0.1 E1

$$A^{\mu\nu}S_{\mu\nu} = A^{\rho\sigma}S_{\rho\sigma} = A^{\rho\sigma}S_{\sigma\rho} = -A^{\sigma\rho}S_{\sigma\rho} = -A^{\mu\nu}S_{\mu\nu}. \tag{1}$$

Thus $A^{\mu\nu}S_{\mu\nu} = 0$.

0.2 E2

By definition :

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \quad \partial^\mu = \frac{\partial}{\partial x_\mu}. \tag{2}$$

It follows that

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \frac{\partial x_\nu}{\partial x^\mu}\frac{\partial}{\partial x_\nu} = \frac{\partial(\eta_{\nu\rho}x^\rho)}{\partial x^\mu}\frac{\partial}{\partial x_\nu} = \eta_{\nu\rho}\delta^\rho_\mu\frac{\partial}{\partial x_\nu} = \eta_{\nu\mu}\frac{\partial}{\partial x_\mu} = \eta_{\nu\mu}\partial^\nu \tag{3}$$

Thus :

$$\partial_\mu = \eta_{\nu\mu}\partial^\nu \quad \text{and} \quad \partial^\mu = \eta^{\nu\mu}\partial_\nu. \tag{4}$$

We can prove by brute force (good exercice) that ∂_μ transforms as a covariant vector and that ∂^μ transforms as a contravariant vector. We can also guess that the reason why differentiating with respect to a contravariant vector yields a covariant vector is that we *remove* a contravariant vector when differentiating. Therefore, an elegant solution is to consider the scalar $S = x^\nu y_\nu$ and to act with ∂_μ :

$$\partial_\mu S = \partial_\mu(x^\nu y_\nu) = \delta^\nu_\mu y_\nu = y_\mu. \tag{5}$$

y_μ being a covariant vector and S a scalar, ∂_μ must be a covariant vector.

0.3 E3

T is a tensor, then :

$$T'_{\mu\nu} = \Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}T_{\mu'\nu'}. \tag{6}$$

For its trace we get :

$$T'^\nu{}_\nu = \eta^{\nu\mu}T'_{\mu\nu} = \eta^{\nu\mu}\Lambda_\mu{}^{\mu'}\Lambda_\nu{}^{\nu'}T_{\mu'\nu'} = \eta_{\nu\mu}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}T^{\mu'\nu'} = \eta_{\nu'\mu'}T^{\mu'\nu'} \tag{7}$$

and thus

$$T'^\nu{}_\nu = T^\nu{}_\nu \tag{8}$$

which means that the trace is invariant as expected for contracted indices.

0.4 E4

In the limit of very small velocity v, we find that $x' = x - vt + O(v^2)$ which means that O' is moving with a velocity v with respect to O in the x direction.

We rewrite the Lorentz transformation in a vectorial form

$$\begin{cases} t' = t + (\gamma - 1)(t - vx) - vx \\ x' = x + (\gamma - 1)x - \gamma vt \\ y' = y \\ z' = z \end{cases} \tag{9}$$

We use the fact that for this particular transformation : $x = (\vec{r}.\vec{v})\,\vec{v}/\vec{v}^{\,2}$ and $\vec{v}$ is colinear with $\vec{r}$. This yields :

$$\Rightarrow \begin{cases} t' = t + (\gamma - 1)(t - \vec{v}.\vec{r}) - \vec{v}.\vec{r} \\ \vec{x}^{\,\prime} = \vec{r} + \dfrac{(\gamma - 1)}{\vec{v}^{\,2}}\,(\vec{r}.\vec{v})\,\vec{v} - \gamma t\vec{v} \end{cases} \tag{10}$$

The important point is that rotation invariance implies that the Lorentz boost we consider is completely general, it is our choice of axis which is not. Therefore, once we have been able to rewrite our relation in a manifestly rotation invariant way, what we have found in a particular frame is valid in all frames. We conclude that (10) is the general expression of a Lorentz boost whatever the vector $\vec{v}$ is. Notice that it would have been rather difficult to derive this relation directly. This is one example of the power of tensor calculus.

0.5 E5

The choice of matrices representing S_A is solely constrained by the multiplication table of C_{3v}. Once we have a set of matrices for the rotations and mirror symmetries that reproduce the multiplication table of C_{3v}, this is a representation. We know that S_A^2 is the identity is this is the only constraint that the matrix representing S_A should staisfy and it is satisfied by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

Thus, this is a possible choice of matrix for S_A. The other matrices representing S_B and S_C follows from the choice of matrices representing the rotations and the fact that S_B and S_C can be obtained by composition of S_A with the rotations. For instance : $S_B = S_A.R_{2\pi/3}$. Thus, if we take for the matrix representing S_B the product of the matrices representing S_A and $R_{2\pi/3}$, we obviously preserve the multiplication table of C_{3v} and we therefore obtain a representation.

With the choice (11) for the matrix representing S_A, we find that the coordinates (x, y) of a vector in the ABC plane are transformed under S_A by

$$\begin{aligned} x' &= x \\ y' &= -y. \end{aligned} \tag{12}$$

The choice of axis where this is satisfied obviously corresponds to the x-axis along the OA direction (O being the center of the triangle) and the y-axis perpendicular to the x-axis.

0.6 E6

$$J_i = -i\epsilon_{ijk}|j\rangle\langle k| \tag{13}$$

implies that

$$\begin{aligned} J_1 &= -i\epsilon_{1jk}|j\rangle\langle k| \\ &= -i(\epsilon_{123}|2\rangle\langle 3| + \epsilon_{132}|3\rangle\langle 2|) \\ &= -i(|2\rangle\langle 3| - |3\rangle\langle 2|) \end{aligned} \tag{14}$$

and thus

$$J_1(x|1\rangle + y|2\rangle + z|3\rangle) = i(y|3\rangle - z|2\rangle). \tag{15}$$

and thus

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{16}$$

which is indeed the generator of SO(3) in the $\hat{x}$ direction. We now check the Lie algebra.

$$\begin{aligned} J_i J_j &= -(\epsilon_{ikl}|k\rangle\langle l|)(\epsilon_{jmn}|m\rangle\langle n|) \\ &= -\epsilon_{ikl}\epsilon_{jmn}|k\rangle\langle l|m\rangle\langle n| \\ &= -\epsilon_{ikl}\epsilon_{jln}|k\rangle\langle n| \\ &= \epsilon_{ikl}\epsilon_{jnl}|k\rangle\langle n| \\ &= (\delta_{ij}\delta_{kn} + \delta_{in}\delta_{kj})|k\rangle\langle n| \\ &= \delta_{ij}|k\rangle\langle k| + |j\rangle\langle i|. \end{aligned} \tag{17}$$

Thus, we find

$$[J_i, J_j] = |j\rangle\langle i| - |i\rangle\langle j|. \tag{18}$$

For instance :

$$\begin{aligned} [J_1, J_2] &= |2\rangle\langle 1| - |1\rangle\langle 2| \\ &= iJ_3 \\ &= i\epsilon_{123}J_3 \\ &= i\epsilon_{12k}J_k. \end{aligned} \tag{19}$$

This is trivially generalized to the other indices and thus the Lie algebra of SO(3) is satisfied as expected.

0.7 E7

See the notes (in french) “Un soupçon de théorie des groupes”.

0.8 E8

In a particular frame

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{20}$$

In a rotated frame, we obtain by definition of a tensor :

$$\begin{aligned} \epsilon'_{\alpha\beta} &= U_{\alpha\gamma} U_{\beta\delta} \epsilon_{\gamma\delta} \\ &= U_{\alpha\gamma} \epsilon_{\gamma\delta} \left({}^t U\right)_{\delta\beta}. \end{aligned} \tag{21}$$

from a matrix point of view, this means that

$$\epsilon' = U \epsilon {}^t U. \tag{22}$$

Taking the general parametrization of a SU(2) matrix

$$U = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \tag{23}$$

with $|a|^2 + |b|^2 = 1$, we easily find that $\epsilon'_{\alpha\beta} = \epsilon_{\alpha\beta}$. Thus ϵ is an invariant tensor.

(ii) The Lorentz matrices are defined by

$$\eta_{\mu\nu} \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} = \eta_{\mu'\nu'}. \tag{24}$$

and

$$\left(\Lambda^{-1}\right)^{\mu}{}_{\mu'} = \Lambda_{\mu'}{}^{\mu} \tag{25}$$

where, by definition, the indices in $\Lambda_{\mu'}{}^{\mu}$ have been raised and lowered from $\Lambda^{\mu}{}_{\nu}$ with the metric. By considering the metric as a (covariant) tensor and using Eq.(24), we find :

$$\begin{aligned} \eta'_{\mu\nu} &= \Lambda_{\mu}{}^{\mu'} \Lambda_{\nu}{}^{\nu'} \eta_{\mu'\nu'} \\ &= \Lambda_{\mu}{}^{\mu'} \Lambda_{\nu}{}^{\nu'} \eta_{\mu''\nu''} \Lambda^{\mu''}{}_{\mu'} \Lambda^{\nu''}{}_{\nu'} \\ &= \eta_{\mu\nu}. \end{aligned} \tag{26}$$

Thus, η is an invariant tensor. This is actually the meaning of the conservation of the distance ds^2 under Lorentz transformation because when we set : $\eta_{\mu\nu} dx^{\mu} dx^{\nu} = \eta_{\mu\nu} dx'^{\mu} dx'^{\nu}$ we already state that the metric is invariant under a Lorentz transformation and since we assume that ds^2 is scalar, $\eta_{\mu\nu}$ must be a tensor. Therefore, it must be an invariant tensor which is the property we have checked above.

(iii) Z = spinor $\Rightarrow \quad Z'_{\alpha} = U_{\alpha\beta} Z_{\beta}$. Thus

$$(Z^{\dagger} Z)' = Z^{\dagger} U^{\dagger} U Z = Z^{\dagger} Z \tag{27}$$

and thus, $Z^{\dagger} Z$ is a scalar.

$$(Z^{\dagger} \sigma_i Z)' = Z^{\dagger} U^{\dagger} \sigma_i U Z = R_{ij} Z^{\dagger} \sigma_j Z \tag{28}$$

and thus, $Z^{\dagger} \vec{\sigma} Z$ is a vector.

