1 Group theory and tensor calculus

1.1 E1

$A_{\mu\nu}$ and $S_{\mu\nu}$ are respectively antisymmetric and symmetric tensors : $A_{\mu\nu} = -A_{\nu\mu}$, $S_{\mu\nu} = S_{\nu\mu}$. Show that $A^{\mu\nu}S_{\mu\nu} = 0$. Show that it follows that if $T^{\mu\nu}$ and $T'^{\mu\nu}$ satisfy $T^{\mu\nu}A_{\mu\nu} = T'^{\mu\nu}A_{\mu\nu}$ for all antisymmetric tensor $A_{\mu\nu}$ then $T_{\mu\nu}$ and $T'_{\mu\nu}$ are not necessarily equal. What can we deduce on the symmetric and antisymmetric parts of $T_{\mu\nu}$ and $T'_{\mu\nu}$?

1.2 E2

Show that $\partial^\mu = \eta^{\mu\nu}\partial_\mu$ and that it transforms as a covariant vector under Lorentz transformations.

1.3 E3

Show that if $T_{\mu\nu}$ is a tensor, its trace $T^\mu{}_\mu$ is a scalar.

1.4 E4

The observer O is considered fixed and O' moves with a velocity v in the x direction. Its coordinates are related to those of O by

$$\begin{aligned} t' &= \gamma(t - vx) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned} \tag{1}$$

with $\gamma = 1/\sqrt{1 - v^2}$. By manipulating these equations, eliminate completely v, x, y, z and rewrite them in terms of t and of the vectors $\vec{r}$ and $\vec{v}$. What can you deduce from the form of the transformations obtained ?

1.5 E5 : equivalent representations

We consider the C_{3v} symmetry group of the equilateral triangle A, B, C. Is it possible to represent the mirror symmetry S_A about the axis going through the point A by the matrix :

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; ? \tag{2}$$

To which basis in the ABC plane does this choice correspond to ? What are the matrices that represent S_B and S_C in this case ? Check that together with the rotation matrices, it is also a representation of C_{3v}.

1.6 E6

Using the bra and ket notation can sometimes yield a simple notation.

We consider a three-dimensional vector space and we call $(|1\rangle, |2\rangle, |3\rangle)$ the orthonormal basis vectors. Check that (with Einstein's convention)

$$J_i = -i\epsilon_{ijk}|j\rangle\langle k| \tag{3}$$

are the generators of the SO(3) representation of the rotation group. With this notation, check the Lie algebra of the rotation group.

1.7 E7 : homomorphism between SU(2) and SO(3)

We explicilty construct in the following the mapping between SU(2) and SO(3) that also shows that SU(2) is not a representation of SO(3) while SO(3) is a representation of SU(2).

1. We consider the set $\mathcal{M}$ of complex matrices M that are 2×2, hermitic and traceless.

(i) How many real parameters do the matrices M depend on ?

(ii) Show that a general parametrization of M is

$$M = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{4}$$

with x, y, z real numbers. We denote $\vec{x} = (x, y, z)$. Conclude that $\mathcal{M}$ is isomorphic to $\mathbb{R}^3$.

2. Check that a basis of the four-dimensional complex vector space of 2×2 matrices is $\sigma_\mu = (\mathrm{I}_2, \sigma_1, \sigma_2, \sigma_3)$ with $\mu = 0, 1, 2, 3$, I_2 is the 2×2 unit matrix and the σ_i's are the Pauli matrices. How $M = M(\vec{x})$ can be decomposed on the previous basis ? Rewrite this as a formal scalar product between $\vec{x}$ and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$.

3. Show that :

$$\frac{1}{2}\mathrm{Tr}(\sigma_\mu \sigma_\nu) = \delta_{\mu\nu}. \tag{5}$$

Show that therefore the coordinates N_μ of a matrix N on the basis σ_μ can be conveniently obtained from a trace. How would you call $\frac{1}{2}\mathrm{Tr}(AB)$ where A and B are two 2×2 matrices ?

4. Compute the determinant of $M(\vec{x})$ in terms of $\vec{x}$. How does this determinant change when a rotation is performed on $\vec{x}$?

5. We now build the homomorphism between SU(2) and SO(3). It will map a matrix $U \in$ SU(2) to a matrix R in SO(3). We use the matrices $M \in \mathcal{M}$ to this aim.

