Particle Accelerators 2-3

Acceleration and longitudinal dynamics

Antoine Chancé

CEA Saclay DRF/IRFU/DACM

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Lecture content

1. **How to accelerate a beam ?**
2. **The RF cavity**
3. **RF Accelerator design**
4. **Longitudinal dynamics in an RF accelerator**

How to accelerate a beam ?

1. How to accelerate a beam ?

Electromagnetic force

- Goal: to give (kinetic) energy to the particle.
- The Lorentz force $\vec{F}$ acts on a particle with charge q and velocity $\vec{v}$ in electromagnetic field $\left(\vec{E}, \vec{B}\right)$:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

- The force is (by definition) the time derivative of the particle momentum $\vec{p}$:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Particle energies

- The particle total energy W is:

$$\begin{aligned} W^2 &= W_0^2 + (\vec{p}\cdot\vec{p})\,c^2 \\ &= \left(\gamma \cdot m \cdot c^2\right)^2 \\ &= (W_0 + T)^2 \end{aligned}$$

- c the speed of light in vacuum.
- $W_0 = mc^2$ the particle rest energy.
- m the particle rest mass.
- γ the particle reduced energy.
- T the particle kinetic energy.

$$\gamma = \frac{W}{W_0} = 1 + \frac{T}{W_0}$$

Energy time evolution

The energy time evolution with time t is:

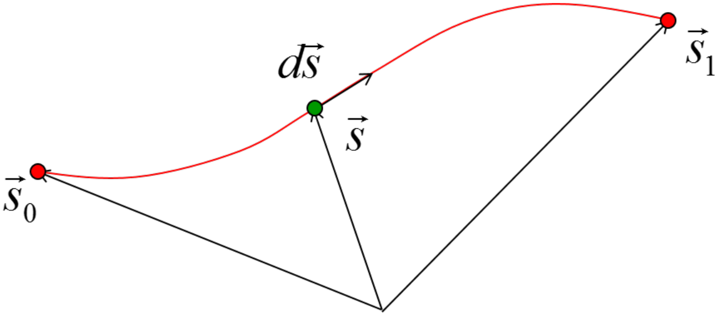
$$\begin{aligned}
\frac{dW}{dt} &= \frac{\vec{p}}{W} \cdot \frac{d\vec{p}}{dt} c^2 \\
&= \frac{\gamma m \vec{v}}{\gamma m c^2} \cdot q\left(\vec{E} + \vec{v} \times \vec{B}\right) c^2 \\
&= q\left(\vec{v} \cdot \vec{E} + \vec{v} \cdot \left(\vec{v} \times \vec{B}\right)\right) \\
\Rightarrow \frac{dW}{dt} &= \frac{dT}{dt} = q \cdot \vec{v} \cdot \vec{E}
\end{aligned}$$

$\Rightarrow$ Only the electric field gives (kinetic) energy to the beam

Energy gain

The energy gain ΔW of a particle over a given path, between positions $\vec{s}_0$ and $\vec{s}_1$ is:

$$\vec{v} = \frac{d\vec{s}}{dt} \qquad \Delta W\left(\vec{s}_0 \rightarrow \vec{s}_1\right) = q\int_{\vec{s}_0}^{\vec{s}_1} \vec{E}\left(\vec{s};t\right)\cdot d\vec{s}$$



s is the curved abscissa on the trajectory.

