Particle Accelerators 5

Introduction to collective effects: space-charge

Antoine Chancé

CEA Saclay DRF/IRFU/DACM

NPAC-2022

Lecture content

1. **Space-charge force**
2. **Linear(ized) motion**
3. **Non-linear effects**
4. **Wall effects**

Space-charge force

1. Space-charge force

Field produced by charge and current densities

General case: Maxwell equations

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{B} = \mu_0 \cdot \mathbf{j} + \dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} \end{cases}$$

These charge ρ and current $\mathbf{j}$ densities are:

- those of the beam (direct space-charge),
- those induced in surrounding material (indirect space-charge).

Two solutions

- Simplified model: static
- Numerical resolution

Static model: application case

- When beam is continuous, one assumes that charge and current distributions at a given position are **stationary**. Fields are then invariant with time and electric and magnetic fields are independent.

$$\rho(\mathbf{r},t),\mathbf{j}(\mathbf{r},t) \longrightarrow \rho(\mathbf{r}),\mathbf{j}(\mathbf{r})$$

- In the bunched beam frame, the particle relative displacements are generally non-relativistic and field is mainly electrostatic.

$$\mathbf{E}^*(\mathbf{r},t),\mathbf{B}^*(\mathbf{r},t) \xrightarrow{\beta^*\ll 1} \mathbf{E}^*(\mathbf{r}),\mathbf{0} \xrightarrow[\text{transform}]{\text{Lorentz}} \mathbf{E}(\mathbf{r},t),\mathbf{B}(\mathbf{r},t)$$

Except in specific cases, the magnetic field is not directly computed but

the magnetic force is deduced from the electric force.

Electrostatic field

Charge distribution

A still **charge density** ρ $[\mathrm{C\,m^{-3}}]$ produces an **electrostatic field**:

$\mathbf{E}$ $[\mathrm{V\,m^{-1}}]$ solution of equations : $\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{E} = \mathbf{0} \end{cases}$

The solution of two coupled equations is not obvious as once we found a solution of the first, it has to satisfy the second one.

It is then easier to solve a unique equation by remarking that $\nabla \times \left(\nabla f\right) = \mathbf{0}$, whatever f.

Defining: $\mathbf{E} = -\nabla\phi$ With ϕ $[\mathrm{V}]$ the scalar **electrostatic potential**.
The system becomes:

$$\nabla \cdot \left(\nabla\phi\right) = \Delta\phi = -\frac{\rho}{\epsilon_0}$$

Magnetostatic field

Current distribution

A **current flux** $\mathbf{j}$ $[\mathrm{A\,m^{-2}}]$ produces a **magnetic field**:

$\mathbf{B}$ $[\mathrm{T}]$ solution of equations : $\begin{cases} \nabla \times \mathbf{B} = \mu_0 \cdot \mathbf{j} \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$

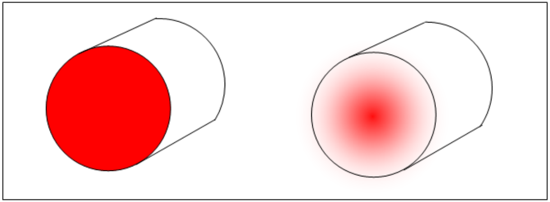
The solution of two coupled equations is not obvious as once we found a solution of the first, it has to satisfy the second one.

It is then easier to solve a unique equation by remarking that $\nabla \cdot (\nabla \times \mathbf{f}) = \mathbf{0}$, whatever $\mathbf{f}$.

Defining: $\mathbf{B} = \nabla \times \mathbf{A}$ With $\mathbf{A}$ $[\mathrm{T\,m}]$ the **magnetic vector potential**.
The system becomes:

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}$$

Cylindrical continuous beam



$$\rho(x, y, z) \to \rho(r)$$
$$\mathbf{j}(x, y, z) \to j(r)\mathbf{e}_z$$
$$r = \sqrt{x^2 + y^2}$$

Gauss theorem: $\oiint \mathbf{E} \cdot \mathbf{dS} = \frac{\iiint \rho d\tau}{\epsilon_0}$

$$E_r(r) = \frac{1}{\epsilon_0 \cdot r} \int_0^r r' \cdot \rho(r') \cdot dr'$$

Ampere theorem: $\oint \mathbf{B} \times \mathbf{dl} = \mu_0 \iiint \mathbf{j} dS$

$$B_\theta(r) = \frac{\mu_0}{r} \int_0^r r' \cdot \mathbf{j}(r') \cdot dr'$$

Cylindrical continuous beams

Some examples

Charge per linear meter:

$$\rho(r) = \frac{\lambda_0}{2\pi r} \cdot \delta(r) \qquad \longrightarrow \qquad E_r(r) = \frac{\lambda_0}{2\pi\epsilon_0} \cdot \frac{1}{r}$$

Uniform beam: electric field linear with r in beam

$$\rho(r) = \begin{cases} \frac{\lambda}{\pi \cdot R_h^2} & \text{if } r < R_h \\ 0 & \text{otherwise} \end{cases} \qquad \longrightarrow \qquad E_r(r) = \begin{cases} \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{r}{R_h^2} & \text{if } r < R_h \\ \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{1}{r} & \text{otherwise} \end{cases}$$

Gaussian beam:

$$\rho(r) = \frac{\lambda}{2\pi\sigma_r^2} \cdot \exp\left(-\frac{r^2}{2\sigma_r^2}\right) \qquad \longrightarrow \qquad E_r(r) = \frac{\lambda}{2\pi\epsilon_0} \cdot \frac{1}{r} \cdot \left(1 - \exp\left(-\frac{r^2}{2\sigma_r^2}\right)\right)$$

- λ $[\mathrm{C\,m^{-1}}]$ is the charge per linear meter: $\lambda = \frac{I}{\bar{\beta} \cdot c}$
- I [A] is the beam current
- $\bar{\beta}$ is the beam particle average velocity

Cylindrical continuous beam

electric – magnetic forces

Assuming that all particles have the same velocity: $\mathbf{v} = \bar{\beta}_z c \cdot \mathbf{u}_z$

$$\mathbf{j}(r) = \rho(r) \cdot \bar{\beta}_z c \cdot \mathbf{u}_z \quad \longrightarrow \quad B_\theta(r) = \frac{\mu_0 \cdot c}{r} \cdot \bar{\beta}_z \cdot \int_0^r r' \cdot \rho(r') \cdot dr' = E_r(r) \cdot \frac{\bar{\beta}_z}{c}$$

The force on each particle with charge q and longitudinal **reduced velocity** β_z is:

$$F_r = q\,(E_r - v_z \cdot B_\theta + v_\theta \cdot B_z) = q\left(E_r - \beta_z c \cdot B_\theta\right)$$
$$F_r = q \cdot E_r(r) \cdot \left(1 - \beta_z \cdot \bar{\beta}_z\right)$$

Paraxial approximation: $\beta^2 = \beta_x^2 + \beta_y^2 + \beta_z^2 \approx \beta_z^2$

$$\boxed{F_r = q \cdot E_r(r) \cdot \left(1 - \beta^2\right) = \frac{q \cdot E_r(r)}{\gamma^2}}$$

F_r scales with $1/\gamma^2$: Laplace force mitigates Coulomb repulsion.