(i) Show that the mapping R_U defined by

$$M \to M' = R_U(M) = UMU^{-1} \tag{6}$$

preserves all the properties of M if $U \in$ SU(2), that is, R_U is a mapping from $\mathcal{M}$ to $\mathcal{M}$. How can we rewrite M' in terms $\vec{\sigma}$ and a vector $\vec{x}'$?

(ii) Conclude that we can consider R_U as a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that $\vec{x}' = R_U(\vec{x})$.

(iii) Compute the determinant of M'. What can you conclude about R_U considered as a mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$? Can it be a mirror symmetry ?

6. Compute $(R_U.R_V)(M)$ and conclude that the mapping

$$\left\{ \begin{array}{ll} SU(2) & \to SO(3) \\ \\ U & \to R_U \end{array} \right. \tag{7}$$

is an homomorphism between the SU(2) and SO(3) groups, that is, SO(3) is a representation of SU(2).

7. Compute the coordinates x'_i of $M' = UMU^{-1}$ in terms of σ_i and conclude that the matrix elements of R_U are given by :

$$(R_U)_{ij} = \frac{1}{2} \mathrm{Tr} \left(\sigma_i U \sigma_j U^{-1} \right) . \tag{8}$$

8. What is the inverse of the matrix $-U$? Conclude from the previous equation that $R_U = R_{-U}$ and that therefore the inverse mapping of Eq.(7) does not exist.

9. We consider the rotation of axis $\hat{z}$ and angle θ. How is it represented in SO(3) ? Check explicitly that if we take

$$U(\hat{z}, \theta) = e^{i\theta\sigma_z/2} \tag{9}$$

for the SU(2) counterpart of this rotation, Eq.(8) is indeed satisfied. Conclude that with a rotation of angle 2π is not associated the unit matrix in SU(2).

1.8 E8 : SU(2), spinors and invariant tensors

We call spinors the two-component complex objects that transform under a rotation of parameters $(\vec{n}, \theta)$ by the SU(2) matrix $U(\vec{n}, \theta) = \exp(i\theta\vec{n}.\vec{\sigma}/2)$ (we could have as well called them SU(2)-vectors or vectors for SU(2) but for historical reasons they have been called spinors). We also call multi-spinors the quantities that transform under a rotation by $T'_{\alpha_1,\cdots,\alpha_n} = U_{\alpha_1,\beta_1} \cdots U_{\alpha_n,\beta_n} T_{\beta_1,\cdots,\beta_n}$ (they are also called SU(2)-tensors or tensors for SU(2)).

(i) Prove that the fully antisymmetric tensor $\epsilon_{\alpha\beta}$ defined in a particular frame by $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ and $\epsilon_{12} = 1$ is an invariant tensor for SU(2), that is, its transformations under SU(2) preserve the numerical value of its matrix elements.

(ii) We define the metric $\eta_{\mu\nu}$ as a tensor. Show that the definition of the Lorentz matrices : $\eta_{\mu\nu}\Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'} = \eta_{\mu'\nu'}$ implies that it is an invariant tensor for the Lorentz group.

(iii) Show that if Z_α is a spinor, then $Z^\dagger Z$ is a scalar and $Z^\dagger \vec{\sigma} Z$ is a real vector for SO(3). It will be useful to use the relation $U^{-1}\sigma_i U = R_{ij}\sigma_j$ where $U \in$ SU(2) and R is the SO(3) matrix associated with U in the mapping from SU(2) to SO(3) (see exercice 7).

1.9 E9 : Eigenvalues and eigenvectors of the SO(3) generators

We recall that

$$J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{10}$$

(i) Compute the eigenvalues and eigenvectors of J_3 and show that the transition matrix to the diagonal basis is

$$N = \begin{pmatrix} -1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{11}$$

with NJ_3N^{-1} a diagonal matrix. What do these eigenvalues remind you ?

(ii) We consider a vector of cartesian coordinates (V_x, V_y, V_z). Find its coordinates in the eigenbasis (notice that they are complex).

(iii) Compute $\vec{J}^{\,2} = J_1^2 + J_2^2 + J_3^2$. What does this remind you ? We call $\vec{J}^{\,2}$ the Casimir operator of the rotation group.