Beam coordinates

- A beam is a set of particles with a non-null average velocity.
- (x, y) is the transverse plan, and z the beam propagation direction (quantified with a curved abscissa s).

$$\frac{dW}{ds} = q \cdot \frac{\vec{v}}{v_z} \cdot \vec{E}$$
$$= q \cdot (E_z + x' \cdot E_x + y' \cdot E_y)$$
$$x'/y' \equiv \frac{v_{x/y}}{v_z} = \frac{dx/y}{ds}$$

Usually, $x', y' \ll 1$
$\Rightarrow$ **The electric field's transverse component has a very low contribution to the energy gain (compared to the longitudinal one).**

Maxwell equations

- We should imagine and develop technological objects homing electric field with at least 1 hole (most often 2) where the beam can enter and exit.
- The electromagnetic field evolution is given by Maxwell equations:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad c^2 \cdot \nabla \times \vec{B} = \frac{\vec{j}}{\epsilon_0} + \frac{\partial \vec{E}}{\partial t}$$

Potential difference

- An electrostatic field $\vec{E}$ can be represented by a potential U:

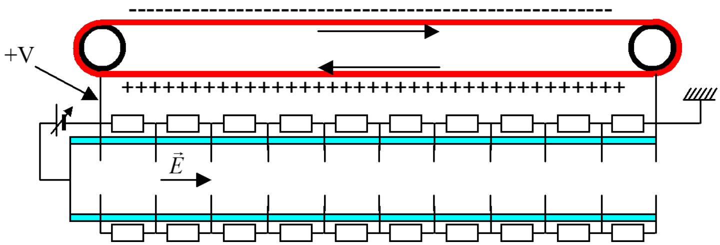
$$\vec{E} = -\nabla \cdot U \qquad \Rightarrow \qquad E_w = -\frac{\partial U}{\partial w} \qquad w = x, y, z$$

- The energy gain of a particle going from point A to point B where potentials are respectively V_A and V_B is:

$$\begin{aligned}\Delta W (A \rightarrow B) &= q \int_{\vec{s_A}}^{\vec{s_B}} \vec{E}(\vec{s}) \cdot d\vec{s} = q \int_{s_A}^{s_B} E_s ds \\ &= -q \cdot (V_B - V_A)\end{aligned}$$

- The integration is done on particle trajectory.
- The energy can be expressed in eV (electron-Volt): 1 eV $= e$ J.
- This is the energy gain of an electron under a potential change of 1 V.

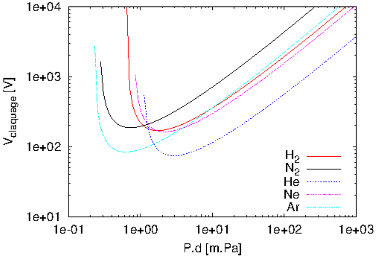
Van de Graaf accelerator



- Charges are deposited at accelerator's end on an insulating belt by friction with a polarized metallic brush.
- They are transported to the accelerator's other end where there are collected.
- The return current through resistances column produces voltage, used to accelerate the beam.

Use-limitations

- When electric field is too high, electrons can be pulled out from the surfaces lowering the voltage (breakdown) or/and consuming energy (leak).
- Ground potential is used as a reference ($U = 0\,\mathrm{V}$).



- For safety reasons, accelerator tank is at ground potential (where operators stand).
- In an electrostatic accelerator, the maximum accessible energy is then limited by its transverse size close to the source where the full voltage is applied.
- Electrostatic accelerator voltage is rarely higher than $10\,\mathrm{MV}$.

Plasma acceleration: 1D model and motivations

- The big advantage of plasma acceleration is that a plasma can manage large acceleration gradient (up to $100\,\mathrm{GV/m!}$), paving the path to compact acceleration (if we consider only the acceleration medium without the laser ;-))
- We will introduce the laser plasma acceleration of electrons.
- Plasma acceleration of ions is also an active research field but out of the scope of this lecture.
- The driver in the plasma (to generate plasma oscillations) can be also a beam.
- Main assumptions:
  - Cold unmagnetised plasma: $T_e = 0$.
  - Ions initially singly charged $(Z = 1)$ with homogeneous background o ion density $n_0$and immobile $(v_i = 0)$.
  - Thermal motion negligible compared to induced motion by laser field $(vosc \gg v_{\mathrm{th},e})$.

