that can be generalized to include them)

$$C_1 z(\vec{q}, s) = \beta_{\vec{q},s} C_0$$
$$0 = \beta_{\vec{q},s} C_1$$
$$C_3 z(\vec{q}, s) = C_2 \beta_{\vec{q},s}$$
$$0 = C_3 \beta_{\vec{q},s}$$
$$\vdots$$

Thus $C_i = 0$, $\forall i$ and $|\Phi\rangle = 0$. The argument can of course be repeated for the particles of type c and for a general state involving both types of particles $\Rightarrow$ no eigenstate of $b_{\vec{q},s}$ and $c_{\vec{q}s}$.

Exercice 3 — We define $\int_{\vec{k}_1} = \int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2\omega_{\vec{k}_1}}$

1.

$$D(x,y) = \langle 0 | \phi(x)\, \phi(y) | 0 \rangle$$

$$= \int_{\vec{k}_1,\vec{k}_2} \langle 0 | \left(a_{\vec{k}_1} e^{-ik_1 \cdot x} \cancel{+ a^\dagger_{\vec{k}_1} e^{ik_1 x}}\right)\left(\cancel{a_{\vec{k}_2} e^{-ik_2 y}} + a^\dagger_{\vec{k}_2} e^{ik_2 y}\right) | 0 \rangle$$

$$= \int_{\vec{k}_1,\vec{k}_2} \langle 0 | a_{\vec{k}_1} a^\dagger_{\vec{k}_2} | 0 \rangle\, e^{-ik_1 x}\, e^{ik_2 y}$$

$$= \int_{\vec{k}_1,\vec{k}_2} (2\pi)^3\, 2\omega_{\vec{k}_2}\, \delta(\vec{k}_1 - \vec{k}_2)\, e^{-ik_1 x + i k_2 y}$$

$$= \int \frac{d^3 k}{(2\pi)^3\, 2\omega_{\vec{k}}}\, e^{-ik\cdot(x-y)}$$

2.

$$D_{\alpha\beta}(x,y) = \langle 0 | \psi_\alpha(x)\, \bar{\psi}_\beta(y) | 0 \rangle$$

$$D_{\alpha\beta}(x,y) = \sum_{s_1,s_2} \int_{\vec{k}_1,\vec{k}_2} \langle 0 | \left(b_{\vec{k}_1,s_1} u_{s_1,\alpha}(\vec{k}_1) e^{-ik_1 x} + \cancel{c^\dagger_{\vec{k}_1,s_1} v_{s_1,\alpha}(\vec{k}_1) e^{ik_1 x}} \right)$$

$$\cdot \left(b^\dagger_{\vec{k}_2,s_2} \bar{u}_{s_2,\beta}(\vec{k}_2) e^{ik_2 y} + \cancel{c_{\vec{k}_2,s_2} \bar{v}_{s_2,\beta}(\vec{k}_2) e^{-ik_2 y}} \right) | 0 \rangle$$

$$= \sum_{s_1,s_2} \int_{\vec{k}_1,\vec{k}_2} \langle 0 | b_{\vec{k}_1,s_1}\, b^\dagger_{\vec{k}_2,s_2} | 0 \rangle\, u_{s_1,\alpha}(\vec{k}_1)\, \bar{u}_{s_2,\beta}(\vec{k}_2)\, e^{-ik_1 x + i k_2 y}$$

$$= \int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2\omega_{\vec{k}_1}} \sum_{s_1} u_{s_1,\alpha}(\vec{k}_1)\, \bar{u}_{s_1,\beta}(\vec{k}_1)\, e^{-ik_1(x-y)}$$

$$= \int \frac{d^3\vec{k}_1}{(2\pi)^3\, 2\omega_{\vec{k}_1}} (\not{k}_1 + m\mathbb{1})_{\alpha\beta}\, e^{-ik_1\cdot(x-y)}$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_{\vec{k}}} (i\not{\partial}_x + m\mathbb{1})_{\alpha\beta}\, e^{-ik(x-y)} \qquad \left(\not{\partial}_x = \gamma^\mu \frac{\partial}{\partial x^\mu} \right)$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3\, 2\omega_{\vec{k}}} (-i\not{\partial}_y + m\mathbb{1})_{\alpha\beta}\, e^{-ik(x-y)}$$

$$= (i\not{\partial}_x + m\mathbb{1})_{\alpha\beta} \int \frac{d^3 k}{(2\pi)^3\, 2\omega_{\vec{k}}}\, e^{-ik(x-y)} = (-i\not{\partial}_y + m\mathbb{1})_{\alpha\beta} \int_{\vec{k}} e^{-ik(x-y)}$$