(iv) We generically call $\mathcal{J}_i$ the generators of a representation of either SU(2) or SO(3). The $\mathcal{J}_i$ are matrices of any dimension. For SU(2) : $\mathcal{J}_i = \sigma_i/2$.

Compute $\vec{\mathcal{J}}^2 = (\sigma_1/2)^2 + (\sigma_2/2)^2 + (\sigma_3/2)^2$. What does it remind you ?

A remark : We could prove that for all representations of SU(2) of dimension $2j+1$ with j an integer (true representations of SO(3)) or half an integer (representations of SU(2) that are not representations of SO(3)),

$$\vec{\mathcal{J}}^2 = j(j+1)I_{2j+1} \tag{12}$$

where I_{2j+1} is the identity matrix of dimension $2j+1$ and the eigenvalues of $\mathcal{J}_3$ are $\{-j, -j+1, \cdots, j-1, j\}$ that are either integers or half-integers.

1.10 E10 : tensor products in SO(3) and reducible representations

We consider a tensor for SO(3) with two indices : T_{ij}.

(i) Prove that the trace of T is a scalar for SO(3). Do you think legitimate to say that this trace spans an irreducible representation of SO(3) ? If T was the tensor product of two vectors $\vec{x}$ and $\vec{y}$, what would be the trace of T in terms of $\vec{x}$ and $\vec{y}$?

(ii) We consider the antisymmetric part of T : $A_{ij} = (T_{ij} - T_{ji})/2$. Show that its transformation under SO(3) involves only A. Same question as above : Do you think legitimate to say that A spans a representation of SO(3) ?

(iii) We define three quantities V_i by : $V_i = \epsilon_{ijk}A_{jk}$. Show that $V_i = \epsilon_{ijk}T_{jk}$. Using infinitesimal transformations for convenience, show that V_i is a vector for SO(3). The following relations will be useful :

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \tag{13}$$

and

$$\delta V_i = V_i' - V_i = -(\vec{\delta\theta} \wedge \vec{V})_i = -\epsilon_{ijk}\delta\theta_j V_k. \tag{14}$$

Deduce that A spans the (irreducible) vector representation of SO(3).

(iv) When T is the tensor product of $\vec{x}$ and $\vec{y}$, what is $\vec{V}$?

(v) We now define the symmetric part of T :

$$S_{ij} = \frac{1}{2}(T_{ij} + T_{ji}). \tag{15}$$

How should we modify the definition of S to make it traceless ? We call $\mathcal{S}_{ij}$ this quantity. How many degrees of freedom does $\mathcal{S}$ involve ?

(vi) Check that the components of S transform under SO(3) among themselves. Is it also true for $\mathcal{S}$? (same question : Do you think legitimate to say that S and $\mathcal{S}$ spans a representation of SO(3) ?)

(vii) We define a 5-uplet by the following linear combinations of the S_{ij} : $s_1 = S_{11} - S_{33}$, $s_2 = S_{12}$, $s_3 = S_{13}$, $s_4 = S_{22} - S_{33}$, $s_5 = S_{23}$. Show that they are independent linear combinations of components of $\mathcal{S}$ (they form a basis).

(viii) Find the matrices representing the generators $\mathcal{J}_i$ in the basis s_i. Are they hermitic ? Comments ?

(ix) Check that the commutation relations of the $\mathcal{J}_i$'s reproduce the Lie algebra of the rotation group and compute $\vec{\mathcal{J}}^2$. What is the "spin" of this representation ?

(x) Conclude about the tensor product of two vectors : What are the different objects made out of the two vectors that span representations in this tensor product ?

1.11 E11 : composition of two spins 1/2

We consider $\mathbb{C}^2$ which is the representation space for $j = 1/2$. We call $\{|\alpha\rangle\}$ the eigenbasis of σ_z in this space which is also called in the literature :

$$\{|\alpha\rangle\} = \{|\frac{1}{2}, \frac{1}{2}\rangle, |\frac{1}{2}, -\frac{1}{2}\rangle\} = \{|+\rangle, |-\rangle\} = \{|\uparrow\rangle, |\downarrow\rangle\}. \tag{16}$$

(i) Find a convenient basis of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ in terms of $\{|\alpha\rangle\}$.