0.9 E9

(i) It is easy to check that $N^\dagger.N =$I and thus N is a unitary matrix. A simple calculation shows that the diagonal matrix ${J_3}^d$ reads :

$$ {J_3}^d = N J_3 N^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{29} $$

The three eigenvalues are therefore $-1, 0, 1$ and we recognize the three eigenvalues of J_z for a spin 1.

(ii) We call $\mathcal{J}_3$ the operator represented by J_3 in the cartesian basis $\{\vec{e}_i\}$ and by ${J_3}^d$ in the diagonal basis $\{\vec{e_i}^d\}$. We call P the matrix such that : $\vec{e_i}^d = P_{ij}\vec{e}_j$. We thus have :

$$ \begin{array}{ll} \mathcal{J}_3\vec{e}_i & = (J_3)_{ij}\,\vec{e}_j \\ \mathcal{J}_3\vec{e_i}^d & = ({J_3}^d)_{ij}\vec{e_j}^d = \lambda_i\vec{e_i}^d \end{array} \tag{30} $$

with λ_i the eigenvalue associated with $\vec{e_i}^d$. We obtain :

$$ \mathcal{J}_3 P_{ij}\vec{e}_j = P_{ij}\,(J_3)_{jk}\,\vec{e}_k = ({J_3}^d)_{ij} P_{jk}\vec{e}_k \tag{31} $$

and thus ${J_3}^d = P\ J_3\ P^{-1} \Rightarrow\ P = N$. We thus conclude that for a vector $\vec{V} = V_i\vec{e}_i = V_-\vec{e_1}^d + V_+\vec{e_2}^d + V_0\vec{e_3}^d$:

$$ \begin{array}{ll} V_- & = -(V_1 - iV_2)/\sqrt{2} \\ V_+ & = (V_1 + iV_2)/\sqrt{2} \\ V_0 & = V_3 \end{array} \tag{32} $$

(iii) A simple calculation shows that $\vec{\mathcal{J}}^2 = 2$I. In the language of quantum mechanics we say that we are dealing with the spin one representation of the rotation group : $j = 1$, that $\vec{\mathcal{J}}^2 = j(j+1)\,\mathrm{I}$ and that the basis vector are $|j, m\rangle$ with $m = -1, 0, 1$.

(iv) For SU(2), we find $\vec{\sigma}^2/4 = 3/4\,\mathrm{I}$ and thus $j = 1/2\ \Rightarrow$ it is the spin 1/2 representation.

0.10 E10

(i) T is a tensor $\Rightarrow T'_{ij} = R_{ik}R_{jl}T_{kl}$. Thus

$$ T'_{ii} = R_{ik}R_{il}T_{kl} = ({}^tRR)_{kl}T_{kl} = T_{kk}. \tag{33} $$

and thus $\mathrm{Tr}\,T$ is a scalar. It spans the (irreducible) scalar representation of SO(3).

If $T = \vec{x}\otimes\vec{y}\ \Rightarrow\ T_{ij} = x_i y_j$, then $\mathrm{Tr}\,T = \vec{x}\cdot\vec{y}$.

(ii)

$$ A'_{ij} = (R_{ik}R_{jl}T_{kl} - R_{jk}R_{il}T_{kl})/2 = R_{ik}R_{jl}(T_{kl} - T_{lk})/2 = R_{ik}R_{jl}A_{kl} \tag{34} $$

and thus A is a tensor $\Rightarrow$ it spans a representation of SO(3).

(iii)

$$V_i = \epsilon_{ijk} A_{jk} = \epsilon_{ijk} \left(A_{jk} + (T_{jk} + T_{kj})/2\right) = \epsilon_{ijk} T_{jk} \tag{35}$$

where the second equality comes from the fact that $\epsilon_{ijk}(T_{jk} + T_{kj}) = 0$, see exercice 1.

Under infinitesimal rotations we get :

$$\begin{aligned} V_i' &= \epsilon_{ijk} \left(\delta_{ja} + \epsilon_{jal}\delta\theta_l\right)\left(\delta_{kb} + \epsilon_{kbl}\delta\theta_l\right) T_{ab} \\ &= \epsilon_{iab} + \left(\epsilon_{iak}\epsilon_{kbl} + \epsilon_{ijb}\epsilon_{jal}\right)\delta\theta_l T_{ab} \\ &= V_i + (\delta_{ib}\delta_{al} - \delta_{ia}\delta_{bl})\delta\theta_l T_{ab} \\ &= V_i - \epsilon_{ilk}\epsilon_{abk}\delta\theta_l T_{ab} \\ &= V_i - \epsilon_{ilk}\delta\theta_l V_k. \end{aligned} \tag{36}$$

Thus $\vec{V}$ transforms as a vector and A spans therefore the (irreducible) vector representation of SO(3).

(iv) If $T = \vec{x} \otimes \vec{y} \;\Rightarrow\; \vec{V} = \vec{x} \wedge \vec{y}$. It shows that $\vec{V}$ is a pseudo-vector, that is, is a vector for SO(3) but is invariant under the inversion : $\vec{x} \to -\vec{x}$ and $\vec{y} \to -\vec{y}$ contrarily to a true vector.

(v) $\mathrm{Tr}\, S = \mathrm{Tr}\, T$ which implies that

$$\mathcal{S}_{ij} = \frac{1}{2}(T_{ij} + T_{ji}) - \frac{1}{3}\delta_{ij}\mathrm{Tr}\, T \tag{37}$$

is symmetric and traceless. As any 3×3 symmetric matrix, it involves 6 independent degrees of freedom.

(vi) We know that the trace of a tensor is invariant under SO(3). Thus, $\mathcal{S}'$ is also traceless. Let us show that it is symmetric.

$$\begin{aligned} (T_{ij} + T_{ji})' &= R_{ik}R_{jl}T_{kl} + R_{jk}R_{il}T_{kl} \\ &= R_{ik}R_{jl}T_{kl} + R_{jl}R_{ik}T_{lk} \\ &= R_{ik}R_{jl}(T_{kl} + T_{lk}) \end{aligned} \tag{38}$$

and thus

$$\mathcal{S}'_{ij} = R_{ik}R_{jl}\mathcal{S}_{kl}. \tag{39}$$

We conclude that both S and $\mathcal{S}$ span a representation of SO(3).

(vii) We trivially have $s_1 = \mathcal{S}_{11} - \mathcal{S}_{33}$, $s_2 = \mathcal{S}_{12}$, $s_3 = \mathcal{S}_{13}$, $s_4 = \mathcal{S}_{22} - \mathcal{S}_{33}$, $s_5 = \mathcal{S}_{23}$. There are five components in s and five undependent degrees of freedom in $\mathcal{S}$. Since all components of s involve either diagonal components of $\mathcal{S}$ or components in its upper right part and since none of them is redundant, we conclude that all components of s are independent.

(viii) Since S is a tensor we get in an infinitesimal rotation around $\hat{x}$:

$$\begin{array}{ll} S'_{ij} & = S_{ij} + i\delta\theta(-i)\left(\delta_{ik}\epsilon_{1jl} + \delta_{jl}\epsilon_{1ik}\right) \\ S'_{11} & = S_{11} \\ S'_{12} & = S_{12} + i\delta\theta(-i)S_{13} \\ S'_{13} & = S_{13} + i\delta\theta(-i)\left(-S_{12}\right) \\ S'_{22} & = S_{22} + i\delta\theta(-i)\left(2S_{23}\right) \\ S'_{23} & = S_{23} + i\delta\theta(-i)\left(-S_{22} + S_{33}\right) \\ S'_{33} & = S_{33} + i\delta\theta(-i)\left(-2S_{23}\right) \end{array} \tag{40}$$

We thus find

$$\mathcal{J}_1 = -i \begin{pmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}. \tag{41}$$

Similar calculations for rotations around $\hat{y}$ and $\hat{z}$ yield :

$$\mathcal{J}_2 = -i \begin{pmatrix} 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{42}$$

and

$$\mathcal{J}_3 = -i \begin{pmatrix} 0 & 2 & -4 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}. \tag{43}$$

Notice that these generators are not hermitic. By an equivalence we could make them hermitic.

(ix) The Lie algebra of SO(3) is trivially reproduced and we find that

$$\vec{\mathcal{J}}^2 = 6\,\mathrm{I}. \tag{44}$$

We thus conclude that this representation corresponds to $j = 2$: it is the "spin 2" (irreducible) representation of SO(3).

(x) If $T = \vec{x} \otimes \vec{y}$, we conclude that

- $\vec{A} \cdot \vec{B}$ is a scalar (spin 0) ;
- $\vec{A} \wedge \vec{B}$ is a (pseudo-)vector (spin 1) ;
- $\frac{1}{2}(A_i B_j + A_j B_i) - \frac{1}{3}\delta_{ij}(\vec{A} \cdot \vec{B})$ is a spin 2 tensor.

Notice that a general tensor T_{ij} involves nine degrees of freedom and we retrieve them in the scalar (1), vector (3) and spin-2 tensor (5) irreducible representations.

0.11 E11

(i) A basis in $\mathbb{C}^2 \otimes \mathbb{C}^2$ is $\{|\alpha\rangle \otimes |\beta\rangle\}$, that is, $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$.