Plasma acceleration 1D: motion + Maxwell equations

$$\frac{d\mathbf{p}}{dt} = \frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{p} = -e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \qquad \mathbf{p} = \gamma m_e \mathbf{v}$$

$$\nabla\cdot\mathbf{E} = \frac{e}{\epsilon_0}(n_0 - n_e) \qquad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{B} = 0 \qquad c^2\nabla\times\mathbf{B} = -\frac{e}{\epsilon_0}n_e\mathbf{v} + \frac{\partial \mathbf{E}}{\partial t}$$

Motion invariant by translation along y and z: $\partial_y = \partial_z = 0$.
Linearly polarized laser with plane-wave geometry:

$$\mathbf{E}_L = E_y\mathbf{e}_y = -\frac{\partial A_y}{\partial t}\mathbf{e}_y \qquad \mathbf{B}_L = B_z\mathbf{e}_z \qquad p_y = eA_y$$

By using $\mathbf{B} = \nabla\times\mathbf{A}$ and $\mathbf{E} = -\nabla\cdot\phi - \frac{\partial \mathbf{A}}{\partial t}$

$$c^2\nabla\times(\nabla\times\mathbf{A}) + \frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{\mathbf{J}}{\epsilon_0} - \nabla\frac{\partial \phi}{\partial t}$$

Plasma acceleration 1D: potential vector

Let us split $\mathbf{J}$ with a rotational (solenoidal) part and irrotational (longitudinal) part:

$$\mathbf{J} = \mathbf{J}_\perp + \mathbf{J}_\parallel = \nabla \times \Pi + \nabla \psi$$

$$c^2 \nabla \times (\nabla \times \mathbf{A}) + \frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{\mathbf{J}}{\epsilon_0} - \nabla \frac{\partial \phi}{\partial t} \Rightarrow \qquad \frac{\mathbf{J}_\parallel}{\epsilon_0} - \nabla \frac{\partial \phi}{\partial t} = 0 \qquad v_x = \frac{\epsilon_0}{e n_e} \frac{\partial E_x}{\partial t}$$

Coulomb's gauge $\nabla \cdot \mathbf{A} = 0$ and $p_y = \gamma m_e v_y = e A_y$ give:

$$\frac{\partial^2 A_y}{\partial t^2} - c^2 \Delta A_y = \frac{J_y}{\epsilon_0} = -\frac{e^2 n_e}{\epsilon_0 m_e \gamma} A_y$$

The right-hand nonlinear source term on the right-hand contains two important bits of physics:

- $n_e = n_0 + \delta n$, coupling the EM wave to plasma waves,
- $\gamma = \sqrt{1 + \mathbf{p}^2 / m_e^2 c^2}$, introducing relativistic effects.

Plasma acceleration 1D: electric field

Motion equation and Poisson's law give:

$$\frac{d}{dt}(\gamma m_e v_x) = -eE_x - \frac{e^2}{2m_e\gamma}\frac{\partial}{\partial x}A_y^2 \qquad v_x = \frac{\epsilon_0}{en_e}\frac{\partial E_x}{\partial t} \qquad n_e = n_0 - \frac{\epsilon_0}{e}\frac{\partial E_x}{\partial x}$$

We make the average on a laser period. Perturbative approach by linearizing the plasma fluid quantities:

$$n_e \approx n_0 + n_1 \ldots \qquad v_x \approx \quad v_1 + \ldots \qquad \gamma \approx \gamma_0 + \gamma_1 \ldots$$

$$\omega_p = \sqrt{\frac{e^2 n_0}{\epsilon_0 m_e}} \qquad \frac{e\left\langle Ay^2\right\rangle}{m_e c} = \frac{a_0^2}{2} \qquad \gamma_0 = \sqrt{1 + \frac{a_0^2}{2}}$$

$$\left(\frac{\gamma_0}{\omega_p^2}\frac{\partial^2}{\partial t^2} + 1\right) eE_x = -\frac{e^2}{2m_e\gamma_0}\frac{\partial A_y^2}{\partial x}$$

$$\langle F_x\rangle = -\frac{e^2}{2m_e\gamma_0}\frac{\partial A_y^2}{\partial x}$$: relativistic ponderomotive force