(ii) We call Z a spinor and we write it as

$$Z = Z_\alpha|\alpha\rangle = Z_1|\uparrow\rangle + Z_2|\downarrow\rangle. \tag{17}$$

We consider the tensor product of two spinors Y and Z : $W = Y \otimes Z$. How many (complex) components does W have ? Determine the generators $\mathcal{J}_i$ of the (representation of the) rotation group acting on $Y \otimes Z$.

(iii) Show that $\mathcal{J}^2$ is not diagonal in the basis $\{|\alpha\rangle \otimes |\beta\rangle\}$ and find the diagonal basis. What are the eigenvalues of $\mathcal{J}^2$ and of $\mathcal{J}_3$?

(iv) Find the components of W in the diagonal basis and show that the spin 0 part of $W_{\alpha\beta}$ is proportional to the invariant tensor $\epsilon_{\alpha\beta}$.

1.12 E12 : Lie algebra of the Lorentz group

We define the generators of the Lie algebra of SO(3,1) by

$$\Lambda = \mathrm{I} - \frac{1}{2} i \epsilon_{\alpha\beta} J^{\alpha\beta} \quad \Rightarrow \quad \Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} - \frac{1}{2} i \epsilon_{\alpha\beta} \left(J^{\alpha\beta}\right)^{\mu}{}_{\nu} \tag{18}$$

where $\epsilon^{\alpha\beta}$ is antisymmetric and involves the infinitesimal parameters of the Lorentz transformations.

(i) How many independent $J^{\alpha\beta}$ matrices exist ?

(ii) From what you know of the infinitesimal Lorentz transformations, show that

$$\left(J_{\alpha\beta}\right)^{\mu}{}_{\nu} = i \left(\delta^{\mu}_{\alpha} \eta_{\nu\beta} - \delta^{\mu}_{\beta} \eta_{\nu\alpha}\right) \tag{19}$$

and that

$$\begin{cases} J_{0i} & = -i K_i \\ J_{ij} & = \epsilon_{ijk} J_k . \end{cases} \tag{20}$$

(iii) Show that the $J_{\alpha\beta}$ satisfy the following Lie algebra :

$$[J_{\mu\nu}, J_{\rho\sigma}] = i \left(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\nu\sigma} J_{\mu\rho}\right) \tag{21}$$

(iv) Retrieve this algebra by setting $J_{\mu\nu} = i \left(|\mu\rangle\langle\nu\,| - |\nu\rangle\langle\mu\,|\right)$ with (of course) $\langle\mu\,|\,\nu\rangle = \eta_{\mu\nu}$ and check that the matrices $J_{\mu\nu}$ defined this way are indeed the generators of the Lorentz group.

1.13 E13 : About the range of the exponential in sl(2,$\mathbb{C}$)

We want to show that the matrix

$$M = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \tag{22}$$

that belongs to SL(2,$\mathbb{C}$) cannot be obtained by exponentiating an element of the Lie algebra of SL(2,$\mathbb{C}$) (called $sl(2,\mathbb{C})$). A first possibility is to compute $\exp(i\vec{\alpha}\cdot\vec{\sigma})$ with α_i complex numbers and show directly that it cannot be equal to M whatever the α_i's. Another and more clever strategy consists in the following steps :

(i) Compute the eigenvalues of M. Then, using a proof by contradiction, assume that M can be diagonalized and find a contradiction.

(ii) Assume that there exists a matrix $m \in sl(2,\mathbb{C})$ such that $M = \exp(m)$. What is the trace of m ? Show that therefore m can be diagonalized and find a contradiction. Conclude.

1.14 E14 : Dirac bi-spinors and gamma matrices

We consider four Weyl spinors : $\psi_L, \psi_R, \chi_L, \chi_R$.

(i) Using for simplicity the M_1 and M_2 matrices, compute the transformation of $\chi_L^\dagger \psi_L$ under a Lorentz boost in the $\hat{x}$ direction. Perform the same calculation for $\chi_L^\dagger \vec{\sigma} \psi_L$. Conclude that $(\chi_L^\dagger \psi_L, \chi_L^\dagger \vec{\sigma} \psi_L)$ are the covariant components of a 4-vector.

(ii) Perform the same calculations for $(\chi_R^\dagger \psi_R, \chi_R^\dagger \vec{\sigma} \psi_R)$ and conclude that they are the contravariant components of a 4-vector.

(iii) From the 4-vectors above, find a true 4-vector and pseudo 4-vector.