Elliptical uniform continuous beam

$$\rho(x,y,z) = \begin{cases} \frac{\lambda}{\pi \cdot X \cdot Y} & \text{if } \frac{x^2}{X^2} + \frac{y^2}{Y^2} < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{cases} E_x(x,y) = \dfrac{\lambda}{2\pi^2 \cdot \epsilon_0 \cdot X \cdot Y} \displaystyle\int_{-Y}^{Y} dy' \cdot \int_{-X\sqrt{1-y^2/Y^2}}^{X\sqrt{1-y^2/Y^2}} dx' \dfrac{x-x'}{\left((x-x')^2 + \left(y-y'\right)^2\right)^{1/2}} \\ E_y(x,y) = \dfrac{\lambda}{2\pi^2 \cdot \epsilon_0 \cdot X \cdot Y} \displaystyle\int_{-X}^{X} dx' \cdot \int_{-Y\sqrt{1-x^2/X^2}}^{Y\sqrt{1-x^2/X^2}} dy' \dfrac{y-y'}{\left((x-x')^2 + \left(y-y'\right)^2\right)^{1/2}} \end{cases}$$

$$\boxed{\begin{cases} E_x(x,y) = \dfrac{\lambda}{\pi \cdot \epsilon_0} \cdot \dfrac{1}{X+Y} \cdot \dfrac{x}{X} \\ E_y(x,y) = \dfrac{\lambda}{\pi \cdot \epsilon_0} \cdot \dfrac{1}{X+Y} \cdot \dfrac{y}{Y} \end{cases}}$$

(In beam)

Electric field linear with position

Numerical methods

The space-charge field produced by a set of N particles can be calculated with different **space-charge routines**:

- **PPI** (Particle-Particle Interactions) methods
- **PIC** (Particles in cells) methods
  - direct,
  - FFT,
  - relaxation.
- Functional methods

PPI (Particle-Particle Iteraction) Method

At each time step, the field contribution of all particles is calculated at the position of each particle:

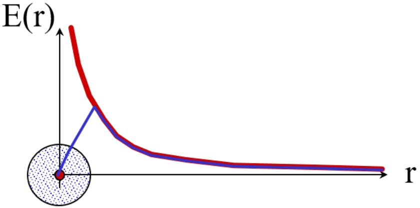
$$\mathbf{E}(\mathbf{r}_i) = \sum_{j \neq i}^{N} \frac{q_j}{4\pi\epsilon_0} \frac{\mathbf{r}_i - \mathbf{r}_j}{\left\|\mathbf{r}_i - \mathbf{r}_j\right\|^3}$$

Advantages

- No mesh
- Easy to compute

Drawbacks

- Long ($\propto N^2$),
- Artificially colliding



PIC (Particles in Cells) method

- Particles are counted in a mesh with n lattices
- In the direct method, the influence of the density in each lattice is calculated on each mesh node.

Advantages

- Low noise (charge smoothing on the mesh)

Drawbacks

- Long ($\propto n^2$),
- No image charge.

PIC FFT method

$$\phi(x_0, y_0, z_0) = \frac{1}{4\pi\epsilon_0} \iiint_{\text{space}} \frac{\rho(x, y, z)}{\sqrt{(x - x_0)^2 + \left(y - y_0\right)^2 + (z - z_0)^2}} dx \cdot dy \cdot dz$$

$$= \left(\rho * G\right)(x, y, z)$$

$$\text{With } : G = \frac{1}{4\pi\epsilon_0} \cdot \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

Then: $\phi(x, y, z) = FFT^{-1}\left(FFT(\rho) \times FFT(G)\right)$

Advantages

- Fast ($\propto n \cdot \log(n)$)

Drawbacks

- Noisy.
- No image charge.

PIC relaxation method

Illustration in 1D : $\frac{\partial^2\phi}{\partial x^2} = -\frac{\rho(x)}{\epsilon_0} = \rho'(x)$

On each lattice: $\phi_{i+1} - 2\phi_i + \phi_{i-1} = \rho'_i \cdot \delta^2$

Iterative process k: $\phi_i^{k+1} = \phi_i^k + \alpha\left(\frac{\phi_{i+1}^k + \phi_{i-1}^k - \rho'_i \cdot \delta^2}{2} - \phi_i^k\right)$

Advantages

- Could be fast ($\propto n \cdot \log(n)$, for multigrid)
- Image charge

Drawbacks

- Limit condition should be defined (or assumed).

Functional method

$$\rho(\mathbf{r}) = \sum_j A_j \cdot P_j(\mathbf{r}) \qquad \text{with:} \qquad A_j = \sum_{i=1}^{N} F\left(\rho\left(\mathbf{r}_i\right), P_j\left(\mathbf{r}_i\right)\right)$$

with P_j such as: $\Delta\Gamma_j(\mathbf{r}) = P_j(\mathbf{r})$

$$\Rightarrow \quad \phi(\mathbf{r}) = \sum_j A_j \cdot \Gamma_j(\mathbf{r})$$

Advantages

- Mathematically smart.

Drawbacks

- Very long,
- Noisy,
- No image charge

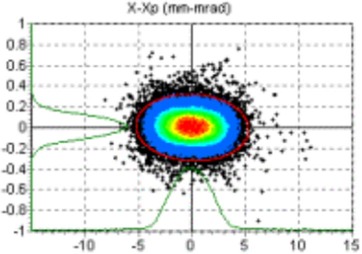
Linear(ized) motion

2. Linear(ized) motion

Statistical representation

Beam: Set of billions (N) of particles evolving with an independent variable s (time, curved abscissa...)

Macro-particle model: → Set of n macro-particles ($n < N$)



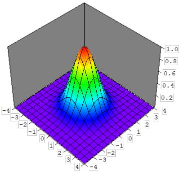
6 coordinates :

- 3 for position: $\mathbf{r}$ (Cartesian, cylindrical...)
- 3 for motion: $\mathbf{p}$ (velocity, momentum, energy, slope...)

Distribution function model: → function

$$f(\mathbf{r},\mathbf{p}) \cdot d^3r \cdot d^3p$$

Expected number of particles
between $\mathbf{r}$ and $\mathbf{r}+d\mathbf{r}$
between $\mathbf{p}$ and $\mathbf{p}+d\mathbf{p}$



RMS sizes

Average value of $A\left(\mathbf{r},\mathbf{r}'\right)$ over the beam:

$$\left\langle A\left(\mathbf{r},\mathbf{r}'\right)\right\rangle = \frac{1}{n}\sum_{i=1}^{n} A\left(\mathbf{r}_i,\mathbf{r}'_i\right) = \frac{1}{N}\iint d^3 f\left(\mathbf{r},\mathbf{r}'\right)\cdot A\left(\mathbf{r},\mathbf{r}'\right) d^3\mathbf{r}'$$

- Examples:

C.o.g position: $\left(\langle u\rangle,\langle u'\rangle\right)$

RMS size: $u_{\text{rms}} = \sqrt{\sigma_u} = \sqrt{\left\langle (u-\langle u\rangle)^2\right\rangle}$

RMS slope: $u'_{\text{rms}} = \sqrt{\sigma_{u'}} = \sqrt{\left\langle (u'-\langle u'\rangle)^2\right\rangle}$

RMS coupling: $uu'_{\text{rms}} = \sigma_{uu'} = \left\langle (u-\langle u\rangle)\cdot\left(u'-\langle u'\rangle\right)\right\rangle$

RMS emittance: $\epsilon_{u,\text{rms}} = \sqrt{u^2_{\text{rms}}\cdot u'^2_{\text{rms}} - \left(uu'_{\text{rms}}\right)^2}$

The ellipse matching the best the beam distribution is:

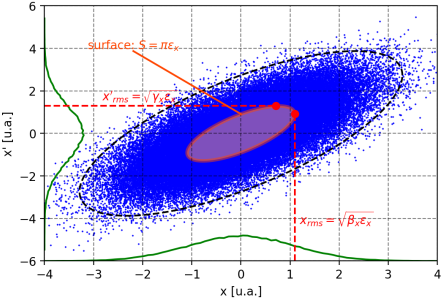
$$\gamma_{t,u} \cdot u^2 + 2\alpha_{t,u} \cdot u \cdot u' + \beta_{t,u} \cdot u'^2 = A_u$$

Such as:

$$\beta_{t,u} = \frac{u^2_{\text{rms}}}{\epsilon_{u,\text{rms}}} = \frac{\sigma_u}{\epsilon_{u,\text{rms}}}$$

$$\gamma_{t,u} = \frac{u'^2_{\text{rms}}}{\epsilon_{u,\text{rms}}} = \frac{\sigma_{u'}}{\epsilon_{u,\text{rms}}}$$

$$\alpha_{t,u} = -\frac{uu'_{\text{rms}}}{\epsilon_{u,\text{rms}}} = -\frac{\sigma_{uu'}}{\epsilon_{u,\text{rms}}}$$



Are the beam's Twiss Parameters.