(ii) For the tensor product, we get $W = Y \otimes Z = W_{\alpha\beta}\, |\alpha\rangle \otimes |\beta\rangle = Y_\alpha Z_\beta\, |\alpha\rangle \otimes |\beta\rangle$. It involves 2×2 complex degrees of freedom, that is 8 real degrees of freedom. Under an SU(2) infinitesimal transformation, $W_{\alpha\beta}$ transforms as :

$$W'_{\alpha\beta} = \left[e^{id\vec{\theta}\cdot\vec{\sigma}/2}\right]_{\alpha\gamma} \left[e^{id\vec{\theta}\cdot\vec{\sigma}/2}\right]_{\beta\rho} W_{\gamma\rho}. \tag{45}$$

The SU(2) transformation is thus given on $W_{\alpha\beta}$ by

$$e^{id\vec{\theta}\cdot\vec{\sigma}/2} \otimes e^{id\vec{\theta}\cdot\vec{\sigma}/2} = \mathrm{I} \otimes \mathrm{I} = id\vec{\theta} \cdot \left(\frac{\vec{\sigma}}{2} \otimes \mathrm{I} + \mathrm{I} \otimes \frac{\vec{\sigma}}{2}\right) \tag{46}$$

and the generators are therefore :

$$\mathcal{J}_i = \frac{\sigma_i}{2} \otimes \mathrm{I} + \mathrm{I} \otimes \frac{\sigma_i}{2}. \tag{47}$$

(iii) In the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis :

$$\begin{aligned} \sigma_1|\uparrow\rangle &= |\downarrow\rangle, \quad & \sigma_1|\downarrow\rangle &= |\uparrow\rangle \\ \sigma_2|\uparrow\rangle &= i|\downarrow\rangle, \quad & \sigma_2|\downarrow\rangle &= -i|\uparrow\rangle \\ \sigma_3|\uparrow\rangle &= |\uparrow\rangle, \quad & \sigma_3|\downarrow\rangle &= -|\downarrow\rangle. \end{aligned} \tag{48}$$

thus

$$\begin{aligned} \vec{\mathcal{J}}^2|\uparrow\downarrow\rangle &= |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \\ \vec{\mathcal{J}}^2|\downarrow\uparrow\rangle &= |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle \end{aligned} \tag{49}$$

and $\vec{\mathcal{J}}^2$ is therefore not diagonal in the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$. It is trivial to find the diagonal basis since

$$\begin{aligned} \vec{\mathcal{J}}^2\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) &= 0 \\ \vec{\mathcal{J}}^2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right) &= 2\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right). \end{aligned} \tag{50}$$

Moreover

$$\begin{aligned} \vec{\mathcal{J}}^2|\uparrow\uparrow\rangle &= 2|\uparrow\uparrow\rangle \\ \vec{\mathcal{J}}^2|\downarrow\downarrow\rangle &= 2|\downarrow\downarrow\rangle. \end{aligned} \tag{51}$$

The four (complex) dimensional space can therefore be splitted into two parts according to the eigenvalues of $\vec{\mathcal{J}}^2$:

- a three-dimensional space spanned by $\{|\uparrow\uparrow\rangle, \frac{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle}{\sqrt{2}}, |\downarrow\downarrow\rangle\}$ where $\vec{\mathcal{J}}^2 = 2\,\mathrm{I}$,
- a one-dimensional space (orthogonal to the 3D space) spanned by $\frac{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle}{\sqrt{2}}$ where $\vec{\mathcal{J}}^2 = 0$.

In these spaces, it is trivial to check that $\mathcal{J}_3$ is diagonal and that its eigenvalues are $1, 0, -1$ in the 3d space and 0 in the 1d space. These two spaces are of course respectively the Hilbert space for a spin one and the Hilbert space for a spin 0. The basis vectors are often denoted $\{|1,1\rangle, |1,0\rangle, |1,-1\rangle\}$ and $|0,0\rangle$.

(iv) On the spherical basis (this is its name) :

$$\begin{aligned}\mathcal{W} = \quad & Y_1 Z_1 |\uparrow\uparrow\rangle + Y_2 Z_2 |\downarrow\downarrow\rangle + \frac{1}{\sqrt{2}}(Y_1 Z_2 + Y_2 Z_1)\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}} \\ & + \frac{1}{\sqrt{2}}(Y_1 Z_2 - Y_2 Z_1)\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}.\end{aligned} \tag{52}$$

The spin 0 part, that is, the antisymmetric part is of course proportional to $\epsilon_{\alpha\beta}$: $Y_\alpha Z_\beta - Y_\beta Z_\alpha = (Y_2 Z_1 - Y_1 Z_2)\epsilon_{\alpha\beta}$.

0.12 E12

(i) $J^{\alpha\beta}$ is an antisymmetric tensor of generators of the Lorentz group (each non vanishing component $J^{\alpha\beta}$ is a generator) in much the same way as $\vec{J}$ is a vector of generators of SO(3). Being antisymmetric and since the indices run on 4 values it has $3 + 2 + 1 = 6$ independent non vanishing components.

(ii) We of course define $J_{\alpha\beta}$ from $J^{\alpha\beta}$ by manipulating its indices with the metric $\eta_{\mu\nu}$. Then, $\left(J^{\alpha\beta}\right)^{\mu}{}_{\nu}$ is the matrix element ${}^{\mu}{}_{\nu}$ of the generator $J^{\alpha\beta}$: It is a number. Let's plug

$$\left(J_{\alpha\beta}\right)^{\mu}{}_{\nu} = i\left(\delta^{\mu}_{\alpha}\eta_{\nu\beta} - \delta^{\mu}_{\beta}\eta_{\nu\alpha}\right) \tag{53}$$

into the right hand side of

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - \frac{1}{2}i\epsilon^{\alpha\beta}\left(J_{\alpha\beta}\right)^{\mu}{}_{\nu}. \tag{54}$$

We obtain :

$$\begin{aligned}\Lambda^{\mu}{}_{\nu} \quad &= \delta^{\mu}_{\nu} - \tfrac{1}{2}i\epsilon^{\alpha\beta} i\left(\delta^{\mu}_{\alpha}\eta_{\nu\beta} - \delta^{\mu}_{\beta}\eta_{\nu\alpha}\right) \\ &= \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{\nu}.\end{aligned} \tag{55}$$

This is indeed what we obtain when performing infinitesimal Lorentz transformations. It shows a posteriori that the expression in Eq.(53) is the right one.

From Eq.(53) we obtain that the only nonvanishing matrix elements of J_{01} are for $\mu = 0, \nu = 1$ with $\left(J_{01}\right)^{0}{}_{1} = -i$ and $\mu = 1, \nu = 0$ with $\left(J_{01}\right)^{1}{}_{0} = -i$. A direct comparison with the matrix elements of K_1 shows that the two matrices are proportional and that $J_{01} = -iK_1$. The same holds true for all J_{0i}.

For J_{ij}, we obtain as the only nonvanishing matrix elements are :
$\left(J_{ij}\right)^{i}{}_{j} = -i$ and $\left(J_{ij}\right)^{j}{}_{i} = i$. Here again a direct comparison with the matrices J_i shows the result.

(iii) The tedious way consists in evaluating the left and right hand sides of the Lie algebra with Eq.(53) and showing that they are the same :

$$\left[J_{\mu\nu}, J_{\rho\sigma}\right]_{\alpha\beta} = i\left(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\rho}J_{\nu\sigma} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\sigma}J_{\mu\rho}\right)_{\alpha\beta}. \tag{56}$$

A more clever way consists in realizing (i) that the right hand side of the Lie algebra is linear in the generators, (ii) must be antisymmetric in (μ, ν), in (ρ, σ) and in the exchange $(\mu, \nu) \leftrightarrow (\rho, \sigma)$ and (iii) can involve only $\eta_{\mu\nu}$, $J_{\mu\nu}$ and $\epsilon_{\mu\nu\rho\sigma}$ which is the totally antisymmetric tensor. Actually, $\epsilon_{\mu\nu\rho\sigma}$ cannot appear because of parity reasons. Then, since the right-hand side is linear in $J_{\mu\nu}$, it must also be linear in $\eta_{\mu\nu}$.

We start for instance from $\eta_{\nu\rho}J_{\mu\sigma}$ and by exchanging μ and ν we generate $-\eta_{\mu\rho}J_{\nu\sigma}$. Then, from these two terms, we generate two new ones by exchanging ρ and σ that are the last two terms in Eq.(56). We then check that the result is automatically antisymmetric in the exchange : $(\mu, \nu) \leftrightarrow (\rho, \sigma)$. The only thing to determine is now the overall normalization. We consider :
$[J_{01}, J_{02}] = -[K_1, K_2] = -iJ_3 = -\alpha\eta_{00}J_{12} = -\alpha J_3$
which implies that $\alpha = i$.

(iv) We compute the matrix elements of $i\,(|\mu\rangle\langle\nu\,| - |\nu\rangle\langle\mu\,|)$:

$$i\langle\rho\,|\,(|\mu\rangle\langle\nu\,| - |\nu\rangle\langle\mu\,|)\,|\sigma\rangle = i\,(\eta_{\rho\mu}\eta\nu\sigma - \eta_{\rho\nu}\eta\mu\sigma) \tag{57}$$

which is indeed identical to Eq.(53) in covariant coordinates :
$(J_{\mu\nu})_{\rho\sigma} = i\,(\eta_{\rho\mu}\eta_{\nu\sigma} - \eta_{\rho\nu}\eta_{\mu\sigma})$.

The Lie algebra is then simple to reproduce from :

$$\begin{aligned} J_{\mu\nu}J_{\rho\sigma} &= -\,(|\mu\rangle\langle\nu\,| - |\nu\rangle\langle\mu\,|)\,(|\rho\rangle\langle\sigma\,| - |\sigma\rangle\langle\rho\,|) \\ &= -\eta_{\nu\rho}|\mu\rangle\langle\sigma\,| + \eta_{\nu\sigma}|\mu\rangle\langle\rho\,| + \eta_{\mu\rho}|\nu\rangle\langle\sigma\,| - \eta_{\mu\sigma}|\nu\rangle\langle\rho\,|. \end{aligned} \tag{58}$$

0.13 E13

(i) The eigenvalues of M are $-1, -1$ (twice degenerate). If M can be diagonalized then there is a matrix P such that

$$PMP^{-1} = -\mathrm{I}. \tag{59}$$

But then by multiplying the previous equation by P^{-1} on the left and P on the right, we find that M is $-\mathrm{I}$. Thus, M cannot be diagonalized.

(ii) If $m \in sl(2, \mathbb{C})$ then $m = \vec{\alpha}\cdot\vec{\sigma}$ with $\alpha_i \in \mathbb{C}$. Then, Tr $m = 0$. We conclude that

$$m = \begin{pmatrix} k & a \\ b & -k \end{pmatrix}. \tag{60}$$

This matrix has two different eigenvalues and is diagonalizable. Then its exponential is diagonalizable which contradicts (i).

Another proof consists in showing that m^2 is proportional to the identity, to resum the series of $\exp(m)$ and to show that it cannot be M.