(iv) We define :

$$\gamma^0 = \begin{pmatrix} & \mathrm{I}_2 \\ \mathrm{I}_2 & \end{pmatrix} \tag{23}$$

and

$$\gamma^i = \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix}. \tag{24}$$

Show that the true 4-vector found in (iii) is $\bar{\chi}\gamma^\mu\psi$ and the pseudo-vector $\bar{\chi}\gamma^\mu\gamma^5\psi$ where $\bar{\psi} = \psi^\dagger\gamma^0$ and ψ is the 4-component Dirac bi-spinor made out of ψ_L and ψ_R.

1.15 E15 : Lorentz transformations of the Dirac bi-spinors

We define the matrix S of transformations of the bi-spinors by $\chi' = S(\Lambda)\chi$ where Λ is a Lorentz matrix. For an infinitesimal transformation, we know that $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \epsilon^\mu{}_\nu$ with $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. We thus define the generators $\sigma^{\mu\nu}$ by :

$$S = S(\epsilon) = \mathrm{I}_4 - \frac{i}{4}\epsilon_{\mu\nu}\sigma^{\mu\nu}. \tag{25}$$

with $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$. Each $\sigma^{\mu\nu}$ is a 4×4 matrix with matrix elements $[\sigma^{\mu\nu}]_{\alpha\beta}$, $\alpha, \beta = 1, 2, 3, 4$. This is similar to the generators of rotations J_i, each of which being a 3×3 matrix.

(i) For infinitesimal Lorentz transformations, rewrite the transformation of the bi-spinor χ in terms of the rapidity $d\vec{\phi}$ and rotation (pseudo-)vector $d\vec{\theta}$.

(ii) Our task is to rewrite this transformation in terms of $\sigma^{\mu\nu}$. Consider the special case of an infinitesimal rotation around $\hat{x}$ of angle $d\theta_1$. Rewrite the transformation of χ in terms of the infinitesimal parameter ϵ_{32} (instead of $d\theta_1$) and the commutator of σ_2 and σ_3. Finally, rewrite the transformation of χ in terms of the commutator of γ_2 and γ_3.

(iii) For the particular rotation considered in (ii), rewrite the transformation of χ in terms of $\epsilon_{\mu\nu}$ and $\sigma^{\mu\nu}$. Conclude about a general infinitesimal transformation of parameter $\epsilon_{\mu\nu}$.

1.16 E16 : Translations and Lorentz transformations on classical or quantum fields

We call x^μ and x'^μ the coordinates of the same event in Minskowski space in two different frames (passive point of view) and $f(x)$ and $f'(x')$ (with $x = (x^0, x^1, x^2, x^3)$) two (classical) fields representing the same physical quantity.

(i) How x^μ and x'^μ are related when the frames are translated, rotated or Lorentz transformed ?

(ii) Recall how $f(x)$ and $f'(x')$ are related when the frames are translated. Same question for rotations and Lorentz transformations when f is a scalar, spinor, vector or tensor field.

(iii) We call $\delta f = f' - f$ the change of function when the transformations considered above are infinitesimal. Find what δf is for translations, rotations and Lorentz transformations in the case of a scalar, spinor or vector field. What are the infinitesimal generators of these transformations (more precisely, the representation of these generators in the space of fields). What is the commutator algebra of these generators ?

(iv) We now consider a quantum field $\phi(x)$. Recall how the translations and Lorentz transformations act on the expectation values of ϕ and what this implies on the Wigner operators representing the symmetries in the Fock space.

(v) We (abusively) give the same name to the generators of translations and Lorentz transformations in the Fock space and to the generators acting on classical fields : For instance, for infinitesimal translations, we expand the Wigner operator of the translations at first order and call P_μ the generator. Deduce from (iv) the commutator between P_μ and $\phi(x)$. Same question for Lorentz transformations.

2 Noether, Clifford, commutation relations

2.1 E17 : Noether charges and generators in the quantum case : integer spins

We call P_μ^{Noether} the Noether charge of translations. We shall check that the canonical equal time commutation relations :

$$[\phi(x), \phi(x')]^{=} = [\Pi(x), \Pi(x')]^{=} 0 \quad \text{and} \quad [\phi(x), \Pi(x')]^{=} = i\delta^{(3)}(\vec{x} - \vec{x}\,'). \tag{26}$$

imply that P_μ^{Noether} is the generator of translations in the Fock space.