6D model

Particle 6D phase-space coordinates can be:

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} \qquad \begin{pmatrix} x \\ p_x \\ y \\ p_y \\ z \\ p_z \end{pmatrix} \text{ or } \begin{pmatrix} x \\ x' \\ y \\ y' \\ \varphi \\ E \end{pmatrix} \qquad \text{for example}$$

Beam distribution can be modelled by a variance-covariance matrix:

$[\sigma]$ such as: $\sigma_{ij} = \langle v_i \cdot v_j \rangle$ The sigma matrix.

One has:

$$[\sigma]_e = [T_{e\leftarrow s}] \cdot [\sigma]_s \cdot [T_{e\leftarrow s}]^T$$

$T_{e\leftarrow s}$ is the transfer matrix from s to e.

Transverse motion equation

Particle dynamics:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{r}, \mathbf{p}; t) \qquad \Rightarrow \qquad \frac{d\mathbf{p}}{ds} = \frac{\mathbf{F}(\mathbf{r}, \mathbf{p}; s)}{\beta_z \cdot c}$$

Magnetic field, no acceleration, transverse motion :

$$\Rightarrow \begin{cases} \dfrac{d\left(x' \cdot \beta_z\right)}{ds} = \dfrac{F_x(\mathbf{r}, \beta, s)}{\gamma \cdot \beta_z \cdot m \cdot c^2} \\ \dfrac{d\left(y' \cdot \beta_z\right)}{ds} = \dfrac{F_y(\mathbf{r}, \beta, s)}{\gamma \cdot \beta_z \cdot m \cdot c^2} \end{cases}$$

Linac + paraxial approximation: $\beta_x^2 + \beta_y^2 \ll \beta_z^2 \approx \beta^2$

$$\Rightarrow \begin{cases} \dfrac{dx'}{ds} = \dfrac{F_x(\mathbf{r}, \beta, s)}{\gamma \cdot \beta_z^2 \cdot m \cdot c^2} = F'_x(\mathbf{r}, \beta, s) \\ \dfrac{dy'}{ds} = \dfrac{F_y(\mathbf{r}, \beta, s)}{\gamma \cdot \beta_z^2 \cdot m \cdot c^2} = F'_y(\mathbf{r}, \beta, s) \end{cases}$$

Envelope equation (1)

$$\tilde{x}^2 = \left\langle x^2 \right\rangle$$

$$\tilde{x}\tilde{x}' = \left\langle x x' \right\rangle$$

$$\tilde{x}'^2 + \tilde{x}\tilde{x}'' = \left\langle x'^2 \right\rangle + \left\langle x x'' \right\rangle$$

$$\tilde{x}'' = \frac{\left\langle x'^2 \right\rangle + \left\langle x x'' \right\rangle}{\tilde{x}} - \frac{\tilde{x}'^2}{\tilde{x}}$$

$$\tilde{x}'' = \frac{\left\langle x'^2 \right\rangle + \left\langle x x'' \right\rangle}{\tilde{x}} - \frac{\left\langle x x' \right\rangle^2}{\tilde{x}^3}$$

$$\tilde{x}'' = \frac{\left\langle x'^2 \right\rangle + \left\langle x x'' \right\rangle}{\left\langle x^2 \right\rangle^{1/2}} - \frac{\left\langle x x' \right\rangle^2}{\left\langle x^2 \right\rangle^{3/2}}$$

Envelope equation (2)

RMS size evolution:

$$\tilde{x}'' = x''_{\text{rms}} = \frac{\left\langle x'^2 \right\rangle + \left\langle x \cdot x'' \right\rangle}{\left\langle x^2 \right\rangle^{1/2}} - \frac{\left\langle x \cdot x' \right\rangle^2}{\left\langle x^2 \right\rangle^{3/2}}$$

By noting that:

$$\left\langle x \cdot x'' \right\rangle = \left\langle x \cdot F'_x \left(\mathbf{r}, \beta, s\right) \right\rangle$$

One gets:

$$\tilde{x}'' - \tilde{K}_x \cdot \tilde{x} - \frac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0$$

$$\tilde{\epsilon}_x = \sqrt{\left\langle x'^2 \right\rangle \cdot \left\langle x^2 \right\rangle - \langle x \cdot x' \rangle^2}$$

The horizontal rms emittance

$$\tilde{K}_x = \frac{\left\langle x \cdot F'_x \left(\mathbf{r}, \beta, s\right) \right\rangle}{\tilde{x}^2}$$

The force linearisation coefficient

The linearised force can be applied to the envelope equation or as a transfer matrix (with sigma matrix).

Motion linearisation

$$F_x\left(\mathbf{r},\mathbf{p},s\right) \xrightarrow{\text{linearisation}} k_x \cdot x$$

Interest:

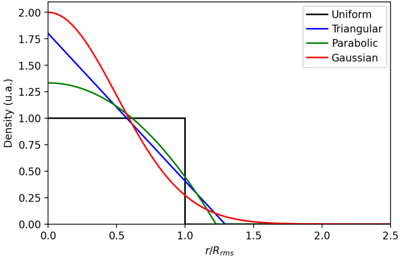
- Easy
- Fast
- Efficient

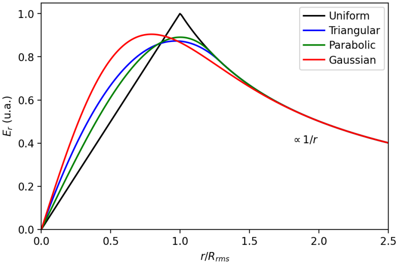
$\Rightarrow$ Equivalent uniform beam

Equivalent beams

Two beams are said " equivalent " when they carry the same current (continuous) or charge (bunched) and they have the same sigma matrix.

Example of distribution of continuous equivalent beams:





RMS emittance evolution

$$\begin{aligned}\frac{d\tilde{\epsilon}_x^2}{ds} &= \frac{d\left\langle x'^2\right\rangle}{ds}\cdot\left\langle x^2\right\rangle+\left\langle x'^2\right\rangle\cdot\frac{d\left\langle x^2\right\rangle}{ds}-2\left\langle x\cdot x'\right\rangle\cdot\frac{d\left\langle x\cdot x'\right\rangle}{ds}\\ &=2\left\langle x'\cdot x''\right\rangle\cdot\left\langle x^2\right\rangle+2\left\langle x'^2\right\rangle\cdot\left\langle x\cdot x'\right\rangle-2\left\langle x\cdot x'\right\rangle\cdot\left(\left\langle x'^2\right\rangle+\left\langle x\cdot x''\right\rangle\right)\\ &=2\left(\left\langle x'\cdot x''\right\rangle\cdot\left\langle x^2\right\rangle-\left\langle x\cdot x'\right\rangle\cdot\left\langle x\cdot x''\right\rangle\right)\end{aligned}$$

If the force is linear:

$$x''=k\cdot x$$

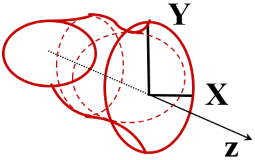
The emittance is constant:

$$\frac{d\tilde{\epsilon}_x^2}{ds}=2\cdot k\cdot\left(\left\langle x'\cdot x\right\rangle\cdot\left\langle x^2\right\rangle-\left\langle x\cdot x'\right\rangle\cdot\left\langle x^2\right\rangle\right)=0$$

Rms emittance is conserved in linear force, otherwise it can increase or decrease !