0.14 E14

(i) For a boost in the $\hat{x}$ direction :

$$\begin{aligned} (\chi_L^\dagger\psi_L)' &= \chi_L^\dagger M_1^\dagger M_1\psi_L \\ &= \chi_L^\dagger e^{d\phi\sigma_1}\psi_L \\ &= \chi_L^\dagger\psi_L + d\phi\chi_L^\dagger\sigma_1\psi_L. \end{aligned} \tag{61}$$

For the space components :

$$
\begin{aligned}
(\chi_L^\dagger \sigma_i \psi_L)' &= \chi_L^\dagger e^{d\phi\frac{\sigma_1}{2}} \sigma_i e^{d\phi\frac{\sigma_1}{2}} \psi_L \\
&= \chi_L^\dagger \left(\mathrm{I} + d\phi\frac{\sigma_1}{2}\right) \sigma_i \left(\mathrm{I} + d\phi\frac{\sigma_1}{2}\right) \psi_L \\
&= \chi_L^\dagger \sigma_i \psi_L + d\phi\, \delta_{i1}\, \chi_L^\dagger \psi_L .
\end{aligned} \tag{62}
$$

In matrix form, this is :

$$
\begin{pmatrix} \chi_L^\dagger \psi_L \\ \chi_L^\dagger \sigma_1 \psi_L \\ \chi_L^\dagger \sigma_2 \psi_L \\ \chi_L^\dagger \sigma_3 \psi_L \end{pmatrix}' = \begin{pmatrix} 1 & d\phi & & \\ d\phi & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \chi_L^\dagger \psi_L \\ \chi_L^\dagger \sigma_1 \psi_L \\ \chi_L^\dagger \sigma_2 \psi_L \\ \chi_L^\dagger \sigma_3 \psi_L \end{pmatrix} \tag{63}
$$

This is nothing but the transformations of the covariant coordinates of a 4-vector. Of course, this can be done with the other Lorentz boosts and with similar results. Thus, $(\chi_L^\dagger \psi_L, \chi_L^\dagger \sigma_i \psi_L)$ are the covariant coordinates of a 4-vector.

(ii) For $(\chi_R^\dagger \psi_R, \chi_R^\dagger \sigma_i \psi_R)$ the same kind of calculations would lead to the conclusion that they are the contravariant coordinates of a 4-vector.

(iii) Since under parity : $\chi_L \leftrightarrow \chi_R$ and since both $(\chi_R^\dagger \psi_R, \chi_R^\dagger \sigma_i \psi_R)$ and $(\chi_L^\dagger \psi_L, -\chi_L^\dagger \sigma_i \psi_L)$ are contravariant coordinates of two 4-vectors we find that :

a. $(\chi_L^\dagger \psi_L + \chi_R^\dagger \psi_R, -\chi_L^\dagger \sigma_i \psi_L + \chi_R^\dagger \sigma_i \psi_R)$ is a true 4-vector because its time component does not change sign under parity while its space components do ;

b. $(\chi_L^\dagger \psi_L - \chi_R^\dagger \psi_R, -\chi_L^\dagger \sigma_i \psi_L - \chi_R^\dagger \sigma_i \psi_R)$ is a pseudo 4-vector because its time component changes sign under parity while its space components do not.

(iv) The above 4-vectors can be conveniently rewritten in a manifestly covariant way with the set of γ^μ matrices :

$$
\begin{aligned}
\bar{\chi}\gamma^0\psi &= \chi^\dagger \gamma^0 \gamma^0 \psi \\
&= \left(\chi_L^\dagger, \chi_R^\dagger\right) \begin{pmatrix} & \mathrm{I}_2 \\ \mathrm{I}_2 & \end{pmatrix} \begin{pmatrix} & \mathrm{I}_2 \\ \mathrm{I}_2 & \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\
&= \chi_L^\dagger \psi_L + \chi_R^\dagger \psi_R
\end{aligned} \tag{64}
$$

and

$$
\begin{aligned}
\bar{\chi}\gamma^i\psi &= \chi^\dagger \gamma^0 \gamma^i \psi \\
&= \left(\chi_L^\dagger, \chi_R^\dagger\right) \begin{pmatrix} & \mathrm{I}_2 \\ \mathrm{I}_2 & \end{pmatrix} \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\
&= -\chi_L^\dagger \sigma_i \psi_L + \chi_R^\dagger \sigma_i \psi_R .
\end{aligned} \tag{65}
$$

These are the contra-variant coordinates of the true 4-vector of question (iii).

For the pseudo-vector :

$$
\begin{aligned}
\bar{\chi}\gamma^0\gamma^5\psi &= \chi^\dagger \gamma^5 \psi \\
&= \left(\chi_L^\dagger, \chi_R^\dagger\right) \begin{pmatrix} \mathrm{I}_2 & \\ & -\mathrm{I}_2 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\
&= \chi_L^\dagger \psi_L - \chi_R^\dagger \psi_R
\end{aligned} \tag{66}
$$

and

$$\begin{aligned}
\bar{\chi}\gamma^i\gamma^5\psi &= \chi^\dagger\gamma^0\gamma^i\gamma^5\psi \\
&= \left(\chi_L^\dagger, \chi_R^\dagger\right) \begin{pmatrix} & \mathrm{I}_2 \\ \mathrm{I}_2 & \end{pmatrix} \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \begin{pmatrix} \mathrm{I}_2 & \\ & -\mathrm{I}_2 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \\
&= -\chi_L^\dagger\sigma_i\psi_L - \chi_R^\dagger\sigma_i\psi_R.
\end{aligned} \tag{67}$$

These are the contra-variant coordinates of the pseudo 4-vector of question (iii).

0.15 E15

For an infinitesimal Lorentz transformation we obtain :

$$\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}' = \left[\mathrm{I}_2 + i \begin{pmatrix} \left(d\vec{\theta} - id\vec{\phi}\right).\vec{\sigma}/2 & 0 \\ 0 & \left(d\vec{\theta} + id\vec{\phi}\right).\vec{\sigma}/2 \end{pmatrix}\right] \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} \tag{68}$$

(ii) For a rotation around $\hat{x}$:

$$\chi' = \left[\mathrm{I}_4 + \frac{i}{2}d\theta_1 \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix}\right] \chi. \tag{69}$$

Because $\epsilon_{32} = d\theta_1$, we can rewrite this expression in terms of commutators :

$$\frac{i}{2}\epsilon_{32} \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix} = \frac{1}{4}\epsilon_{32} \begin{pmatrix} [\sigma_2, \sigma_3] & \\ & [\sigma_2, \sigma_3] \end{pmatrix}. \tag{70}$$

Since

$$\gamma^2\gamma^3 = -\begin{pmatrix} \sigma_2\sigma_3 & \\ & \sigma_2\sigma_3 \end{pmatrix} \tag{71}$$

we find :

$$\frac{i}{2}d\theta_1 \begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix} = \frac{1}{4}\epsilon_{32}[\gamma^3, \gamma^2] = \frac{1}{8}\left(\epsilon_{32}[\gamma^3, \gamma^2] + \epsilon_{23}[\gamma^2, \gamma^3]\right). \tag{72}$$

For this particular transformation, all other components of $\epsilon_{\mu\nu}$ vanish and thus

$$\chi' = \left(\mathrm{I}_4 + \frac{1}{8}\epsilon_{\mu\nu}[\gamma^\mu, \gamma^\nu]\right)\chi. \tag{73}$$

Since this equation is written in a covariant form, it is valid in any frame and thus it holds for any Lorentz transformation. This is the general infinitesimal transformation of a Dirac bi-spinor. We thus find that the generators of the Lorentz group acting on Dirac bi-spinors can be written : $\sigma^{\mu\nu}/2 = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$.

0.16 E16

(i) Under a translation, $x'_\mu = x_\mu + a_\mu$. Notice that this means that the frame with coordiantes x'_μ is translated by a_μ since its origin corresponds to $x'_\mu = 0$ and thus $x_\mu = -a_\mu$. For a rotation, $x'_i = R_{ij}x_j$ with $R \in$ SO(3) and for a Lorentz transformation : $x'_\mu = \Lambda_\mu{}^\nu x_\nu$ with $\Lambda \in$ SO(3,1).

(ii) For a translation and in the active point of view "the new field at the new point is the same as the old field at the old point". In the passive point of view, this means that $f(x_\mu) = f'(x'_\mu)$ whatever the tensor nature of f.

• For **rotations** and in the active point of view "the new field at the new point is rotated with respect to the old field at the old point". Denoting x for (x_1, x_2, x_3), we have :
- for a scalar (under SO(3)) field : $f'(x') = f(x)$,
- for a spinor field : $f'_\alpha(x') = U_{\alpha\beta} f_\beta(x)$ with $U \in$ SU(2),
- for a vector field $f'_i(x') = R_{ij} f_j(x)$,
- for a tensor field $f'_{i_1 i_2 \cdots i_n}(x') = R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} f_{j_1 j_2 \cdots j_n}(x)$
where $x'_i = R_{ij}(\vec{\theta})x_j$ and $U = U(\vec{\theta}) = \exp\left(i\vec{\theta}.\vec{\sigma}/2\right)$.

• The same kind of relations exist for **Lorentz transformations** where now x means (x_0, x_1, x_2, x_3) :
- for a scalar (under SO(3,1)) field : $f'(x') = f(x)$,
- for a Dirac spinor field : $f'_\alpha(x') = S(\Lambda)_{\alpha\beta} f_\beta(x)$,
- for a 4-vector field $f'_\mu(x') = \Lambda_\mu{}^\nu f_\nu(x)$,
- for a tensor field $f'_{\mu_1\mu_2\cdots\mu_n}(x') = \Lambda_{\mu_1}{}^{\nu_1} \Lambda_{\mu_2}{}^{\nu_2} \cdots \Lambda_{\mu_n}{}^{\nu_n} f_{\nu_1\nu_2\cdots\nu_n}(x)$.
where $x'_\mu = \Lambda_\mu{}^\nu x_\nu$ and for infinitesimal transformations where $\Lambda_\mu{}^\nu = \delta_\mu{}^\nu + \epsilon_\mu{}^\nu$ with $\epsilon_{\mu\nu}$ antisymmetric, $S(\Lambda) = S(\epsilon) = \mathrm{I}_4 - \frac{i}{4}\epsilon_{\mu\nu}\sigma^{\mu\nu}$ and $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ (see exercice E15).