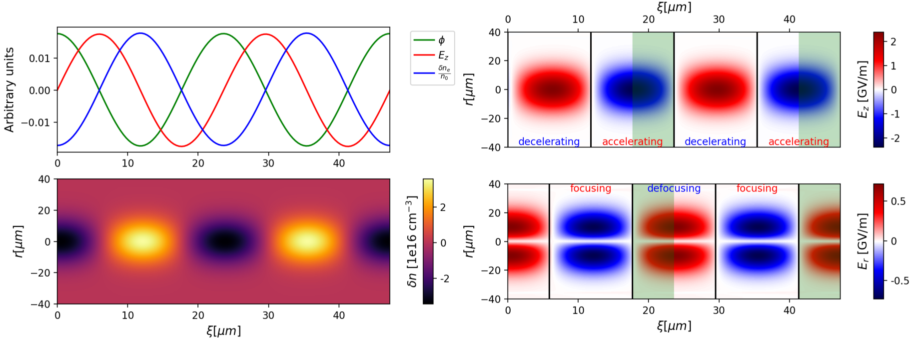
Plasma acceleration (1): linear regime

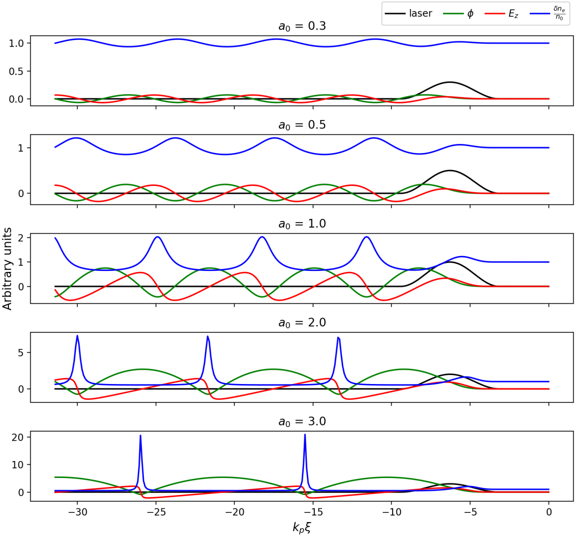
$a_0 \equiv \frac{eE_0}{m_e \omega c}$: normalized potential. $a_0^2 \approx 0.73 \cdot \lambda^2\,[\mu\mathrm{m}] \cdot I_0\,[1 \times 10^{18}\,\mathrm{W\,cm^{-2}}]$.

$a_0 \approx 1$: quasi-linear regime. $(I_0 = 2 \times 10^{18}\,\mathrm{W\,cm^{-2}}, \lambda = 0.8\,\mu\mathrm{m})$

If external electrons are injected at the right moment, they can be trapped in a plasma wave either in the linear or non linear regime.

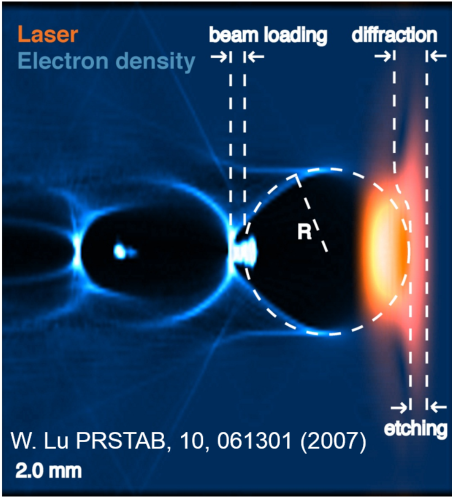
$a_0 = 0.15$.