Uniform continuous beam

$$\rho(x,y) = \begin{cases} \rho_0 & \text{if } \frac{x^2}{X^2} + \frac{y^2}{Y^2} < 1 \\ 0 & \text{otherwise} \end{cases}$$



We have: $\begin{cases} \tilde{x} = X/2 \\ \tilde{y} = Y/2 \end{cases}$ and: $\rho_0 = \dfrac{I}{\pi \cdot X \cdot Y \cdot v}$

Space-charge force:
$$\begin{cases} \tilde{K}_{\mathrm{SC},x} = \dfrac{q \cdot I}{2\pi\epsilon_0 m (\gamma\beta c)^3} \cdot \dfrac{2}{X \cdot (X+Y)} = \dfrac{2 \cdot K}{X \cdot (X+Y)} \\ \tilde{K}_{\mathrm{SC},y} = \dfrac{q \cdot I}{2\pi\epsilon_0 m (\gamma\beta c)^3} \cdot \dfrac{2}{Y \cdot (X+Y)} = \dfrac{2 \cdot K}{Y \cdot (X+Y)} \end{cases}$$

$$K = \frac{q \cdot I}{2\pi\epsilon_0 m (\gamma\beta c)^3}$$

The beam generalized perveance.

Continuous beam envelope equations

$$\begin{cases} X'' + k_{x,0}^2(s)\cdot X - \dfrac{2K}{X+Y} - \dfrac{\epsilon_{x,\text{eff}}^2}{X^3} = 0 \\ Y'' + k_{y,0}^2(s)\cdot Y - \dfrac{2K}{X+Y} - \dfrac{\epsilon_{y,\text{eff}}^2}{Y^3} = 0 \end{cases}$$

These are the beam 2D envelope equations

$\epsilon_{x,\text{eff}} = 4\cdot\tilde{\epsilon}_x$ the effective emittance of the continuous beam.

Or, valid whatever the elliptical beam transverse distribution:

$$\begin{cases} \tilde{x}'' + k_{x,0}^2(s)\cdot \tilde{x} - \dfrac{K/2}{\tilde{x}+\tilde{y}} - \dfrac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0 \\ \tilde{y}'' + k_{y,0}^2(s)\cdot \tilde{y} - \dfrac{K/2}{\tilde{x}+\tilde{y}} - \dfrac{\tilde{\epsilon}_y^2}{\tilde{y}^3} = 0 \end{cases}$$

These are the beam RMS 2D envelope equations.

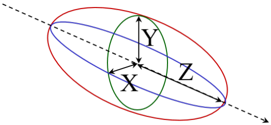
A few words about the envelope equation

$$\begin{cases} \tilde{x}'' + k_{x,0}^2(s)\cdot\tilde{x} - \dfrac{K/2}{\tilde{x}+\tilde{y}} - \dfrac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0 \\ \tilde{y}'' + k_{y,0}^2(s)\cdot\tilde{y} - \dfrac{K/2}{\tilde{x}+\tilde{y}} - \dfrac{\tilde{\epsilon}_y^2}{\tilde{y}^3} = 0 \end{cases}$$

The 2D envelope equation has 3 contributors to the dynamics:

- $k_{x,0}^2(s)\cdot\tilde{x}$: the external force contributor.
- $-\frac{K/2}{\tilde{x}+\tilde{y}}$: the space-charge contributor.
  - The effect is defocusing (negative sign).
  - The effect is proportional to the generalized perveance $K = \frac{q\cdot I}{2\pi\epsilon_0 m(\gamma\beta c)^3}$: it is thus decreasing with energy.
  - The effect decreases with the beam size: the slope of the electric field at the core depends on the beam size.
- $-\frac{\tilde{\epsilon}_x^2}{\tilde{x}^3}$: the emittance contribution. This term increases when the beam size is decreasing: that is even the driver for very small beam sizes (even stronger than space charge).

Bunched uniform beam



$$\rho(x,y)=\begin{cases}\rho_0 & \text{if } \frac{x^2}{X^2}+\frac{y^2}{Y^2}+\frac{z^2}{Z^2}<1\\ 0 & \text{otherwise}\end{cases}$$

We have: $\begin{cases}\tilde{x}=X/\sqrt{5}\\ \tilde{y}=Y/\sqrt{5}\\ \tilde{z}=Z/\sqrt{5}\end{cases}$ and: $\rho_0=\dfrac{Q}{\frac{4}{3}\pi\cdot X\cdot Y\cdot Z}$

Space-charge force: $\begin{cases}F'_{\mathrm{SC},x}=\dfrac{q}{mc^2}\cdot\dfrac{3}{4\pi\epsilon_0}\cdot\dfrac{Q}{\beta^2\gamma}\cdot\dfrac{1-f}{(X+Y)Z}\cdot\dfrac{x}{X}\\ F'_{\mathrm{SC},y}=\dfrac{q}{mc^2}\cdot\dfrac{3}{4\pi\epsilon_0}\cdot\dfrac{Q}{\beta^2\gamma}\cdot\dfrac{1-f}{(X+Y)Z}\cdot\dfrac{y}{Y}\\ F'_{\mathrm{SC},z}=\dfrac{q}{mc^2}\cdot\dfrac{3}{4\pi\epsilon_0}\cdot\dfrac{Q}{\beta^2\gamma}\cdot\dfrac{f}{XY}\cdot\dfrac{z}{Z}\end{cases}$

$f=f\left(\frac{X}{Y},\frac{\gamma Z}{\sqrt{XY}}\right)$ is a form factor of the ellipsoid.

$K_3=\frac{q}{5^{3/2}mc^2}\cdot\frac{3}{4\pi\epsilon_0}\cdot\frac{Q}{\beta^2\gamma^3}$ the 3-D space charge parameter.

Bunched beam envelope equations

$$\begin{cases} X'' + k_{x,0}^2(s)\cdot X - \dfrac{K_3(1-f)5^{3/2}}{(X+Y)Z} - \dfrac{\epsilon_{x,\text{eff}}^2}{X^3} = 0 \\ Y'' + k_{y,0}^2(s)\cdot Y - \dfrac{K_3(1-f)5^{3/2}}{(X+Y)Z} - \dfrac{\epsilon_{y,\text{eff}}^2}{Y^3} = 0 \\ Z'' + k_{z,0}^2(s)\cdot Z - \dfrac{K_3 f 5^{3/2}}{XY} - \dfrac{\epsilon_{z,\text{eff}}^2}{Z^3} = 0 \end{cases}$$

These are the beam 3D envelope equations

$\epsilon_{x,\text{eff}} = 5\cdot\tilde{\epsilon}_x$ the effective emittance of the bunched beam.

Or, valid whatever the ellipsoidal beam distribution:

$$\begin{cases} \tilde{x}'' + k_{x,0}^2(s)\cdot \tilde{x} - \dfrac{K_3(1-f)}{(\tilde{x}+\tilde{y})\tilde{z}} - \dfrac{\tilde{\epsilon}_x^2}{\tilde{x}^3} = 0 \\ \tilde{y}'' + k_{y,0}^2(s)\cdot \tilde{y} - \dfrac{K_3(1-f)}{(\tilde{x}+\tilde{y})\tilde{z}} - \dfrac{\tilde{\epsilon}_y^2}{\tilde{y}^3} = 0 \\ \tilde{z}'' + k_{z,0}^2(s)\cdot \tilde{z} - \dfrac{K_3 f}{\tilde{x}\tilde{y}} - \dfrac{\tilde{\epsilon}_z^2}{\tilde{z}^3} = 0 \end{cases}$$

These are the beam RMS 3D envelope equations.