(iii) We can now find the generators acting on classical fields :

• For a **translation** :

$$\begin{aligned} f(x) &= f'(x') \\ &= f'(x+a) \\ &= f'(x) + da^\mu \partial_\nu f'(x) + O(da^2) \\ &= f'(x) + da^\mu \partial_\mu f(x) + O(da^2). \end{aligned} \tag{74}$$

Thus

$$\delta f = -da^\mu \partial_\mu f = i da^\mu (i\partial_\mu f) = i da^\mu P_\mu f \quad \text{with} \quad P_\mu = i\partial_\mu \ . \tag{75}$$

Notice that if we call P^i the coordinates of $\vec{P}$ and $\partial_i = \frac{\partial}{\partial x^i}$ the coordinates of $\vec{\bigtriangledown}$, we find that $\vec{P} = -i\vec{\bigtriangledown}$.

• For an infinitesimal **rotation** around $\hat{z}$ and for a **scalar** field :

$$\begin{aligned} f(x) &= f'(x') \\ &= f'(R.x) \\ &= f'(x + d\theta y, y - d\theta x, z) + O(d\theta^2) \\ &= f'(x) + d\theta\,(y\partial_x - x\partial_y)\, f(x) + O(d\theta^2)\ . \end{aligned} \tag{76}$$

We thus find

$$\delta f = d\theta \left(x\partial_y - y\partial_x \right) f = d\theta_3 \left(x_1\partial_2 - x_2\partial_1 \right) f = d\theta_i \epsilon_{ijk} x_j \partial_k f \qquad (77)$$

where the last equality comes from the fact that for this particular rotation $d\theta_1 = d\theta_2 = 0$. The last equality being written in a covariant (for SO(3)) form is actually valid for any rotation. We introduce the generators L_i of the rotations (more precisely, the representation of the generators of SO(3) acting on scalar fields) :

$$L_i = -i\epsilon_{ijk} x_j \partial_k \quad \Rightarrow \quad \vec{L} = -i\vec{x} \wedge \vec{\bigtriangledown} = \vec{x} \wedge \vec{P} \qquad (78)$$

and we therefore have :

$$\delta f = i d\vec{\theta} \cdot \vec{L}. \qquad (79)$$

The Lie algebra of SO(3) is of course retrieved for the L_i's. From :

$$\begin{aligned} L_i L_j f(x) &= -\epsilon_{imn} x_m \partial_n \left(\epsilon_{jpq} x_p \partial_q f(x) \right) \\ &= -\epsilon_{imn}\epsilon_{jpq} x_m \left(\delta_{np}\partial_q f(x) + x_p \partial_n \partial_q f(x) \right) \\ &= \epsilon_{imn}\epsilon_{jqn} x_m \partial_q f(x) - \epsilon_{imn}\epsilon_{jpq} x_m x_p \partial_n \partial_q f(x) \\ &= \left(\delta_{ij}\delta_{mq} - \delta_{iq}\delta_{mj} \right) x_m \partial_q f(x) - \epsilon_{imn}\epsilon_{jpq} x_m x_p \partial_n \partial_q f(x) \\ &= \delta_{ij} \left(\vec{x} \cdot \vec{\bigtriangledown} \right) f(x) - x_j \partial_i f(x) - \epsilon_{imn}\epsilon_{jpq} x_m x_p \partial_n \partial_q f(x) \end{aligned} \qquad (80)$$

we find :

$$\begin{aligned} [L_i, L_j] f(x) &= -(x_j \partial_i - x_i \partial_j) f(x) \\ &= -\left(\delta_{jl}\delta_{im} - \delta_{il}\delta_{jm} \right) x_l \partial_m f(x) \\ &= \epsilon_{ijk}\epsilon_{lmk} x_l \partial_m f(x) \\ &= i\epsilon_{ijk} L_k f(x). \end{aligned} \qquad (81)$$

Since this holds true for all functions f we obtain : $[L_i, L_j] = i\epsilon_{ijk} L_k$ which is the Lie algebra of the SO(3) group.

- For an infinitesimal **rotation** around $\hat{z}$ and for a **vector** field :

$$\begin{aligned} f_i(x) &= \left(R^{-1} \right)_{ij} f'_j(x') \\ &= \left(\mathrm{I}_3 - id\theta S_z \right)_{ij} \left(f'_j(x) + d\theta \left(y\partial_x - x\partial_y \right) f_j(x) \right) \\ &= f'_i(x) - id\theta \left[(S_z)_{ij} + i \left(y\partial_x - x\partial_y \right) \delta_{ij} \right] f_j(x) + O(d\theta^2) \ . \end{aligned} \qquad (82)$$

where the S_i's are the three 3×3 matrix generators of SO(3) in the vector representation and previously denoted J_i when we did not consider fields, that is, when the orbital part of the generator – the L_i's – is absent. We thus obtain :

$$\delta f_i(x) = id\theta \left[(S_z)_{ij} + i \left(y\partial_x - x\partial_y \right) \delta_{ij} \right] f_j(x). \qquad (83)$$

The generators of the rotation group acting on vector fields are therefore :

$$J_i = L_i + S_i. \qquad (84)$$

They are made of an orbital part and a "spin" part, the orbital part taking care of the change of coordinates of the point where the field is computed and the “spin” part being responsible for the rotaion of the field itself. Of course, this latter part was absent for the scalar field.

• The generalization to **spinor** fields is now straightforward : the generators of SO(3) acting on a spinor field $Z_\alpha(x)$ are again of the form $J_i = L_i + S_i$ but where now the “spin” part is adapted to the spinor nature of the field $S_i \to \sigma_i/2$. For a tensor field, it would proceed in the same way : the orbital part of the generators would be the same as above and the “spin” part would be the generators of SO(3) in the representation corresponding to the tensor nature of the field.

Of course, here again, the commutator algebra of the J_i's is the Lie algebra of SO(3).

• For Lorentz transformations, we proceed in exactly the same way. We have ${\Lambda_\mu}^\nu = {\delta_\mu}^\nu + {\epsilon_\mu}^\nu$. Thus, for a bf scalar field

$$\delta f = -\frac{1}{2} i\, \epsilon_{\mu\nu} i \left(x_\mu \partial_\nu - x_\nu \partial_\mu \right) f. \tag{85}$$

We define

$$L_{\mu\nu} = i \left(x_\mu \partial_\nu - x_\nu \partial_\mu \right) \Rightarrow \begin{cases} L_{0i} & = i(t\partial_i + x_i \partial_t) = -iK_i \\ \\ L_{ij} & = i(x_i \partial_j - x_j \partial_i) = i\epsilon_{ijk} L_k \end{cases} \tag{86}$$

where the L_i's have been defined above. The commutator algebra of the $L_{\mu\nu}$ can be obtained straightforwardly :

$$[L_{\mu\nu}, L_{\rho\sigma}] = i \left(\eta_{\nu\rho} L_{\mu\sigma} - \eta_{\mu\rho} L_{\nu\sigma} + \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\sigma} L_{\mu\rho} \right). \tag{87}$$

It is of course the Lie algebra of SO(3,1) as can be seen on exercice 12, Eq.(56).

• For Lorentz transformations on a **4-vector** field, the calculation is completely similar to that for rotations : Instead of Eq.(82) we have now :

$$f'_\mu \left(\left({\delta_\rho}^\tau + {\epsilon_\rho}^\tau \right) x_\tau \right) \quad = \left({\delta_\mu}^\nu + {\epsilon_\mu}^\nu \right) f_\nu(x) \tag{88}$$

and we of course find that the generators $J_{\mu\nu}$ of the Lorentz group are made of an orbital part which is $L_{\mu\nu}$ and a “spin” part $S_{\mu\nu}$ which is nothing but the 4×4 matrix generators of SO(3,1) in the 4-vector representation.

• The generalization to Lorentz transformations on a **Dirac spinor** field is straightforward and we again find that the generators are the sum of the orbital part $L_{\mu\nu}$ and of a “spin” part $S_{\mu\nu} = \sigma_{\mu\nu}/2$ with $\sigma_{\mu\nu} = i/2[\gamma^\mu, \gamma^\nu]$, see Exercice 15.

• The full commutator algebra now involves also the commutators of the P_μ and of $L_{\mu\nu}$. It is straightforward to show that :

$$[P_\mu, L_{\rho\sigma}] = i \left(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho \right). \tag{89}$$

Of course, there is also : $[P_\mu, P_\nu] = 0$.

(iv) We choose to transform the expectation values of the fields (and of their conjugate momentum) as their classical couterparts :

$$x'_\mu = \Lambda_\mu{}^\nu x_\nu + a_\mu \quad \Rightarrow \quad \langle\phi_\alpha(x')\rangle' = \mathcal{S}_{\alpha\beta}(\Lambda)\langle\phi_\beta(x)\rangle \tag{90}$$

where $S(\Lambda)$ is the "spin" part of the Lorentz transformation and is therefore adapted to the tensor or spinor nature of the field ϕ_α.