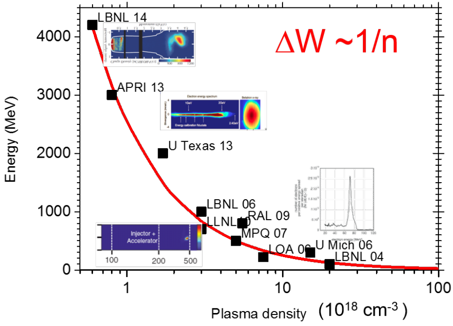
- Linear $\xrightarrow{a_0 \nearrow}$ non linear.
- When a_0 becomes very large ($a_0 > 2$), the electron motion becomes turbulent. The electron trajectory can cross the axis: wavebreaking.
- Electrons from the plasma can be trapped in the plasma wave in extreme a_0: blowout regime.

Plasma acceleration (3): blowout regime



- Pulse compression and self-focussing.
- Electrons are expelled from high laser intensity area and leave behind a cavity (bubble filled with ions).
- Electrons self-injected at the back of the bubble and accelerated.
- Injected electrons modify the back of the bubble (beam loading).

Plasma acceleration (4): limitations



- Dephasing between the driver (laser or beam) and accelerated electrons.
  - Limitations on the accelerating length.
  - Requires several plasma stages to go beyond 10 GeV.
- Energy depletion of the driver.
- Focusing length of the driver.
- Other hot topics: preserving beam quality, reducing momentum spread, reproducibility, ...

The RF cavity

2. The RF cavity

RF resonant cavity

- Goal : Give kinetic energy to the beam
- Basic principle
  - Conductor enclosing a close volume,
  - Maxwell equations + *Boundary conditions* allow possible electromagnetic field E_n/B_n configurations each oscillating with a given frequency f_n : **a resonant mode**. The field is a weighted superposition of these modes.
  - The wanted (accelerating) mode is excited at the good frequency and position from a RF power supply through a power coupler,
  - The phase of the electric field is adjusted to accelerate the beam.

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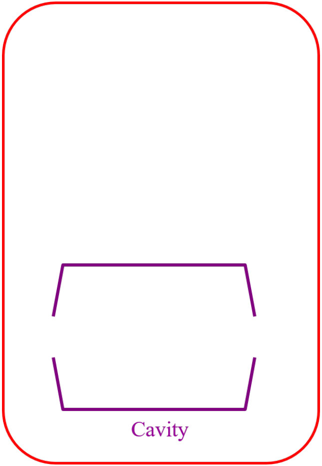

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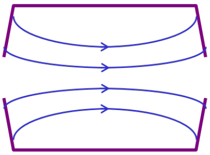

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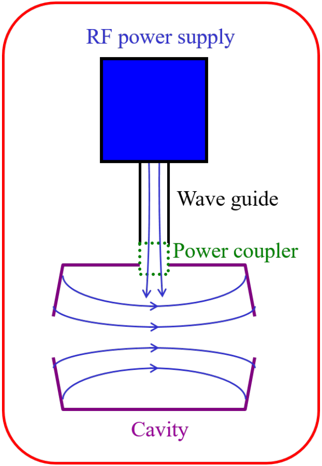

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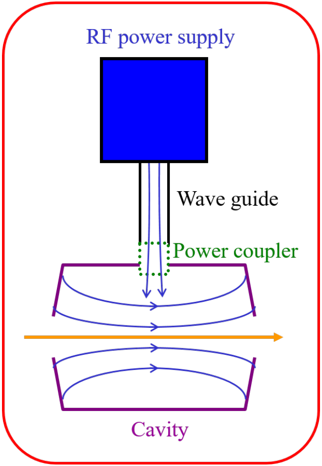


Mode calculation (1)