Space-charge tune depression

Replacing the periodic focusing force by a continuous force.
The particle motion without space-charge is:

$$\frac{d^2x}{ds^2} = -\left(\frac{\sigma_{x,0}}{L}\right)^2 \cdot x = -k_{x,0}^2 \cdot x; \qquad k_{x,0} = \left(\frac{\sigma_{x,0}}{L}\right)$$

Phase advance per meter.

$$\Rightarrow x(s) = x_0 \cdot \cos\left(k_{x,0} \cdot s + \varphi\right)$$

The particle motion with linearised space-charge is:

$$\frac{d^2x}{ds^2} = -\left(k_{x,0}^2 - \tilde{K}_{\mathsf{SC},x}\right) \cdot x = -\tilde{k}_x^2 \cdot x$$

$$\tilde{k}_x = \sqrt{k_{x,0}^2 - \tilde{K}_{\mathsf{SC},x}} = \tilde{\eta} \cdot k_{x,0}$$

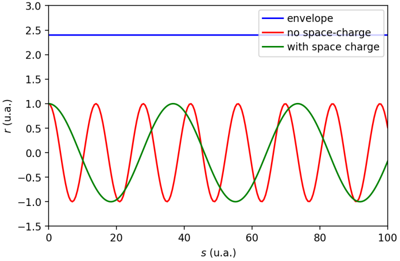
RMS Phase advance par meter with linear space-charge

$$\eta = \frac{\tilde{k}_x}{k_{x,0}}$$

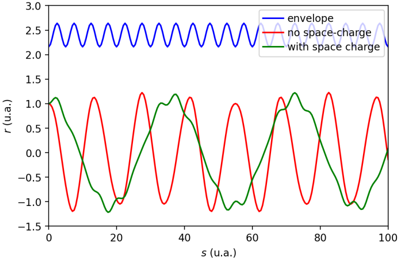
Space-charge tune depression

Space-charge tune depression (2)

Continuous focusing channel



Periodic focusing channel



No space-charge:

$$x_{\mathsf{nsc}}(s) = x_0 \cdot \cos\left(k_{x,0} \cdot s + \varphi\right) \qquad x_{\mathsf{nsc}}(s) = \sqrt{\beta_0 \cdot U} \cdot \cos\left(k_{x,0} \cdot s + \varphi\right)$$

Linear space-charge:

$$x_{\mathsf{sc}}(s) = x_0 \cdot \cos\left(\tilde{\eta} \cdot k_{x,0} \cdot s + \varphi\right) \qquad x_{\mathsf{sc}}(s) = \sqrt{\beta_{\mathsf{sc}} \cdot U} \cdot \cos\left(\tilde{\eta} \cdot k_{x,0} \cdot s + \varphi\right)$$

Non-linear effects

3. Non-linear effects

Non-linear motion equation

Motion equation in a linear continuous external force

$$\frac{dx'}{ds} = -k_{x,0}^2 \cdot x + F'_{x,\mathsf{SC}}(\mathbf{r}, s)$$

The space-charge force can be decomposed:

$$F'_{x,\mathsf{SC}}(\mathbf{r}, s) = \sum_{i>0} k_{x,\mathsf{SC},i} \cdot x^i$$

We obtain then:

$$\frac{dx'}{ds} = \underbrace{-\left(k_{x,0}^2 - k_{x,\mathsf{SC},1}\right) \cdot x}_{\text{Linear force}} \underbrace{- \sum_{i>1} k_{x,\mathsf{SC},i} \cdot x^i}_{\text{Non-linear part}}$$

Tune dispersion

- At small oscillation amplitude:

$$\frac{dx'}{ds} = -\left(k_{x,0}^2 - k_{x,\mathrm{SC},1}\right) \cdot x = -\left(\eta_{x,c} \cdot k_{x,0}\right)^2 \cdot x$$

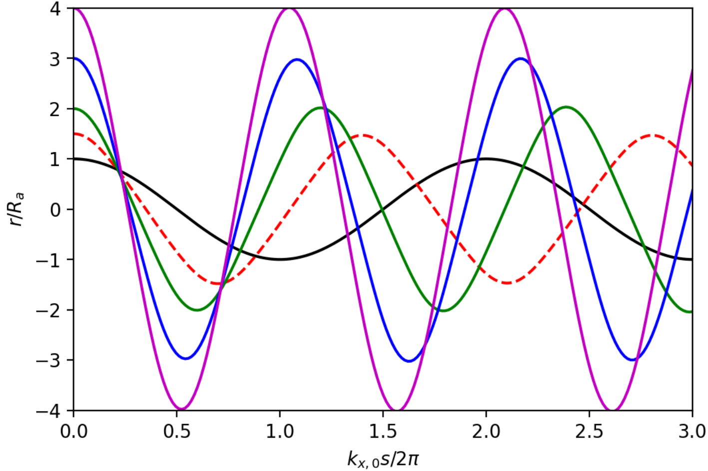
$\eta_{x,c}$: Core space-charge depression.

- At very large amplitudes:
  The particle is often far from the beam it feels essentially the external force. Its oscillation frequency k_x tends to $k_{x,0}$.
- At intermediate amplitude, the particle oscillation frequency depends on its amplitude: this is the space-charge tune dispersion.

$$\boxed{\eta_{x,c} \cdot k_{x,0} \leq k_x < k_{x,0}}$$

Tune depression: example

Particle trajectories around a uniform beam for various amplitudes.



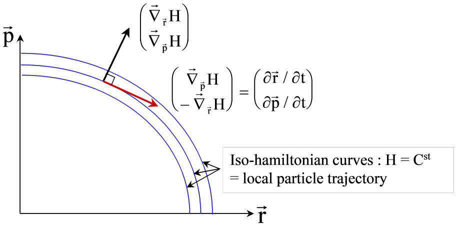
Question : What is the space-charge tune depression here?

Motion Hamiltonian

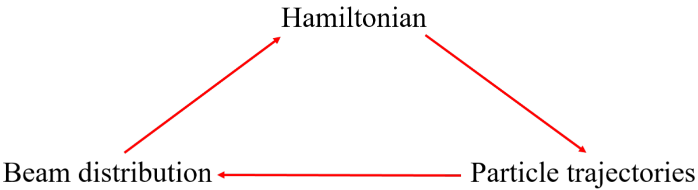
The particle motion can be described with an Hamiltonian H:

$$\begin{cases} \dfrac{d\mathbf{r}}{dt} = \dfrac{\partial H}{\partial \mathbf{p}} = \nabla_{\mathbf{p}} \cdot H \\ \dfrac{d\mathbf{p}}{dt} = -\dfrac{\partial H}{\partial \mathbf{r}} = -\nabla_{\mathbf{r}} \cdot H \end{cases}$$

Particles have phase-space trajectories on which the Hamiltonian is constant (orthogonal to the Hamiltonian gradient)



Perfect matching



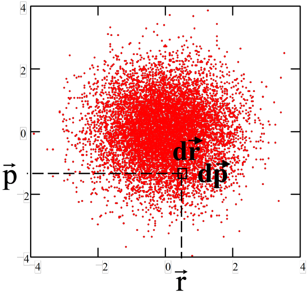
Perfect matching: The beam distribution is stationary

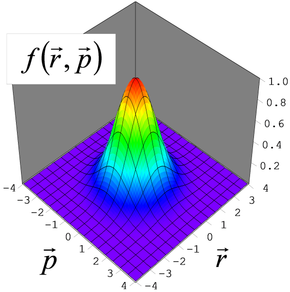
Distribution function

$$f\left(\mathbf{r},\mathbf{p},t\right)\cdot d\mathbf{r}\cdot d\mathbf{p}$$

is the number of particle at time t in a small phase-space hyper volume $d\mathbf{r}\cdot d\mathbf{p}$ at position $\left(\mathbf{r},\mathbf{p}\right)$.
Its evolution is given by Vlasov equation:

$$\frac{\partial f}{\partial t}+\frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{r}}f+q\left(\mathbf{E}+\frac{\mathbf{p}}{m}\times\mathbf{B}\right)\cdot\nabla_{\mathbf{p}}f=0$$





