Correction – Accelerator Physics Tutorials

Master NPAC (Nuclei, Particles, Astroparticles, Cosmology)

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Exercise 1: Cyclotron

1. Give the exit kinetic energy T of ions?

   The maximum magnetic rigidity $(B\rho)_{\max}$ is $BD/2$=1 T m. The maximum kinetic energy is deduced from magnetic rigidity:

$$pc = \gamma\beta mc^2 = \sqrt{\gamma^2 - 1}E_0$$

$$\gamma = \sqrt{1 + \left(\frac{pc}{E_0}\right)^2}, \qquad B\rho = \frac{p}{q}, \qquad T = (\gamma - 1)\,E_0$$

$$T = \left(\sqrt{1 + \left(\frac{(B\rho)qc}{E_0}\right)^2} - 1\right)E_0$$

$$T_{\max} \approx 4.019\,\mathrm{MeV}$$

2. Give the average energy gain per accelerating gap.

   At each turn, there are 2 crossings through the accelerating gap. The energy gain per turn is then:

$$\Delta E = \frac{T_{\max} - T_{\mathrm{inj}}}{2 \times 30} \approx 66.8\,\mathrm{keV}$$

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3. Give the RF frequency.

   The half-turn time Δt for an ion is:

$$\Delta t = \frac{\pi \rho}{\beta c} = \frac{\pi p}{Bq\beta c} = \frac{\pi \gamma m_0}{Bq}$$

   The cyclotron pulsation is:

$$\omega = \frac{\pi}{\Delta t} = \frac{qB}{\gamma m_0}$$

   The RF is synchronized with the ion pulsation when $\omega = \omega_{\mathrm{RF}}$ (the dephasing of the cavity is π when the ion makes one half-turn). The particle is on phase when the curvature radius is 0.5 m corresponding to a magnetic rigidity of $B\rho$=0.5 T m. By using

$$T = E_0 \left(\sqrt{1 + \left(\frac{(B\rho)qc}{E_0} \right)^2} - 1 \right) = 1.005\,\mathrm{MeV}$$

   We find $\gamma_{\mathrm{RF}} - 1 \approx 8.991 \times 10^{-5}$. The RF frequency is thus:

$$f_{\mathrm{RF}} = \frac{\omega}{2\pi} = \frac{qB}{2\pi \gamma_{\mathrm{RF}} m_0}$$
$$f_{\mathrm{RF}} \approx 1.280\,\mathrm{MHz}$$

4. Calculate the injection phase and the energy gain in each gap.

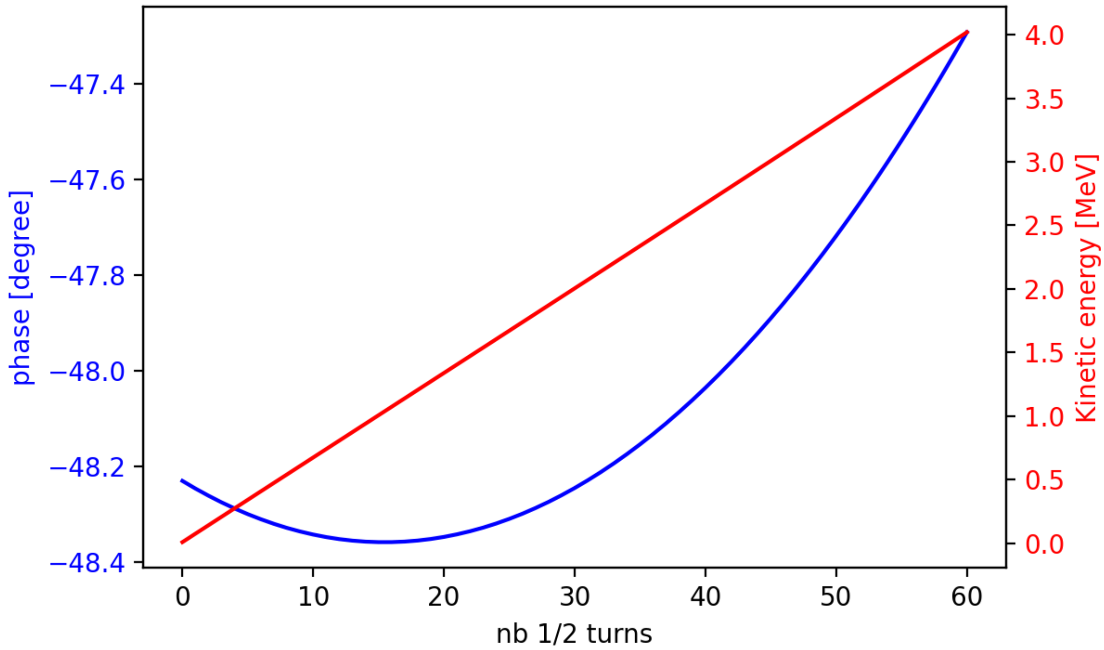
   The energy gain per gap is given by $qV_{\mathrm{RF}} \cos\phi$. The dephasing $\Delta\phi$ of any particle in one half-turn is:

$$\Delta\phi = \pi \left(\omega_{RF} \frac{\gamma m_0}{Bq} - 1 \right)$$
$$\Delta\phi = \pi \left(\frac{\gamma}{\gamma_{RF}} - 1 \right)$$

   We have thus the following sequence:

$$\gamma_{n+1} = \gamma_n + q\frac{V_{\mathrm{RF}}}{E_0} \cos(\phi_n)$$
$$\phi_{n+1} = \phi_n + \pi \left(\frac{\gamma_n}{\gamma_{RF}} - 1 \right)$$
$$\gamma_0 = 1 + \frac{T_{\mathrm{inj}}}{E_0}$$

   We have to adjust ϕ_0 to get $(\gamma_{60} - 1)\, E_0 = T_{\max}$ with $V_{\mathrm{RF}} = 100\,\mathrm{kV}$. We find $\phi_0 \approx -48.24°$.



Exercise 2: 3 gap cavity

1. Give a first guess of the v_{opt} for an optimum acceleration.

   A first guess is to have the maximum acceleration when both gaps are with a dephasing of π.

$$\begin{aligned} v_{\text{opt}} &= \frac{\omega d}{\pi} = 2fd \\ &= 2 \times 10^7 \,\mathrm{m\,s^{-1}} \\ \beta_{opt} &= 0.06667 \end{aligned}$$

2. Give the transit time factor T of the system.

   By definition, $T = \dfrac{\left| \int E(z) \exp\left(\imath \dfrac{\omega z}{\beta c} \right) dz \right|}{\int |E(z)|\, dz}$.

   It is straightforward that $\int |E(z)|\, dz = E_0(g_1 + 2g_2)$.

$$T = \frac{1}{g_1 + 2g_2} \left| \int_{-g_1/2}^{g_1/2} \exp\left(\imath \frac{\omega z}{\beta c}\right) dz - \int_{-d-g_2/2}^{-d+g_2/2} \exp\left(\imath \frac{\omega z}{\beta c}\right) dz - \int_{d-g_2/2}^{d+g_2/2} \exp\left(\imath \frac{\omega z}{\beta c}\right) dz \right|$$

$$T = \frac{1}{g_1 + 2g_2} \frac{2\beta c}{\omega} \left[\sin\left(\frac{\omega g_1}{2\beta c}\right) - 2\cos\left(\frac{\omega d}{\beta c}\right) \sin\left(\frac{\omega g_2}{2\beta c}\right) \right]$$

3. Give the associated $T = T_{\text{opt}}$.

After simplification with $d = g_1 = 2g_2 = 2g$:

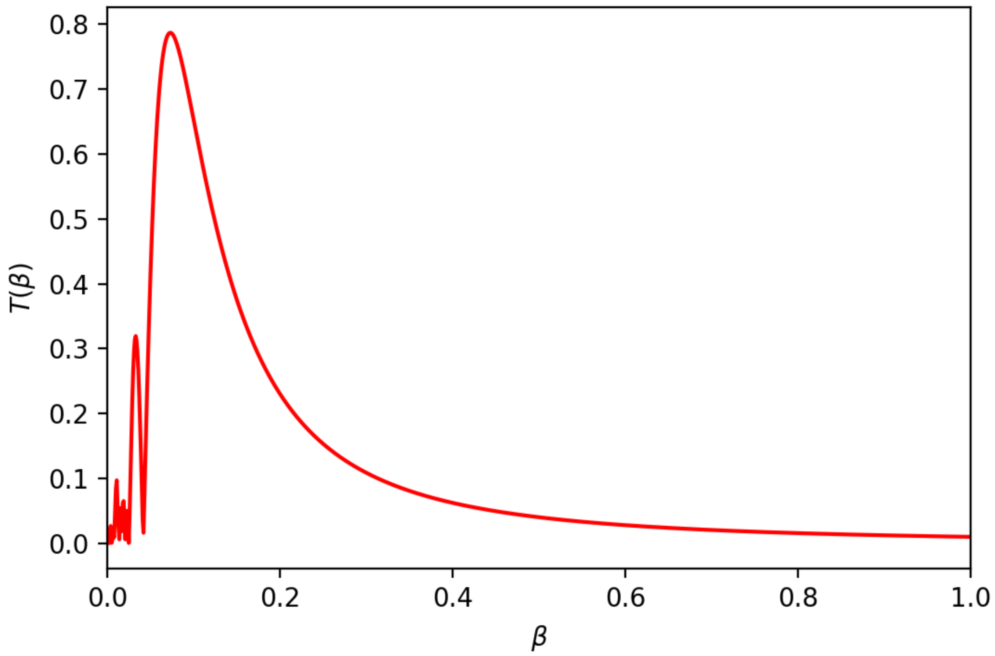
$$T = \frac{\beta c}{d\omega} \left[\sin\left(\frac{\omega d}{2\beta c}\right) - 2\cos\left(\frac{\omega d}{\beta c}\right) \sin\left(\frac{\omega d}{4\beta c}\right) \right]$$

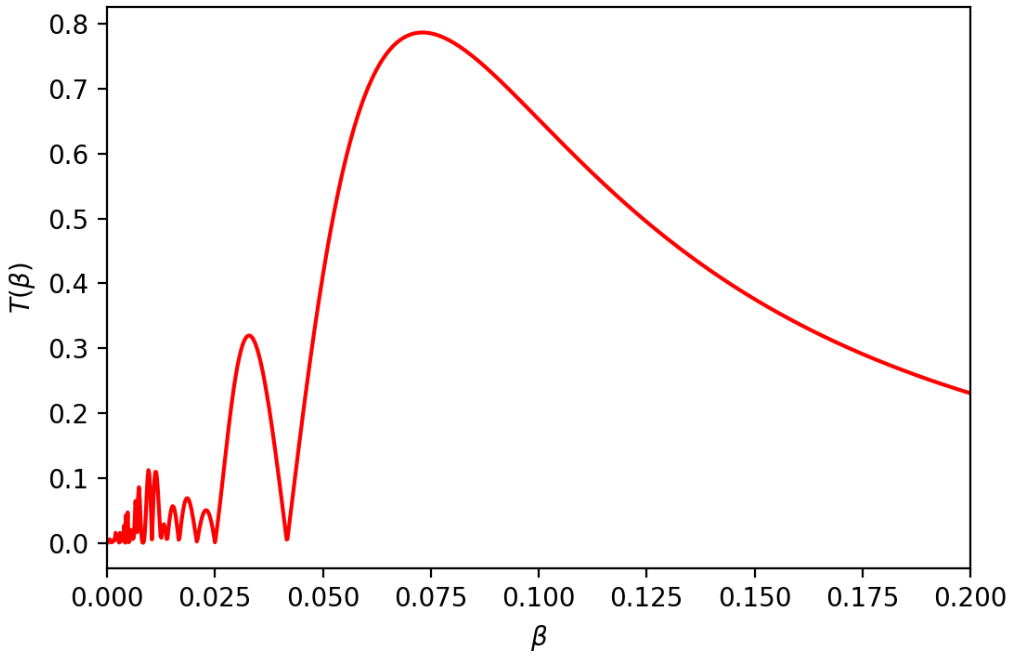
At the guessed velocity $\beta_{\text{opt}} c = \frac{\omega d}{\pi} = \frac{2\omega g}{\pi}$, we get:

$$T_{\text{opt}} = \frac{1 + \sqrt{2}}{\pi} \approx 0.769$$

A numerical solving of $T'(\beta) = 0$ gives the optimum values, near the first guess:

$$\beta_{\text{opt}} \approx 0.073 \qquad\qquad T_{\text{opt}} \approx 0.7866$$





Exercise 3: Storage ring

1. Calculate the curvature radius in dipoles.

$$\rho = \frac{B\rho}{B} = \frac{E_0\sqrt{\gamma^2 - 1}}{B \cdot q \cdot c}$$
$$\approx 4.613\,\mathrm{m}$$

2. Calculate the electron energy loss per turn.

$$\Delta E(\mathrm{keV}) = 88.4 \frac{E(\mathrm{GeV})^4}{\rho_m} \approx 409.5\,\mathrm{keV}$$

3. Calculate the harmonic number h and the synchronous phase φ_s.

$$f_{\mathrm{rev}} = \frac{c}{C} = 892\,\mathrm{kHz}$$
$$h = \frac{f_{RF}}{f_{\mathrm{rev}}} = 560$$

Energy gain per turn: $\Delta E = |q| V \sin\phi_s$

$\gamma_{tr} = \frac{1}{\sqrt{\alpha}} = 42.5$. $\gamma > \gamma_{tr} \Rightarrow$ We are above the transition then $\varphi_s \in [90°, 180°]$.

$\varphi_s = \pi - \arcsin \frac{\Delta E}{E} = 2.94\,\text{rad} = 168°$

4. What is the maximum energy acceptance $(\Delta E/E)$?

   $\eta = \frac{1}{\gamma^2} - \alpha = -5.5 \times 10^{-4}$

   Maximum energy:

$$\delta E_{\text{max}} = \sqrt{2qE_0T\left(\cos\varphi_s - \left(\frac{\pi}{2} - \varphi_s\right)\sin\varphi_s\right) \cdot \frac{\beta_s^3\gamma_s mc^2\lambda_{\text{RF}}}{\pi\eta}}$$

   Energy acceptance:

$$\begin{aligned}\frac{\Delta E}{E} &= \sqrt{2qE_0T\left(\cos\varphi_s - \left(\frac{\pi}{2} - \varphi_s\right)\sin\varphi_s\right) \cdot \frac{\beta_s^3\lambda_{\text{RF}}}{\pi\eta E_{\text{tot}}}} \\ &= 3.67\%\end{aligned}$$

5. Calculate the cavity effective voltage V to set $\Delta E/E = \pm 4\%$

$$\begin{aligned}V &= V_0\left(\frac{\Delta E}{E}\right)^2 / \left(\frac{\Delta E}{E}\right)_0^2 \\ &= 2.39 MV\end{aligned}$$

6. Calculate the rms energy dispersion of a matched beam with longitudinal rms emittance $430\pi\,°\,\text{MeV}$.

$$\begin{aligned}H(\phi, \delta E) &= \frac{\pi\eta}{\lambda_{\text{RF}}}\frac{\delta E^2}{\beta_s^3\gamma_s mc^2} - qE_0T\left(\sin\varphi_s\left(\phi - \sin\phi\right) - \cos\varphi_s\left(1 - \cos\phi\right)\right) \\ \delta E &= \sqrt{\frac{\beta_s^3\gamma_s mc^2 qE_0T\lambda_{\text{RF}}}{\pi\eta}\left(\sin\varphi_s\left(\phi_m - \sin\phi_m\right) - \cos\varphi_s\left(1 - \cos\phi_m\right)\right)}\end{aligned}$$

   Numerically find ϕ such as $\phi(°) \cdot \delta E(\phi)(\text{MeV}) = 430\,°\,\text{MeV}$. Solution: $\phi = 18.444°$