Boundary conditions close to the surface:

$$E_{\parallel} = 0$$

$$\vec{B}_{\perp} = \vec{0}$$

Mode calculation:

$$\Delta \vec{E}_n + \frac{\omega_n^2}{c^2} \vec{E}_n = \vec{0}$$

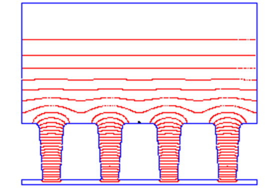
$\omega_n = 2\pi f_n$

c : speed of light

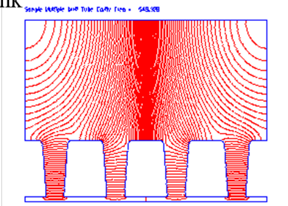
Electric field:

$$\vec{E}(r,t) = \sum_n e_n(t) \vec{E}_n(\vec{r})$$

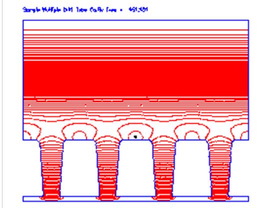
Ex : Drift Tube Linac (DTL) tank



TM_{010} : f=352.2 MHz

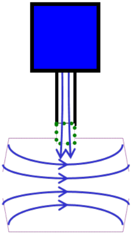


TM_{011} : f=548 MHz



TM_{020} : f=952 MHz

Mode calculation (2)



$$\frac{d^2 e_n}{dt^2} + \omega_n^2 e_n = \qquad ?$$

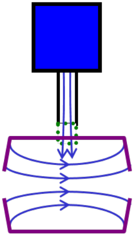
1. Joule losses in conductor $P_{\text{Joule}} = -\frac{\omega_{\text{RF}}}{Q_{0,n}} \cdot \frac{de_n}{dt}$, S_1: conductor surface
2. Energy exchange with outside S_2: open surface

$$P_{\text{exchange}} = \underbrace{-\frac{\omega_{\text{RF}}}{Q_{\text{ex},n}} \cdot \frac{de_n}{dt}}_{\text{losses}} + \underbrace{S_n(t) e^{\imath(\omega_{\text{RF}} t + \phi_0)}}_{\text{feed}}$$

3. Energy exchange with beam : Beam loading V: enclosed volume

$$P_{\text{beam-loading}} = k_n \underline{I}(t)$$

Mode calculation (2)



$$\frac{d^2 e_n}{dt^2} + \omega_n^2 e_n = \qquad -\frac{\omega_n^2}{\sqrt{\epsilon\mu}} \cdot \int_{S_1} \left(\vec{E} \times \vec{H}_n\right) \cdot \vec{n}\, dS_1$$

1. Joule losses in conductor $P_{\text{Joule}} = -\frac{\omega_{\text{RF}}}{Q_{0,n}} \cdot \frac{de_n}{dt}$, $\quad S_1$: conductor surface
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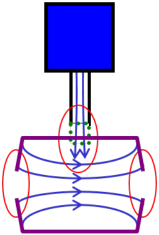

$$\frac{d^2 e_n}{dt^2} + \omega_n^2 e_n = -\frac{\omega_n^2}{\sqrt{\epsilon\mu}} \cdot \int_{S_1} \left(\vec{E} \times \vec{H}_n\right) \cdot \vec{n}\, dS_1$$

$$+\frac{1}{\epsilon} \cdot \frac{d}{dt} \int_{S_2} \left(\vec{H} \times \vec{E}_n\right) \cdot \vec{n}\, dS_2$$

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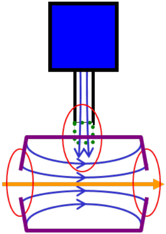

$$\frac{d^2 e_n}{dt^2} + \omega_n^2 e_n = \quad -\frac{\omega_n^2}{\sqrt{\epsilon\mu}} \cdot \int_{S_1} \left(\vec{E} \times \vec{H}_n\right) \cdot \vec{n}\, dS_1$$

$$+\frac{1}{\epsilon} \cdot \frac{d}{dt} \int_{S_2} \left(\vec{H} \times \vec{E}_n\right) \cdot \vec{n}\, dS_2 \quad -\frac{1}{\epsilon} \cdot \frac{d}{dt} \int_V \left(\vec{J}(\vec{r}, t) \cdot \vec{E}_n(\vec{r})\right) \cdot dV$$