Perfect matching

$$\frac{\partial f}{\partial t} = 0 \qquad \Rightarrow \qquad f(\mathbf{r},\mathbf{p}) \cdot d\mathbf{r} \cdot d\mathbf{p} = g(H(\mathbf{r},\mathbf{p})) \cdot d\mathbf{r} \cdot d\mathbf{p}$$

But the Hamiltonian depends on the electrostatic potential ϕ and thus on the beam distribution $(\Delta\phi = -\rho/\epsilon_0)$. With:

$$\rho(\mathbf{r}) = \int f(\mathbf{r},\mathbf{p})\, d\mathbf{p}$$

The perfectly matched distribution is then solution of the implicit equation:

$$f(\mathbf{r},\mathbf{p}) = g(H(\mathbf{r},\mathbf{p},f(\mathbf{r},\mathbf{p})))$$

Case study

- Cylindrical continuous beam.
- Radial dynamics only.
- Continuous radial linear focusing channel.

$$\begin{cases} \dfrac{dr}{ds} = r' = \dfrac{\partial H\left(r, r', s\right)}{\partial r'} \\ \dfrac{dr'}{ds} = -k_0^2 \cdot r + F'_{\mathsf{SC}}(r,s) = -\dfrac{\partial H(r, r', s)}{\partial r} \end{cases}$$

$$H(r, r', s) = \frac{1}{2} \cdot r'^2 + \frac{1}{2} \cdot k_0^2 \cdot r^2 + V_{\mathsf{SC}}(r, s)$$

$$V_{\mathsf{SC}}(r, s) \equiv \frac{q\phi(r)}{\beta^2 \gamma^3 mc^2}$$

$$\rho(r) = \int\limits_0^{a'(r)} \int_0^{2\pi} f(H(r, r'))r' dr' d\psi = 2\pi \int\limits_0^{\frac{1}{2}a'(r)^2} f(H(r, r'))d\left(\frac{1}{2}r'^2\right)$$

$$H(r, r') = \frac{1}{2} r'^2 + W(r) \qquad W(r) \equiv \frac{1}{2} k_0^2 \cdot r^2 + V_{\mathsf{SC}}(r, s)$$

$$\frac{1}{2} r'^2 = H(r, r') - W(r) \qquad \frac{1}{2} a'(r)^2 = W(a) - W(r)$$

Case study (2)

$$\rho(r) = 2\pi \int_{W(r)}^{W(a)} f(H)\,dH$$

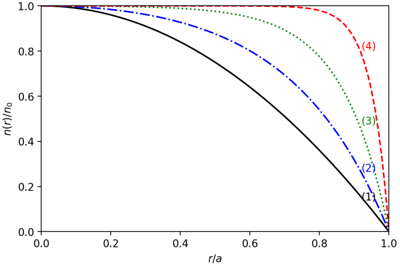
$$\forall r < a; \quad \Delta\phi = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\phi(r)}{dr}\right) = -\frac{2\pi}{\epsilon_0}\int_{W(r)}^{W(a)} f(H)\,dH$$

Whatever $f(H)$:

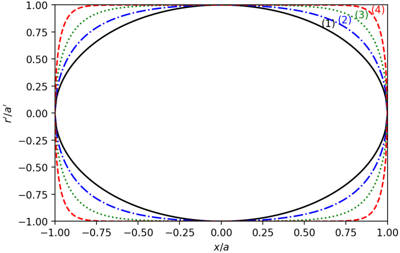
- If emittance dominated ($\eta_c \approx 1$; $V_{\mathrm{SC}} \ll k_0^2 r^2$) ("hot" beam),
  - the radial profile depends on f,
  - the particle phase-space trajectories are ellipses.
- If space-charge dominated ($\eta_c \approx 0$; $W(r) \approx 0$ for $r < a$) ("cold" beam),
  - the radial profile tends to uniform,
  - the particle phase-space trajectories tends to rectangular.

Case study – illustration

Water-bag beam: $$f(H) = \begin{cases} \frac{1}{H_0} & \text{if } H \leq H_0 \\ 0 & \text{if } H > H_0 \end{cases}$$



Radial density



Phase-space trajectories or distribution contour-plot

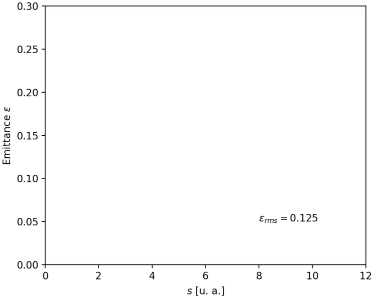
(1):	No space-charge
(2)–(4):	growing space-charge

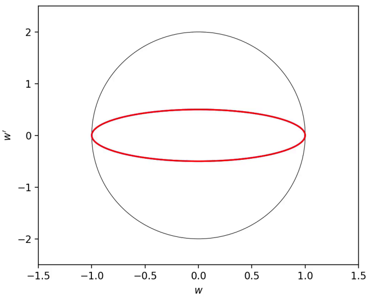
Mismatch - filamentation

The space-charge force is before all non-linear.

A mismatched beam goes filament, and particles are filling gradually the swept phase-space volume.

With associated RMS emittance growth.





Mismatch – 1D mode

Hypothesis: Cylindrical uniform beam.

Envelope equation: $\tilde{x}'' + k_{x,0}^2(s) \cdot \tilde{x} - \frac{K}{4\tilde{x}} - \frac{\tilde{\epsilon}_x}{\tilde{x}^3} = 0$

Mismatched beam: $\tilde{x} = \tilde{x}_a (1 + \delta)$

$$\delta'' + \left(k_{x,0}^2(s) + \frac{K}{4 \cdot \tilde{x}_a^2} + 3\frac{\tilde{\epsilon}_x}{\tilde{x}_a^4}\right) \cdot \delta = 0$$

$$\delta(s) = M \cdot \cos\left(k_{d,r}\, s + \varphi\right)$$

With: $k_{d,r} = \sqrt{k_{x,0}^2(s) + \frac{K}{4 \cdot \tilde{x}_a^2} + 3\frac{\tilde{\epsilon}_x}{\tilde{x}_a^4}} = k_{x,0}\sqrt{2 \cdot \left(1 + \tilde{\eta}_x^2\right)}$

The mismatch mode frequency.

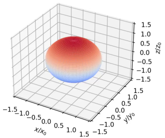
Mismatch – 2D-3D mode

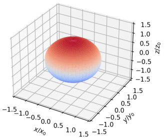
A continuous beam in a quadrupolar channel:

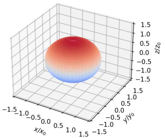
$\Rightarrow$ 2 coupled envelope equations: 2 modes

A bunched beam in a quadrupolar channel and cavities:

$\Rightarrow$ 3 coupled envelope equations: 3 modes



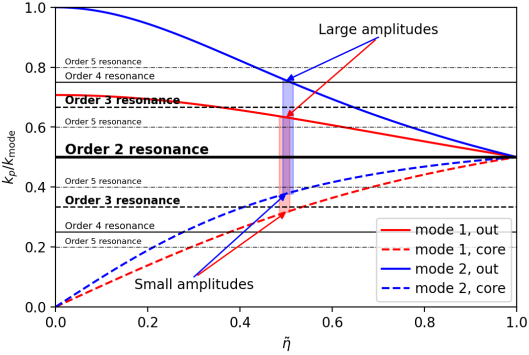




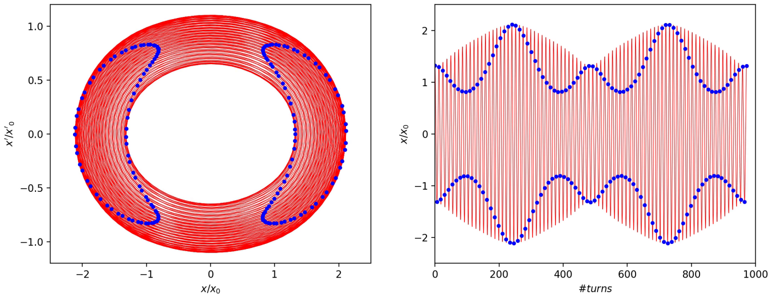
Second order mismatch-mode parametric resonance