7. Calculate the synchrotron oscillation pulsation Ω_s.

   By definition:

$$\begin{aligned}\Omega_s &= 2\pi f_r\sqrt{\frac{\eta\mathcal{C}^2}{2\pi\lambda_{\text{RF}}}\frac{qE_0T}{\beta_s^3\gamma_s mc^2}\cos\varphi_S} \\ &= 40\,867\,\text{rad}\,\text{s}^{-1}\end{aligned}$$

8. What is the ratio between the betatron wave numbers and the longitudinal one.

   Synchrotron tune: $\nu_z = \frac{\Omega_s}{2\pi f_r} \approx 0.0458$

$$\frac{\nu_x}{\nu_z} \approx 400 \qquad \frac{\nu_y}{\nu_z} \approx 181$$

Exercise 4: Space-charge

1. Write down the electric components $E_g(r)$ et $E_u(r)$.

   The beam has a cylinder symmetry and is continuous (thus invariant by translation along longitudinal axis). The symmetry conditions imply a radial electric field only depending on r(where r is the radius in cylindrical coordinates).

   Let us consider a cylinder of radius r and arbitrary length L. Let be Q_{tot} the total charge in the cylinder. We have:

$$I = \frac{dQ_{\text{tot}}}{dt} \tag{1}$$

   Since the beam is continuous, l is constant and $dz = \beta c dt$, which gives:

$$Q_{\text{tot}} = \frac{IL}{\beta c} \tag{2}$$

   The Gauss theorem gives:

$$\oiint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{int}}}{\epsilon_0} \tag{3}$$

$$Q_{\text{int}} = \iiint \rho d\tau \tag{4}$$

   Since $\mathbf{E} = E(r)\mathbf{e}_r$, we have:

$$\oiint \mathbf{E} \cdot d\mathbf{S} = 2\pi r L E(r) \tag{5}$$

$$Q_{\text{int}} = 2\pi L \int_0^r u\rho(u)du \tag{6}$$

$$E(r) = \frac{1}{r\epsilon_0} \int_0^r u\rho(u)du \tag{7}$$

   a) Case of a uniform beam

      We have:

$$\int_0^r u\rho_u(u)du = \begin{cases} \rho_u(0)\frac{r^2}{2} & \text{si } r < R \\ \rho_u(0)\frac{R^2}{2} & \text{si } r \geq R \end{cases}$$

We have: $Q_{\text{tot}} = \frac{IL}{\beta c}$ with $Q_{\text{tot}} = 2\pi L \int_0^\infty u\rho(u)du$. Thus:

$$\rho_u(0) = \frac{I}{\beta c \pi R^2} \tag{8}$$

Finally we get:

$$E_u(r) = \frac{I}{2\pi\epsilon_0 \beta c} \begin{cases} \frac{r}{R^2} & \text{si } r < R \\ \frac{1}{r} & \text{si } r \geq R \end{cases} \tag{9}$$

b) Case of a Gaussian beam

We have:

$$\int_0^r u\rho_g(u)du = \rho_g(0) \int_0^r u \exp\left(-\frac{u^2}{r_0^2}\right) du \tag{10}$$

$$= \frac{\rho_g(0) r_0^2}{2} \left(1 - \exp\left(-\frac{r^2}{r_0^2}\right)\right) \tag{11}$$

We have: $Q_{\text{tot}} = \frac{IL}{\beta c}$ with $Q_{\text{tot}} = 2\pi L \int_0^\infty u\rho(u)du$. We get:

$$\rho_g(0) = \frac{I}{\beta c \pi r_0^2} \tag{12}$$

Finally, we get:

$$E_g(r) = \frac{I}{2\pi\epsilon_0 \beta c} \frac{1 - \exp\left(-\frac{r^2}{r_0^2}\right)}{r} \tag{13}$$

2. Write down the RMS beam size

   By definition, we have:

$$\sigma_r = \sqrt{\frac{\iiint r^2 \rho(r) d\tau}{\iiint \rho(r) d\tau}} \tag{14}$$

By using the symmetry, we get:

$$\sigma_r = \sqrt{\frac{\int_0^\infty r^3 \rho(r) dr}{\int_0^\infty r \rho(r) dr}} \tag{15}$$

a) Case of a uniform beam.