1. Joule losses in conductor $P_{\text{Joule}} = -\frac{\omega_{\text{RF}}}{Q_{0,n}} \cdot \frac{de_n}{dt}$, S_1: conductor surface
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$$P_{\text{beam-loading}} = k_n \underline{I}(t)$$

Mode calculation (3)

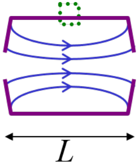
The last equation can be modelled by :

$$\frac{d^2 e_n}{dt^2} + \frac{\omega_{\mathrm{RF}}}{Q_n}\frac{de_n}{dt} + \omega_n^2 e_n = S_n(t)e^{\imath(\omega_{\mathrm{RF}} t + \phi_0)} + k_n \underline{I}(t)$$

Which is a damped harmonic oscillator in a forced regime.

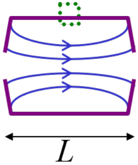
With	$\frac{1}{Q_n} = \frac{1}{Q_{0,n}} + \frac{1}{Q_{\mathrm{ex},n}}$	the quality factor of the cavity
	$\tau = 2\frac{Q_n}{\omega_{\mathrm{RF}}}$	the filling time of the cavity
	$S_n(t)e^{\imath(\omega_{\mathrm{RF}} t + \phi_0)}$	the RF source
	$k_n \underline{I}(t)$	the beam loading.

Shunt impedance per cavity



- Cavity length : L
- Cavity voltage V_0 : $V_0 = \int \widehat{E}_z(z)dz$
- Dissipated power P_d : Mean power dissipated in conductor over one RF period.
- Shunt impedance R : $R = \dfrac{V_0^2}{2P_d}$ $\boxed{P_d = \dfrac{1}{2}\dfrac{V_0^2}{R}}$
- Transit time factor T (calculated later) : $\Delta W_{\max} = q \cdot V_0 \cdot T$
- $\Delta W_{\max}$: Maximum energy that can be gained by a particle in the cavity
- Effective shunt impedance : RT^2 $\boxed{RT^2 = \dfrac{\Delta W_{\mathbf{max}}^2}{2q^2 P_d}}$

Shunt impedance per unit length



- Cavity mean electric field E_0 : $E_0 = \frac{V_0}{L} = \frac{1}{L}\int \widehat{E}_z(z)dz$
- Dissipated power per unit length P'_d : Mean power dissipated per unit length in conductor over one RF period.
- Shunt impedance per unit length Z : $Z = \frac{E_0^2}{2P'_d} = \frac{R}{L}$

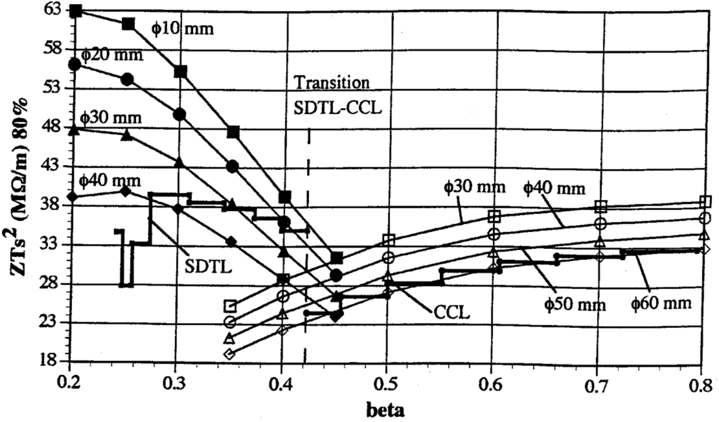
$$P'_d = \frac{1}{2}\frac{E_0^2}{Z}$$

- $\Delta W'_{max}$: Maximum energy that can be gained per unit length by a particle with charge q in the cavity $\Delta W'_{max} = q \cdot E_0 \cdot T$
- Effective shunt impedance per unit length : ZT^2

$$ZT^2 = \frac{\Delta W'^2_{\mathbf{max}}}{2q^2 P'_d}$$

Illustration of shunt impedance

The effective shunt impedance of the structures has been chosen to set the transition energy between sections for TRISPAL project (*C. Bourra, Thomson*).