Due to the tune dispersion, there is always a particle amplitude of which the oscillation frequency is half the mismatch mode frequency.

$$\tilde{\eta}_x^2 \cdot k_x^2 < \left(\frac{k_{d,r}}{2}\right)^2 = k_{x,0}^2 \cdot \frac{1+\tilde{\eta}_x^2}{2} < k_{x,0}^2$$



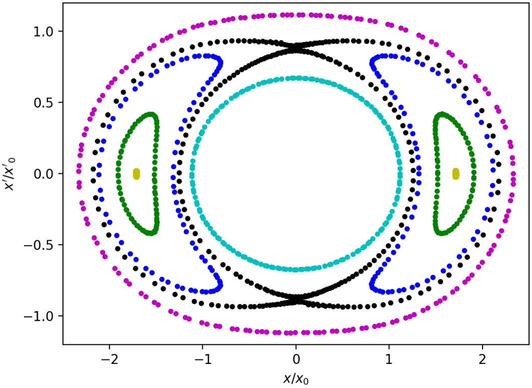
Uniform distribution; Beam mismatch: 10%; $\eta = 0.85$
Particles with an oscillation frequency near half this of a mismatch mode.



Red: particle phase-space trajectory
Blue: stroboscopic viewing at mismatch mode frequency

Second order resonance viewing (2)

Particles with different initial amplitudes (viewing at mismatch frequency):



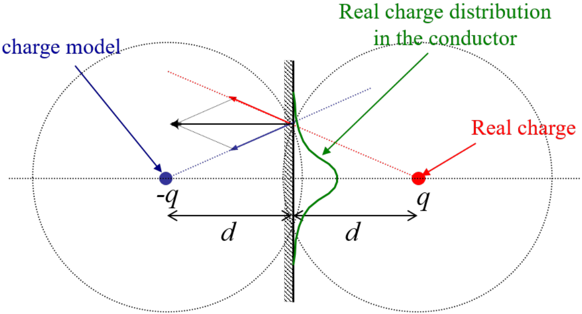
- No perturbation if large (magenta) or small (cyan) initial amplitudes.
- Stability islands (yellow) for particles at the half mismatch frequency.
- Oscillation around stability island for particles in the black region.

Wall effects

Coherent and incoherent motion

- Incoherent motion: the beam consists of many particles, each of which moves inside the beam with its individual betatron amplitude, phase, and even tune Q (under the influence of direct space charge). Amplitude and phase are randomly distributed. The beam and its centre of gravity – and thus the source of the direct space-charge field – do not move (static beam).
- Coherent motion: A static beam is given a transverse fast deflection (< 1 turn) and starts to perform betatron oscillations as a whole. This is readily observed by a position monitor. Note that the source of the direct space charge is now moving: individual particles still continue their incoherent motion around the common coherent trajectory and still experience their incoherent tune shifts as well.

Image charge: plate conductor



The real charge q attracts charges in the plate conductor (at a distance d). This charge distribution sets a constant potential in the conductor. It can be modelled by an image charge $-q$ symmetric of the real charge with respect to the plate.

Demonstration of the image charge (plate conductor)

Let us consider a charge q and a perfectly conductor plate at the distance d. We will use a frame centred on the plate (the position of the charge is thus $(d,0,0)$. Potential generated by the charge:

$$\phi_q = \frac{q}{4\pi\epsilon_0}\frac{1}{\|\mathbf{r}\|} = \frac{q}{4\pi\epsilon_0}\frac{1}{\left((x-d)^2+y^2+z^2\right)^{1/2}}$$

Let be ϕ_w the potential generated by the wall. The total voltage $\phi_T = \phi_q + \phi_w$ on the electric plate is at the ground voltage $V = 0$. We get:

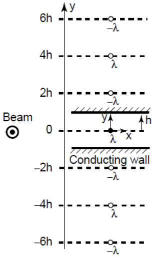
$$\phi_q(x=0) + \phi_w(x=0) = 0$$

$$\phi_w(x=0) = -\frac{q}{4\pi\epsilon_0}\frac{1}{\left(d^2+y^2+z^2\right)^{1/2}} = \frac{-q}{4\pi\epsilon_0}\frac{1}{\left((0-(-d))^2+y^2+z^2\right)^{1/2}}$$

The potential ϕ_w is equivalent to the potential generated by a charge $-q$ at the position $x = -d$.

Electric field if beam between two plates

We consider that the beam pipe is rectangular with a vertical height $2h$ small compared to the width $2w$: $h \ll w$.
We assume a linear distribution. The electric field generated by a linear distribution λ is:



$$\mathbf{E}_\lambda(\mathbf{r}) = \frac{\lambda}{2\pi\epsilon_0}\frac{x\mathbf{e}_x + y\mathbf{e}_y}{x^2+y^2}$$

The sum of the image charges is then $(y \ll h)$:

$$\begin{aligned}
\mathbf{E}(x,y) &= \sum_{n=1}^{\infty}\frac{(-1)^n\lambda}{2\pi\epsilon_0}\left[\frac{x\mathbf{e}_x + (2nh-y)\mathbf{e}_y}{x^2+(2nh-y)^2} + \frac{x\mathbf{e}_x - (2nh+y)\mathbf{e}_y}{x^2+(2nh+y)^2}\right] \\
&\approx \frac{\lambda}{\pi\epsilon_0}\sum_{n=1}^{\infty}(-1)^n\left[\frac{x\mathbf{e}_x - y\mathbf{e}_y}{4n^2h^2} + o(x,y)\right] \\
&\approx \frac{I}{\beta c\pi\epsilon_0}\frac{\pi^2}{48h^2}\left(-x\mathbf{e}_x + y\mathbf{e}_y + o(x,y)\right)
\end{aligned}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

Tune shift if beam between two plates

Example for a uniform elliptical continuous beam:

$$\begin{cases} \dfrac{dx'}{ds} = -k_{x,0}^2 \cdot x + F'_{x,\mathrm{SC}}(\mathbf{r}, s) = -k_{x,0}^2 \cdot x + \dfrac{F_{x,\mathrm{SC}}(\mathbf{r}, s)}{\gamma \beta^2 m_0 c^2} \\ \dfrac{dy'}{ds} = -k_{y,0}^2 \cdot y + F'_{y,\mathrm{SC}}(\mathbf{r}, s) = -k_{y,0}^2 \cdot y + \dfrac{F_{y,\mathrm{SC}}(\mathbf{r}, s)}{\gamma \beta^2 m_0 c^2} \end{cases}$$