The choice is to transform the kets and not the operators. The kets are transformed with the Wigner operators (that depend on the parameters $\Lambda^\mu{}_\nu$ of the Lorentz transformations and on the parameters a_μ of the translations) :

$$|\psi'\rangle = U(\Lambda, a)|\psi\rangle \quad \Rightarrow \quad \langle\psi'\,|\phi_\alpha(x')|\psi'\rangle = \mathcal{S}_{\alpha\beta}(\Lambda)\langle\psi\,|\phi_\beta(x)|\psi\rangle \tag{91}$$

and we conclude that

$$U^\dagger(\Lambda, a)\,\phi_\alpha(x')\,U(\Lambda, a) = \mathcal{S}_{\alpha\beta}(\Lambda)\,\phi_\beta(x). \tag{92}$$

(v) For translations, we have :

$$\begin{cases} x'_\mu = x_\mu + a_\mu \\ |\psi'\rangle = U(a)|\psi\rangle \qquad \text{with} \qquad U(a) = e^{ia^\mu P_\mu} \end{cases} \tag{93}$$

where the last equality is a definition of the generators P_μ acting in the state space. Expanding $U(a)$ at first order we get :

$$(\mathrm{I} - ia^\mu P_\mu)\,(\phi(x) + a^\mu \partial_\mu \phi(x))\,(\mathrm{I} + ia^\mu P_\mu) = \phi(x) \tag{94}$$

which implies

$$[P_\mu, \phi(x)] = -i\partial_\mu \phi(x). \tag{95}$$

For Lorentz transformations we have :

$$\begin{cases} x'_\mu = (\delta^\nu_\mu + \epsilon_\mu{}^\nu)x_\nu \\ \mathcal{S}_{\alpha\beta} = \delta_{\alpha\beta} + \frac{i}{2}\epsilon_{\mu\nu}(S^{\mu\nu})_{\alpha\beta} \\ |\psi'\rangle = U(\epsilon)|\psi\rangle \qquad \text{with} \qquad U(\epsilon) = \mathrm{I} - \frac{i}{2}\epsilon_{\mu\nu}M^{\mu\nu} \end{cases} \tag{96}$$

where $S^{\mu\nu} = 0$ if $\phi(x)$ is a scalar field, $S^{\mu\nu} = \sigma^{\mu\nu}/2$ for a spinor field, ... and the last equality is a definition of the generators $M^{\mu\nu}$ acting in the state space. The calculations are straightforward and we get :

$$[M^{\mu\nu}, \phi_\alpha(x)] = \Big((x^\mu i\partial^\nu - x^\nu i\partial^\mu)\delta_{\alpha\beta} + (S^{\mu\nu})_{\alpha\beta}\Big)\,\phi_\beta(x). \tag{97}$$

0.17 E17

We recall that the conserved charge for the translation invariant systems is :

$$P_\mu^{\rm Noether} = \int d^3x \left(\Pi \partial_\mu \phi - \mathcal{L}\delta_\mu^0\right). \tag{98}$$

We now abbreviate $P_\mu^{\rm Noether}$ by $P_\mu^{\rm N}$ and $\delta^{(3)}(\vec{x}-\vec{x}\,')$ by $\delta_{\vec{x},\vec{x}\,'}$.

We shall often use in the following :

$$\int dx\, f(x)\delta'(x) = -\int dx\, f'(x)\delta(x) \tag{99}$$

and

$$f(x')\delta(x-x') = f(x)\delta(x-x'). \tag{100}$$

(i) We start by $\mu = i$ and since $P_\mu^{\rm Noether}$ is time-independent, we choose the time in the right hand side of Eq.(98) to be identical to the time in $\phi(x)$. We denote $[A(x), B(x')]^=$ the equal-time commutator, that is, the commutator of $A(x)$ and $B(x')$ where $x_0 = x_0'$:

$$\begin{aligned}
[P_i^{\rm N}, \phi(x)] &= \left[\int d^3x'\, \Pi(x')\partial_i'\phi(x'), \phi(x)\right]^= \\
&= \int d^3x' \left([\Pi(x'), \phi(x)]^=\, \partial_i'\phi(t,\vec{x}\,') + \Pi(t,\vec{x}\,')[\partial_i'\phi(x'), \phi(x)]^=\right) \\
&= \int d^3x' \left([\Pi(x'), \phi(x)]^=\, \partial_i'\phi(t,\vec{x}\,') + \Pi(t,\vec{x}\,')\partial_i'[\phi(x'), \phi(x)]^=\right) \\
&= -i\int d^3x' \delta_{\vec{x},\vec{x}\,'}\partial_i'\phi(t,\vec{x}\,') \\
&= -i\,\partial_i\phi(x)
\end{aligned} \tag{101}$$

which is what is expected for the generator of the translations in Fock space.

(ii) We now assume that the lagrangian (density) depends on $\partial_0\phi$ only through $1/2(\partial_0\phi)^2$. Thus :

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi(x))} = \partial_0\phi(x). \tag{102}$$

We also find that

$$\begin{aligned}
\left[\mathcal{L}(x'),\phi(x)\right]^{=} &= \left[\mathcal{L}\big(\phi(x'),\partial_0\phi(t,\vec{x}\,'),\partial_i'\phi(t,\vec{x}\,')\big),\phi(x)\right]^{=} \\
&= \left[\frac{1}{2}(\partial_0\phi(t,\vec{x}\,'))^2,\phi(x)\right]^{=} \\
&= \frac{1}{2}\left([\partial_0\phi(t,\vec{x}\,'),\phi(x)]\partial_0\phi(t,\vec{x}\,')+\partial_0\phi(t,\vec{x}\,')[\partial_0\phi(t,\vec{x}\,'),\phi(x)]\right) \\
&= \frac{1}{2}\left([\Pi(t,\vec{x}\,'),\phi(x)]\partial_0\phi(t,\vec{x}\,')+\partial_0\phi(t,\vec{x}\,')[\Pi(t,\vec{x}\,'),\phi(x)]\right) \\
&= -i\delta_{\vec{x},\vec{x}\,'}\partial_0\phi(t,\vec{x}\,')
\end{aligned} \tag{103}$$

We consider now the $\mu = 0$ case :

$$\begin{aligned}
[P_0^{\mathrm{N}},\phi(x)] &= \left[\int d^3x'\left(\Pi(x')\partial_0\phi(t,\vec{x}\,')-\mathcal{L}(x')\right),\phi(x)\right]^{=} \\
&= \int d^3x'\big([\Pi(x'),\phi(x)]^{=}\,\partial_0\phi(t,\vec{x}\,')+\Pi(t,\vec{x}\,')[\partial_0\phi(t,\vec{x}\,'),\phi(x)]^{=}\big) \\
&\quad + i\partial_0\phi(x) \\
&= -i\,\partial_0\phi(x)
\end{aligned} \tag{104}$$

which is again what is expected for the generator of the translations in time in Fock space.

We conclude that the canonical equal-time commutation relations are sufficient to make the Noether charges of translations become the generators of translations on ϕ. In particular, the generator of the time translation, that is, the Hamiltonian, is :

$$H = P_0^{\mathrm{Noether}} = \int d^3x\,(\Pi\partial_0\phi - \mathcal{L})\,. \tag{105}$$

(iii) We use induction. For $n = 1$, the equality is nothing but the equal-time commutation relation.

We assume that the property holds true for $n-1$ and then :

$$\begin{aligned}
[\phi^n(x),\Pi(x')]^{=} &= \phi^{n-1}(x)[\phi^{n-1}(x),\Pi(x')]^{=} \\
&\quad + [\phi(x),\Pi(x')]^{=}\phi^{n-1}(x) \\
&= in\phi^{n-1}(x)\delta_{\vec{x},\vec{x}\,'}.
\end{aligned} \tag{106}$$

Any function that is expandable in a power series is a sum of monomials and therefore the property follows trivially.

(iii) Let us check that P_μ^{Noether} is also the generator of translation on $\Pi(x)$.

For $\mu = i$:

$$
\begin{aligned}
[P_i^{\rm N}, \Pi(x)] &= \left[\int d^3x'\, \Pi(x')\partial_i'\phi(x'), \Pi(x)\right]^= \\
&= \int d^3x' \big([\Pi(x'), \Pi(x)]^= \partial_i'\phi(t,\vec{x}\,') + \Pi(t,\vec{x}\,')[\partial_i'\phi(x'), \Pi(x)]^=\big) \\
&= \int d^3x' \Pi(t,\vec{x}\,')\partial_i'[\phi(x'), \Pi(x)]^= \\
&= i\int d^3x' \Pi(t,\vec{x}\,')\partial_i'\delta_{\vec{x},\vec{x}\,'} \\
&= -i\int d^3x' \delta_{\vec{x},\vec{x}\,'}\partial_i'\Pi(t,\vec{x}\,') \\
&= -i\,\partial_i\Pi(x)
\end{aligned}
\tag{107}
$$

For $\mu = 0$, we have first to compute $\big[\mathcal{L}(x'), \Pi(x)\big]^=$.

$$
\begin{aligned}
\big[\mathcal{L}(x'), \Pi(x)\big]^= &= \left[\frac{1}{2}\partial_\mu'\phi(x')\partial^{\mu\prime}\phi(x') - V(\phi(x')), \Pi(x)\right]^= \\
&= \frac{1}{2}\partial_\mu'\phi(t,\vec{x}\,')[\partial^{\mu\prime}\phi(x'), \Pi(x)\big]^= + \frac{1}{2}[\partial_\mu'\phi(x'), \Pi(x)\big]^= \partial^{\mu\prime}\phi(t,\vec{x}\,') \\
&\quad - [V(\phi(x')), \Pi(x)\big]^= \\
&= i\,\partial_i\phi(x)\partial^i\delta_{\vec{x},\vec{x}\,'} - iV'(\phi(x))\delta_{\vec{x},\vec{x}\,'}
\end{aligned}
\tag{108}
$$

We can now compute the commutator of $P_0^{\rm N}$ and Π :

$$
\begin{aligned}
[P_0^{\rm N}, \Pi(x)] &= \left[\int d^3x' \big(\Pi(x')\partial_0\phi(t,\vec{x}\,') - \mathcal{L}(x')\big), \Pi(x)\right]^= \\
&= -\int d^3x' \,[\mathcal{L}(x'), \Pi(x)]^= \\
&= -i\int d^3x' \big(\partial_i\phi(x)\partial^i\delta_{\vec{x},\vec{x}\,'} - V'(\phi(x))\delta_{\vec{x},\vec{x}\,'}\big) \\
&= i\big(\partial_i\partial^i\phi(x) + V'(\phi(x))\big) \\
&= -i\partial_0\partial^0\phi(x) + i\big(\partial_\mu\partial^\mu\phi(x) + V'(\phi(x))\big) \\
&= -i\partial_0\Pi(x) + i\big(\partial_\mu\partial^\mu\phi(x) + V'(\phi(x))\big) \\
&= -i\partial_0\Pi(x)
\end{aligned}
\tag{109}
$$

and the (Euler-Lagrange) equations of motion on ϕ, that is :

$$
\partial_\mu\partial^\mu\phi(x) + V'(\phi(x)) = 0. \tag{110}
$$

We conclude that *for the fields satisfying the equations of motion*, $P_\mu^{\rm N}$ is also the generator of translations on $\Pi(x)$. This means in particular that the Hamiltonian can be defined either as the the quantity that generates the translations in time or as the quantity that is conserved when the translations in time are symmetries : these two quantities are one and the same.