(i) For a general lagrangian, compute $[P_i^{\text{Noether}}, \phi(x)]$ and conclude.

(ii) We consider a lagrangian where the only derivative term is $1/2\, \partial_\mu \phi \partial^\mu \phi$:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi(x)). \tag{27}$$

Compute $\Pi(x)$, the conjugate momentum of ϕ. Then, compute $[P_0^{\text{Noether}}, \phi(x)]$ and conclude.

(iii) Show that

$$[\phi^n(x), \Pi(x')]^{=} = in\phi^{n-1}(x)\delta^{(3)}(\vec{x} - \vec{x}\,') \tag{28}$$

and conclude that for any function f of the field $\phi(x)$ that can be expanded in a (convergent) series expansion :

$$[f(\phi(x)), \Pi(x')]^{=} = if'(\phi(x))\delta^{(3)}(\vec{x} - \vec{x}\,'). \tag{29}$$

(iv) Check that for fields satisfying the (Euler-Lagrange) equations of motion :

$$[P_\mu^{\text{Noether}}, \Pi(x)] = -i\, \partial_\mu \Pi(x) \tag{30}$$

What do you conclude about the Hamiltonian ?

2.2 E18 : Noether charges and generators in the quantum case : half integer spins

We consider Dirac bi-spinors ψ and lagrangians $\mathcal{L}$ whose dependence on $\partial_\mu \psi$ is at most linear on $\partial_\mu \psi$, that is, is $i\bar{\psi}\gamma^\mu \partial_\mu \psi$.

(i) Find the conjugate momentum of $\psi_\alpha(x)$.

(ii) The part of the lagrangian $\mathcal{L}$ that does not depend on the derivatives of the field, that is, that does neither depend on $\partial_\mu \psi$ nor on $\partial_\mu \bar{\psi}$ can a priori involve terms depending on ψ and $\bar{\psi}$. Find several examples of such terms (they must be Lorentz invariant and hermitic).

(iii) We now assume that ψ and its conjugate momentum satisfy the canonical equal-time anti-commutation relations :

$$\begin{aligned} \{\psi_\alpha(x), \Pi_\beta(y)\}^{=} &= i\delta_{\alpha\beta}\delta^{(3)}(\vec{x} - \vec{y}) \\ \{\psi_\alpha(x), \psi_\beta(x)(y)\}^{=} &= \{\Pi_\alpha(x), \Pi_\beta(x)(y)\}^{=} = 0. \end{aligned} \tag{31}$$

Compute $[P_\mu^{\text{Noether}}, \psi(x)]$. The following relation can be useful

$$[AB, C] = A\{B, C\} - \{A, C\}B. \tag{32}$$

Conclude.

(iv) Redo the same calculation for $[P_\mu^{\text{Noether}}, \Pi(x)]$. Conclude.

2.3 E19 : Clifford algebra, traces of products of γ matrices and Dirac equation

We define $\not{p} = p_\mu \gamma^\mu$.

(i) Is $\not{p}$ a number ? a 4-vector ? a matrix ?

(ii) Show that $\not{p}^2 = p^2 I_4$ and that $\gamma_\mu \gamma^\mu = 4\text{I}$.

(iii) Compute $\text{Tr}(\gamma^\mu \gamma^5)$, $\text{Tr}(\gamma^\mu \gamma^\nu)$, $\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha)$, $\text{Tr}(\gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta)$. All these traces are very important for the calculation of S-matrix elements. They can (and must !) be performed without having recourse to the explicit values of the γ^μ-matrices : the Clifford algebra together with the γ^5 matrix will be enough. We recall that the γ^5 matrix anti-commutes with all γ^μ matrices and satisfies $\left(\gamma^5\right)^2 = \text{I}$.

(iv) Compute $\text{Tr}(\gamma^5 \gamma^\mu \gamma^\nu)$,

(v) Compute $\text{Tr}(\not{p}\not{k})$, $\text{Tr}(\not{p}\not{k}\not{q})$, $\text{Tr}(\not{p}\not{k}\not{p}'\not{k}')$.

(vi) Multiply on the left the Dirac equation (for a free particle) by $(i\not{\partial} + m)$ and show that the Dirac equation on ψ implies the Klein-Gordon equation on (each component of) ψ.