We get:

$$\int_0^\infty r^3 \rho_u(r) dr = \int_0^R r^3 \rho_u(0) dr = \frac{R^4}{4}\rho_u(0) \tag{16}$$

$$\int_0^\infty r \rho_u(r) dr = \int_0^R r \rho_u(0) dr = \frac{R^2}{2}\rho_u(0) \tag{17}$$

Finally we get:

$$\sigma_u = \frac{R}{\sqrt{2}} \tag{18}$$

b) Case of a Gaussian beam.

We get:

$$\begin{aligned}
\int_0^\infty r^3 \rho_g(r) dr &= \int_0^\infty r^3 \rho_g(0) \exp\left(-\frac{r^2}{r_0^2}\right) dr && (19)\\
&= \rho_g(0) \left\{ \left[-\frac{r_0^2}{2} r^2 \exp\left(-\frac{r^2}{r_0^2}\right)\right]_0^\infty - \int_0^\infty 2r \frac{-r_0^2}{2} \exp\left(-\frac{r^2}{r_0^2}\right) \right\} && (20)\\
&= \rho_g(0) r_0^2 \int_0^\infty r \exp\left(-\frac{r^2}{r_0^2}\right) && (21)\\
&= r_0^2 \int_0^\infty r \rho_g(r) dr && (22)
\end{aligned}$$

Finally we get:

$$\sigma_g = r_0 \tag{23}$$

3. Express $E_g(r)$ as a function of $E_u(r)$ by considering they have the same RMS size and beam current.

We have then $r_0 = \frac{R}{\sqrt{2}}$, which gives:

$$\begin{aligned}
E_g(r) &= \frac{I}{2\pi\epsilon_0 \beta c} \frac{1 - \exp\left(-\frac{2r^2}{R^2}\right)}{r} && (24)\\
&= E_u(r) \left(1 - \exp\left(-\frac{2r^2}{R^2}\right)\right) \begin{cases} \frac{R^2}{r^2} & \text{si } r < R \\ 1 & \text{si } r \geq R \end{cases} && (25)
\end{aligned}$$

4. Express the square of the tune depression $\eta_g^2(r)$ as a function of $\eta_u^2(r)$.

   By definition,

$$\eta = \frac{k_x}{k_{x,0}} \tag{26}$$

   where k_x^2 is the normalized strength with space-charge and $k_{x,0}^2$ with no space-charge. We have $k_x^2 = k_{x,0}^2 - K_{CE}$ with K_{CE} the normalized strength due to space-charge. We get then:

$$\eta = \sqrt{1 - \frac{K_{CE}}{k_{x,0}^2}} \tag{27}$$

   The normalized strength K_{CE} is proportional to the electric field at the particle.Let be α this factor. it does not depend on the distribution. We have then:

$$\eta_u(r) = \sqrt{1 - \alpha\frac{E_u(r)}{k_{x,0}^2}} \tag{28}$$

$$\eta_g(r) = \sqrt{1 - \alpha\frac{E_g(r)}{k_{x,0}^2}} \tag{29}$$

   Thus,

$$\frac{1}{k_{x,0}^2} = \frac{1-\eta_u^2(r)}{\alpha E_u(r)} \tag{30}$$

$$\eta_g^2(r) = 1 - (1-\eta_u^2(r))\frac{E_g(r)}{E_u(r)} \tag{31}$$

$$\eta_g^2(r) = 1 - (1-\eta_u^2(r))\left(1 - \exp\left(-\frac{2r^2}{R^2}\right)\right)\begin{cases}\frac{R^2}{r^2} & \text{si } r < R \\ 1 & \text{si } r \geq R\end{cases} \tag{32}$$

5. What is this value on-axis?

   For $r \ll 1$, we get:

$$\eta_g^2(r) \equiv 1 - 2(1-\eta_u^2(0)) + o(1) \tag{33}$$

   Thus, $\eta_g^2(0) = 2\eta_u^2(0) - 1$

6. For which value of $\eta_u(0)$ are the space charge forces greater than the external focusing strengths?

   That happens when $\eta_g(r) \leq 0$ which corresponds to $\eta_u(0) \leq \frac{1}{\sqrt{2}}$.