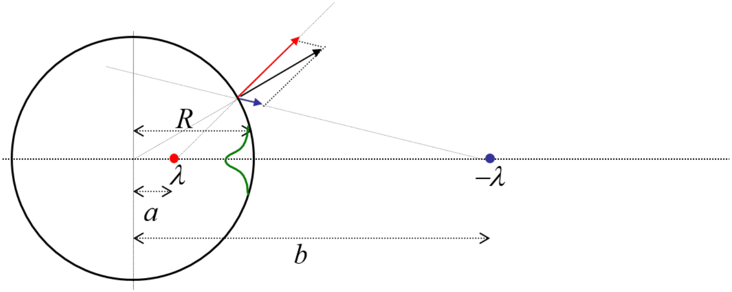
$$\begin{cases} \dfrac{dx'}{ds} + \left[k_{x,0}^2 - \dfrac{q}{\pi \epsilon_0 m_0 c^3} \dfrac{I}{\beta^3 \gamma} \left(\dfrac{1}{\gamma^2 X \cdot (X+Y)} - \dfrac{\pi^2}{48 h^2} \right) \right] \cdot x = 0 \\ \dfrac{dy'}{ds} + \left[k_{y,0}^2 - \dfrac{q}{\pi \epsilon_0 m_0 c^3} \dfrac{I}{\beta^3 \gamma} \left(\dfrac{1}{\gamma^2 Y \cdot (X+Y)} + \dfrac{\pi^2}{48 h^2} \right) \right] \cdot y = 0 \end{cases}$$

$$\begin{cases} k_{x,\mathrm{inc}} = \eta_x k_{x,0} = k_{x,0} \left[1 - \dfrac{q}{\pi \epsilon_0 m_0 c^3} \dfrac{I}{\beta^3 \gamma k_{x,0}^2} \left(\dfrac{1}{\gamma^2 X \cdot (X+Y)} - \dfrac{\pi^2}{48 h^2} \right) \right]^{1/2} \\ k_{y,\mathrm{inc}} = \eta_y k_{y,0} = k_{y,0} \left[1 - \dfrac{q}{\pi \epsilon_0 m_0 c^3} \dfrac{I}{\beta^3 \gamma k_{y,0}^2} \left(\dfrac{1}{\gamma^2 Y \cdot (X+Y)} + \dfrac{\pi^2}{48 h^2} \right) \right]^{1/2} \end{cases}$$

Wall effects against direct space-charge

- The electric image field is vertically defocusing, but horizontally focusing (sign of image term changes), which by the way is not just a feature of this particular geometry, but is typical for most synchrotrons with their rather flattish vacuum pipes;
- The field is larger for small chamber height h;
- Image effects decrease with $1/\gamma$, much slower than the direct space-charge term $(1/\gamma^3)$, and thus are of some concern for electron and high-energy proton machines.
- The incoherent motion can be measured by using a quadrupole lens and by introducing a mismatching. The envelope oscillation period gives the incoherent tune by dividing by 2.

Image charge: cylindrical conductor



The charge distribution on a cylindrical conductor of radius R by a charge per linear meter λ at a distance a from the cylinder center can be modelled by a charge per linear meter $-\lambda$ on the charge-cylinder center axis at distance b such as:

$$a \cdot b = R^2$$

Demonstration of the image charge (cylindrical conductor)

Let us consider a perfectly conductor cylinder of radius R and a linear charge λ at the position $x = a$. The frame center is the center of the cylinder. Electric field generated by the linear charge:

$$\mathbf{E}_q = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz\mathbf{r}}{\|\mathbf{r}\|^3} = \frac{\lambda}{2\pi\epsilon_0}\frac{\mathbf{r} - a\vec{x}}{x^2 + y^2 + a^2 - 2ax}$$

Let be $\mathbf{E}_w$ the potential generated by the cylinder. The total electric field $\mathbf{E}_T$ on the cylinder is normal to the surface. We get:

$$E_{q,\theta}(r = R) + E_{w,\theta}(r = R) = 0$$

$$E_{w,\theta}(r = R) = -\frac{\lambda}{2\pi\epsilon_0}\frac{a\sin\theta}{R^2 + a^2 - 2aR\cos\theta} = \frac{-\lambda}{2\pi\epsilon_0}\frac{R^2/a\sin\theta}{\left(\left(\frac{R^2}{a}\right)^2 + R^2 - 2\frac{R^2}{a}R\cos\theta\right)}$$

The electric field is equivalent to the one generated by a linear charge $-\lambda$ at the position $x = R^2/a$.

Beam dynamics if offset in a circular pipe

Let us consider a linear distribution with an offset of $\mathbf{r_0} = x_0\mathbf{e}_x + y_0\mathbf{e}_y$. The equivalent charge image of the beam pipe is a linear distribution $-\lambda$ at the position $\mathbf{r_1} = \frac{R^2}{r_0^2}\mathbf{r_0}$. The electric field at the beam center is then:

$$\mathbf{E}_\lambda(\mathbf{r_0}) = \frac{\lambda}{2\pi\epsilon_0}\frac{\mathbf{r_1}-\mathbf{r_0}}{\|\mathbf{r_1}-\mathbf{r_0}\|^2} = \frac{\lambda}{2\pi\epsilon_0}\frac{\mathbf{r_0}}{R^2-r_0^2} = \frac{\lambda}{2\pi\epsilon_0}\frac{\mathbf{r_0}}{R^2} + o(x_0, y_0)$$

$$\begin{cases} \dfrac{dx_0'}{ds} = -k_{x,0}^2\cdot x_0 + \dfrac{F_{x,\mathsf{SC}}(\mathbf{r},s)}{\gamma\beta^2 m_0 c^2} = \left[-k_{x,0}^2 + \dfrac{q}{\pi\epsilon_0 m_0 c^3}\dfrac{I}{\beta^3\gamma}\dfrac{1}{2R^2}\right]\cdot x_0 \\ \dfrac{dy_0'}{ds} = -k_{y,0}^2\cdot y_0 + \dfrac{F_{y,\mathsf{SC}}(\mathbf{r},s)}{\gamma\beta^2 m_0 c^2} = \left[-k_{y,0}^2 + \dfrac{q}{\pi\epsilon_0 m_0 c^3}\dfrac{I}{\beta^3\gamma}\dfrac{1}{2R^2}\right]\cdot y_0 \end{cases}$$

$$\begin{cases} k_{x,\mathsf{coh}} = \eta_x k_{x,0} = k_{x,0}\left[1 - \dfrac{q}{\pi\epsilon_0 m_0 c^3}\dfrac{I}{\beta^3\gamma k_{x,0}^2}\dfrac{1}{2R^2}\right]^{1/2} \\ k_{y,\mathsf{coh}} = \eta_y k_{y,0} = k_{y,0}\left[1 - \dfrac{q}{\pi\epsilon_0 m_0 c^3}\dfrac{I}{\beta^3\gamma k_{y,0}^2}\dfrac{1}{2R^2}\right]^{1/2} \end{cases}$$

A few features of the coherent tune shift

- The force is linear in $\bar{\mathbf{r}}$, so there is a coherent tune shift.
- The $1/\gamma$ dependence of the tune shift stems from the fact that the charged particles induce the electrostatic field and thus generate a force proportional to their number, but independent of their mass, whereas the deflection of the beam by this force is inversely proportional to their mass $m_0\gamma$.
- The coherent tune shift is never positive.
- Note that a perfectly conducting beam pipe has been assumed here, for simplicity. The effects of a thin vacuum chamber with finite conductivity are more subtle.
- The coherent tune shift can be measured by deflecting the beam with a transverse kicker (with a gate shorter than one revolution period) and by measuring the position (in a ring, turn after turn or in a linac at different positions) with a beam position monitor.

Summary

- Space charge force comes from the charge and current beam distribution: it decreases with energy. At high energy, wall effects (indirect space charge) are greater than direct space charge.
- Space-charge force is non-linear except for uniform distributions.
- Two beams are equivalent if they carry the same current and has the same covariance matrix.
- The envelope equation gives the evolution of the RMS beam size and has 3 contributors: external force, space-charge effect and emittance.
- Space-charge forces increase the motion period: tune depression.
- The non-linearity makes the tune depend on amplitude: tune dispersion.
- To keep the beam distribution, the beam needs a perfect matching.
- If the beam is not matched, beam-size is oscillating.
- Some resonances can occur with stability islands.