0.18 E18

(i) The conjugate momentum of $\psi_\alpha(x)$ is given by

$$\begin{aligned}\Pi_\alpha(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi_\alpha(x))} &= \frac{\partial(i\bar\psi\gamma^\mu\partial_\mu\psi)}{\partial(\partial_0\psi_\alpha(x))}\\ &= i\bar\psi_\alpha\gamma^0\\ &= i\psi^\dagger_\alpha(x)\end{aligned} \tag{111}$$

(ii) The lagrangian must be a Lorentz scalar. Its (potential) part depends only on ψ and $\bar\psi$. The hermitic invariants that can be built with them are for instance $\bar\psi\psi$ or $(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi)$. Of course all powers of them are invariant. It is possible to add several γ-matrices (the indices of which are contracted) as well as γ^5, if parity is not an issue.

(iii) We can redo all the calculations made in E17 with equal-time anti-commutators instead of equal-time commutators.

In the following calculations, when the spinor indices are not explicitly written, it is understood that there are summed over. For instance $\Pi\partial_i\psi$ means $\Pi_\beta\partial_i\psi_\beta$.

We first start with $\mu = i$:

$$\begin{aligned}[P_i^{\mathrm{N}}, \psi_\alpha(x)] &= \left[\int d^3x'\,\Pi(x')\partial'_i\psi(x'), \psi_\alpha(x)\right]^{=}\\ &= \int d^3x'\big(\Pi(t,\vec x\,')\partial'_i\{\psi(x'),\psi_\alpha(x)\}^{=} - \{\Pi_\beta(x'),\psi_\alpha(x)\}^{=}\,\partial'_i\psi_\beta(t,\vec x\,')\big)\\ &= -i\int d^3x'\delta_{\vec x,\vec x\,'}\partial'_i\psi_\alpha(t,\vec x\,')\\ &= -i\partial_i\psi_\alpha(x)\end{aligned} \tag{112}$$

which is the same result as above.

We now need to recompute the commutator between the lagrangian and the field. We start by the kinetic part of the lagrangian : $i\bar\psi\partial\!\!\!/\psi = i\bar\psi\gamma^\mu\partial_\mu\psi$

$$\begin{aligned}\left[\bar\psi_\alpha(x')\gamma^\mu_{\alpha\beta}\partial'_\mu\psi_\beta(x'),\psi_\sigma(x)\right]^{=} &= \bar\psi_\alpha(t,\vec x\,')\gamma^\mu_{\alpha\beta}\left\{\partial'_\mu\psi_\beta(x'),\psi_\sigma(x)\right\}^{=}\\ &\quad - \gamma^\mu_{\alpha\beta}\left\{\bar\psi_\alpha(x'),\psi_\sigma(x)\right\}^{=}\partial'_\mu\psi_\beta(t,\vec x\,')\\ &= -\gamma^\mu_{\alpha\beta}\left\{\psi^\dagger_\rho(x'),\psi_\sigma(x)\right\}^{=}\gamma^0_{\rho\alpha}\partial'_\mu\psi_\beta(t,\vec x\,')\\ &= i\gamma^\mu_{\alpha\beta}\left\{\Pi_\rho(x'),\psi_\sigma(x)\right\}^{=}\gamma^0_{\rho\alpha}\partial'_\mu\psi_\beta(t,\vec x\,')\\ &= -\gamma^\mu_{\alpha\beta}\gamma^0_{\sigma\alpha}\delta_{\vec x,\vec x\,'}\partial'_\mu\psi_\beta(t,\vec x\,')\\ &= -\gamma^0_{\sigma\alpha}\delta_{\vec x,\vec x\,'}\partial\!\!\!/_{\alpha\beta}\psi_\beta(x).\\ &= -\delta_{\vec x,\vec x\,'}\left(\gamma^0\partial\!\!\!/\psi(x)\right)_\sigma.\end{aligned} \tag{113}$$

Then, for a particular potential term (which is actually the mass term) :

$$
\begin{aligned}
[\bar{\psi}(x')\psi(x'), \psi_\sigma(x)]^= &= \bar{\psi}_\beta(t,\vec{x}\,')\{\psi_\beta(x'), \psi_\sigma(x)\}^= - \{\bar{\psi}_\beta(x'), \psi_\sigma(x)\}^= \psi_\beta(t,\vec{x}\,') \\
&= -\{\psi^\dagger_\alpha(x'), \psi_\sigma(x)\}^= \gamma^0_{\alpha\beta}\psi_\beta(t,\vec{x}\,') \\
&= -\delta_{\vec{x},\vec{x}\,'} \left(\gamma^0\psi\right)_\sigma(x).
\end{aligned}
\tag{114}
$$

We notice that the two results above, Eqs.(113) and (114), can be rewritten as :

$$
\left[A(x'), \psi_\sigma(x)\right]^= = -\delta_{\vec{x},\vec{x}\,'}\gamma^0_{\sigma\alpha}\frac{\partial}{\partial\bar{\psi}_\alpha(x)}A(x) \tag{115}
$$

with $A = \bar{\psi}\gamma^\mu\partial_\mu\psi$ and $A = \bar{\psi}\psi$.

This relation can be trivially generalized :
- to any power of $\bar{\psi}(x')\psi(x')$ since :

$$
\begin{aligned}
\left[\left(\bar{\psi}(x')\psi(x')\right)^n, \psi_\sigma(x)\right]^= &= -\Big(\left(\gamma^0\psi\right)_\sigma\left(\bar{\psi}\psi\right)^{n-1} + \left(\bar{\psi}\psi\right)\left(\gamma^0\psi\right)_\sigma\left(\bar{\psi}\psi\right)^{n-2} + \\
&\qquad \cdots + \left(\bar{\psi}\psi\right)^{n-1}\left(\gamma^0\psi\right)_\sigma\Big)\delta_{\vec{x},\vec{x}\,'} \\
&= -\delta_{\vec{x},\vec{x}\,'}\gamma^0_{\sigma\alpha}\frac{\partial}{\partial\bar{\psi}_\alpha(x)}\left(\bar{\psi}(x)\psi(x)\right)^n
\end{aligned}
\tag{116}
$$

- to $\bar{\psi}\gamma^\mu\psi$ since :

$$
[\bar{\psi}(x')\gamma^\mu\psi(x'), \psi_\sigma(x)]^= = -\delta_{\vec{x},\vec{x}\,'}\left(\gamma^0\gamma^\mu\psi(x)\right)_\sigma. \tag{117}
$$

- to $\left(\bar{\psi}\gamma^\mu\psi\right)\left(\bar{\psi}\gamma_\mu\psi\right)$ since :

$$
\left[\left(\bar{\psi}\gamma^\mu\psi\right)\left(\bar{\psi}\gamma_\mu\psi\right)(x'), \psi_\sigma(x)\right]^= = -\delta_{\vec{x},\vec{x}\,'}\gamma^0_{\sigma\alpha}\left\{\left(\bar{\psi}\gamma^\mu\psi\right)\left(\gamma_\mu\psi\right)_\alpha + \left(\gamma^\mu\psi\right)_\alpha\left(\bar{\psi}\gamma_\mu\psi\right)\right\}. \tag{118}
$$

We conclude that

$$
\left[\mathcal{L}(x'), \psi_\sigma(x)\right]^= = -\delta_{\vec{x},\vec{x}\,'}\gamma^0_{\sigma\alpha}\frac{\partial\mathcal{L}(x)}{\partial\bar{\psi}_\alpha(x)} \tag{119}
$$

We can now compute the commutator between P_0^{N} and ψ :

$$
\begin{aligned}
[P_0^{\mathrm{N}}, \psi_\sigma(x)] &= \left[\int d^3x'\left(\Pi(x')\partial_0'\psi(x') - \mathcal{L}(x')\right), \psi_\sigma(x)\right]^= \\
&= -\int d^3x'\{\Pi_\alpha(x'), \psi_\sigma(x)\}^= \partial_0\psi_\alpha(t,\vec{x}\,') + \gamma^0_{\sigma\alpha}\frac{\partial\mathcal{L}}{\partial\bar{\psi}_\alpha(x)} \\
&= -i\,\partial_0\psi_\sigma(x) + \gamma^0_{\sigma\alpha}\left(\frac{\partial\mathcal{L}}{\partial\bar{\psi}_\alpha(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi}_\alpha(x))}\right)\right)
\end{aligned}
\tag{120}
$$

where the last term has been added because it is vanishing. We conclude that for the fields that satisfy the equations of motion, P_0^{N} is the generator of time translations for $\psi(x)$.

(iv) We redo the calculations above for Π. For $\mu = i$:

$$
\begin{aligned}
[P_i^{\mathrm{N}}, \Pi_\sigma(x)] &= \left[\int d^3x'\, \Pi(x')\partial_i'\psi(x'), \Pi_\sigma(x)\right]^= \\
&= \int d^3x'\, \Pi_\alpha(t,\vec{x}\,')\partial_i'\{\psi_\alpha(x'), \Pi_\sigma(x)\}^= \\
&= i\int d^3x'\, \Pi_\sigma(t,\vec{x}\,')\partial_i'\delta_{\vec{x},\vec{x}\,'} \\
&= -i\,\partial_i\Pi_\sigma(x)
\end{aligned}
\tag{121}
$$

We now have to compute the commutator of $\mathcal{L}$ and Π. We start by

$$
\begin{aligned}
\left[i\bar{\psi}_\alpha(x')\gamma^\mu_{\alpha\beta}\partial_\mu'\psi_\beta(x'), \Pi_\sigma(x)\right]^= &= i\bar{\psi}_\alpha(t,\vec{x}\,')\gamma^\mu_{\alpha\beta}\partial_\mu'\left\{\psi_\beta(x'), \Pi_\sigma(x)\right\}^= \\
&= -\bar{\psi}_\alpha(t,\vec{x}\,')\gamma^\mu_{\alpha\sigma}\partial_\mu'\delta_{\vec{x},\vec{x}\,'} \\
&= -\bar{\psi}_\alpha(t,\vec{x}\,')\gamma^i_{\alpha\sigma}\partial_i'\delta_{\vec{x},\vec{x}\,'} \\
&= \delta_{\vec{x},\vec{x}\,'}\partial_i\bar{\psi}_\alpha(x)\gamma^i_{\alpha\sigma} \\
&= \delta_{\vec{x},\vec{x}\,'}\left(\partial_\mu\bar{\psi}_\alpha(x)\gamma^\mu_{\alpha\sigma} - \partial_0\bar{\psi}_\alpha(x)\gamma^0_{\alpha\sigma}\right) \\
&= \delta_{\vec{x},\vec{x}\,'}\left(\partial_\mu\bar{\psi}_\alpha(x)\gamma^\mu_{\alpha\sigma} - \partial_0\psi^\dagger_\sigma(x)\right) \\
&= i\delta_{\vec{x},\vec{x}\,'}\partial_0\Pi(x) - i\delta_{\vec{x},\vec{x}\,'}\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\alpha(x))}\right)
\end{aligned}
\tag{122}
$$

Then,

$$
\begin{aligned}
[\bar{\psi}(x')\psi(x'), \Pi_\sigma(x)]^= &= \bar{\psi}_\beta(t,\vec{x}\,')\{\psi_\beta(x'), \Pi_\sigma(x)\}^= \\
&= i\delta_{\vec{x},\vec{x}\,'}\bar{\psi}_\sigma(x) \\
&= i\delta_{\vec{x},\vec{x}\,'}\frac{\partial\left(\bar{\psi}(x)\psi(x)\right)}{\partial(\psi_\sigma(x))}.
\end{aligned}
\tag{123}
$$

As we did previously, we can generalize this calculation to all kinds of non-derivative terms (Lorentz-invariant and hermitic) and we therefore find that :

$$
\left[\mathcal{L}(x'), \Pi_\sigma(x)\right]^= = i\delta_{\vec{x},\vec{x}\,'}\partial_0\Pi_\sigma(x) + i\delta_{\vec{x},\vec{x}\,'}\left(\frac{\partial\mathcal{L}}{\partial\psi_\sigma(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\sigma(x))}\right)\right)
\tag{124}
$$

We can now compute the commutator of P_0^{N} and Π :

$$
\begin{aligned}
[P_0^{\mathrm{N}}, \Pi_\sigma(x)] &= \left[\int d^3x'\left(\Pi(x')\partial_0'\psi(x') - \mathcal{L}(x')\right), \Pi_\sigma(x)\right]^= \\
&= \int d^3x'\left(\Pi_\alpha(t,\vec{x}\,')\partial_0'\{\psi_\alpha(x'), \Pi_\sigma(x)\}^= - \left[\mathcal{L}(x'), \Pi_\sigma(x)\right]^=\right) \\
&= -i\,\partial_0\Pi_\sigma(x) - i\left(\frac{\partial\mathcal{L}}{\partial\psi_\sigma(x)} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_\sigma(x))}\right)\right) \\
&= -i\,\partial_0\Pi_\sigma(x)
\end{aligned}
\tag{125}
$$

where, in the last line, we used the equations of motion. We conclude that for the physical fields, P_μ^{N} is the generator of translations for Pi.

0.19 E19

(i) By definition

$$\not{p} = p_\mu \gamma^\mu = p_0 \gamma^0 + \cdots p_3 \gamma^3. \tag{126}$$

It is therefore a 4×4 matrix.

(ii) We use the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \mathrm{I}. \tag{127}$$

to rewrite $\not{p}^2$:

$$\begin{aligned} \not{p}^2 &= p_\mu \gamma^\mu p_\nu \gamma^\nu \\ &= \frac{1}{2}\left(p_\mu \gamma^\mu p_\nu \gamma^\nu + p_\nu \gamma^\nu p_\mu \gamma^\mu\right) \\ &= \frac{1}{2} p_\mu p_\nu \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) \\ &= p_\mu p_\nu \eta^{\mu\nu} \mathrm{I} \\ &= p^2 \mathrm{I}. \end{aligned} \tag{128}$$

We again use the Clifford algebra :

$$\begin{aligned} \gamma_\mu \gamma^\mu &= \eta_{\mu\nu} \gamma^\mu \gamma^\nu \\ &= \frac{1}{2} \eta_{\mu\nu} \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu\right) \\ &= \eta_{\mu\nu} \eta^{\mu\nu} \mathrm{I} \\ &= 4\,\mathrm{I} \end{aligned} \tag{129}$$

(iii) We use $\{\gamma^\mu, \gamma^5\} = 0$ and the fact that the trace is cyclic : $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$:

- $\mathrm{Tr}\left(\gamma^\mu \gamma^5\right)$:

$$\begin{aligned} \mathrm{Tr}\left(\gamma^\mu \gamma^5\right) &= \mathrm{Tr}\left(\gamma^5 \gamma^\mu\right) \\ &= -\mathrm{Tr}\left(\gamma^5 \gamma^\mu\right) \end{aligned} \tag{130}$$

and thus $\mathrm{Tr}\left(\gamma^\mu \gamma^5\right) = 0$.

- $\mathrm{Tr}\left(\gamma^\mu \gamma^\nu\right)$:

$$\begin{aligned} \mathrm{Tr}\{\gamma^\mu, \gamma^\nu\} &= 8\eta^{\mu\nu} \\ &= 2\,\mathrm{Tr}\left(\gamma^\mu \gamma^\nu\right) \end{aligned} \tag{131}$$

and thus

$$\mathrm{Tr}\left(\gamma^\mu \gamma^\nu\right) = 4\eta^{\mu\nu} \tag{132}$$

• $\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\right)$:

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\right) &= \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^5\gamma^5\right)\\
&= -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^5\gamma^\alpha\gamma^5\right)\\
&= -\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^5\right)\\
&= -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^5\gamma^5\right)\\
&= -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\right).
\end{aligned} \tag{133}$$

Thus, $\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\right) = 0$. This is actually true for the trace of any odd number of γ^μ matrices.

• $\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right)$:

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) &= -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha\right) + 2\eta^{\alpha\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right)\\
&= \mathrm{Tr}\left(\gamma^\mu\gamma^\beta\gamma^\nu\gamma^\alpha\right) - 2\eta^{\nu\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\alpha\right) + 2\eta^{\alpha\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right)\\
&= -\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) + 2\eta^{\mu\beta}\mathrm{Tr}\left(\gamma^\nu\gamma^\alpha\right) - 2\eta^{\nu\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\alpha\right)\\
&\quad + 2\eta^{\alpha\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right).
\end{aligned} \tag{134}$$

We therefore obtain :

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) &= \eta^{\mu\beta}\mathrm{Tr}\left(\gamma^\nu\gamma^\alpha\right) - \eta^{\nu\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\alpha\right) + \eta^{\alpha\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right)\\
&= 4\eta^{\mu\beta}\eta^{\nu\alpha} - 4\eta^{\nu\beta}\eta^{\mu\alpha} + 4\eta^{\alpha\beta}\eta^{\mu\nu}
\end{aligned} \tag{135}$$

A Remark : One can notice that the results above were expected since the traces can only be functions of $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu\alpha\beta}$. For instance, there is no nonvanishing tensor of rank 3 that can be built with these two tensors and therefore the trace of an odd number of γ^μ matrices must vanish.

As for $\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right)$, the only thing to check is that $\epsilon^{\mu\nu\alpha\beta}$ does not appear in the right hand side. Let us assume it does :

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = \lambda\epsilon^{\mu\nu\alpha\beta} + \cdots \tag{136}$$

where the dots represent the contribution which is not fully antisymmetric in the indices μ, ν, α, β. Then,

$$\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = \lambda\epsilon^{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\beta} = 4!\,\lambda \tag{137}$$

It is now easy to prove that $\lambda = 0$ since :

$$\begin{aligned}
\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) &= -\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha\right)\\
&= -\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha\right)\\
&= -\epsilon_{\mu\nu\alpha\beta}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right).
\end{aligned} \tag{138}$$

Then, the tensor $\epsilon^{\mu\nu\alpha\beta}$ cannot appear in the right hand side of the trace. Only products of two tensor metric can appear :

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = \lambda\eta^{\mu\beta}\eta^{\nu\alpha} + \rho\eta^{\nu\beta}\eta^{\mu\alpha} + \sigma\eta^{\alpha\beta}\eta^{\mu\nu}. \tag{139}$$

The trace must be symmetric under the simultaneous exchange of $\mu \leftrightarrow \alpha$ and $\nu \leftrightarrow \beta$ since $\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) = \mathrm{Tr}\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right)$ which it automatically is according to Eq.(139). It must also be invariant under the exchange $\mu \to \beta, \nu \to \mu, \alpha \to \nu, \beta \to \alpha$ because the trace is cyclic. This yields : $\lambda = \sigma$. Then, multiplying the trace by $\eta_{\mu\nu}\eta_{\alpha\beta}$ and $\eta_{\mu\alpha}\eta_{\nu\beta}$ we get a system of equations for λ and ρ whose solution is $\lambda = -\rho = 4$.

(iv) For the trace of $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right)$ the only tensor we can have in the right hand side is $\eta^{\mu\nu}$:

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = \lambda\eta^{\mu\nu}. \tag{140}$$

Multiplying both sides of this equation by $\eta_{\mu\nu}$ yields $\lambda = 0$ and thus :

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = 0. \tag{141}$$

(v) We use the results of (iii) :

$$\mathrm{T}r(\not{p}\not{k}) = p_\mu k_\nu \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4p\cdot k. \tag{142}$$

$$\mathrm{T}r(\not{p}\not{k}\not{q}) = p_\mu k_\nu q_\rho \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\right) = 0. \tag{143}$$

$$\begin{aligned}\mathrm{T}r(\not{p}\not{k}\not{p}'\not{k}') &= p_\mu k_\nu p'_\alpha k'_\beta \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right) \\ &= 4p_\mu k_\nu p'_\alpha k'_\beta\left(\eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\nu\beta}\eta^{\mu\alpha} + \eta^{\alpha\beta}\eta^{\mu\nu}\right) \\ &= 4(p\cdot k')(p'\cdot k) - 4(p\cdot p')(k\cdot k') + 4(p\cdot k)(p'\cdot k')\end{aligned} \tag{144}$$

(vi) The Dirac equation reads :

$$\left(i\not{\partial} - m\right)\psi = 0. \tag{145}$$

Thus :

$$\left(i\not{\partial} + m\right)\left(i\not{\partial} - m\right)\psi = -\left(\partial^2 + m^2\right)\psi = 0 \tag{146}$$

which is the Klein-Gordon equation.