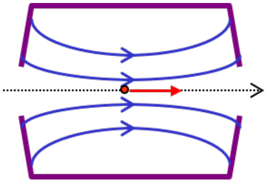


Energy gain

Energy gained by a particle in a cavity of length L :

$$\Delta W = \int qE_z(z) \cdot \cos\phi(s) \cdot ds$$

with: $\phi(s) = \phi_0 + \omega \cdot t = \phi_0 + \frac{\omega}{c} \int_{s_0}^{s} \frac{ds}{\beta_z(s)}$



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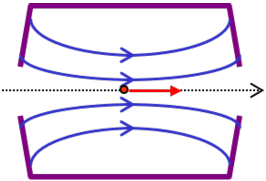

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with: $\phi(s) = \phi_0 + \omega \cdot t = \phi_0 + \frac{\omega}{c} \int_{s_0}^{s} \frac{ds}{\beta_z(s)}$

Assuming a constant velocity : $\overline{\beta}$

$$\Delta W = \int qE_z(z) \cos\left(\phi_0 + \frac{\omega}{\overline{\beta} c}(s - s_0)\right) \cdot ds \Rightarrow \boxed{\Delta W = qV_0 \cdot T\left(\overline{\beta}\right) \cdot \cos\phi_p}$$

Energy gain

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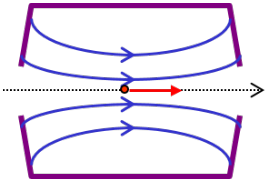

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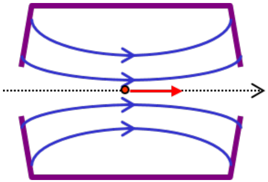
with: $V_0 = \int \left|E_z(s)\right| \cdot ds$

Cavity voltage

Energy gain

Energy gained by a particle in a cavity of length L :



$$\Delta W = \int qE_z(z) \cdot \cos\phi(s) \cdot ds$$

with: $\phi(s) = \phi_0 + \omega \cdot t = \phi_0 + \dfrac{\omega}{c} \displaystyle\int_{s_0}^{s} \frac{ds}{\beta_z(s)}$

Assuming a constant velocity : $\overline{\beta}$

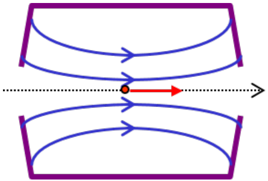
$$\Delta W = \int qE_z(z) \cos\left(\phi_0 + \frac{\omega}{\overline{\beta} c}(s - s_0)\right) \cdot ds \Rightarrow \boxed{\Delta W = qV_0 \cdot T\left(\overline{\beta}\right) \cdot \cos\phi_p}$$

with: $V_0 = \int \left|E_z(s)\right| \cdot ds$ Cavity voltage

$$\phi_p = \arctan\left(\frac{\int E_z(s)\sin\phi(s) \cdot ds}{\int E_z(s)\cos\phi(s) \cdot ds}\right)$$ Synchronous phase

Energy gain

Energy gained by a particle in a cavity of length L :



$$\Delta W = \int qE_z(z) \cdot \cos\phi(s) \cdot ds$$

with: $\phi(s) = \phi_0 + \omega \cdot t = \phi_0 + \frac{\omega}{c} \int_{s_0}^{s} \frac{ds}{\beta_z(s)}$

Assuming a constant velocity : $\overline{\beta}$

$$\Delta W = \int qE_z(z) \cos\left(\phi_0 + \frac{\omega}{\overline{\beta} c}(s - s_0)\right) \cdot ds \Rightarrow \boxed{\Delta W = qV_0 \cdot T\left(\overline{\beta}\right) \cdot \cos\phi_p}$$

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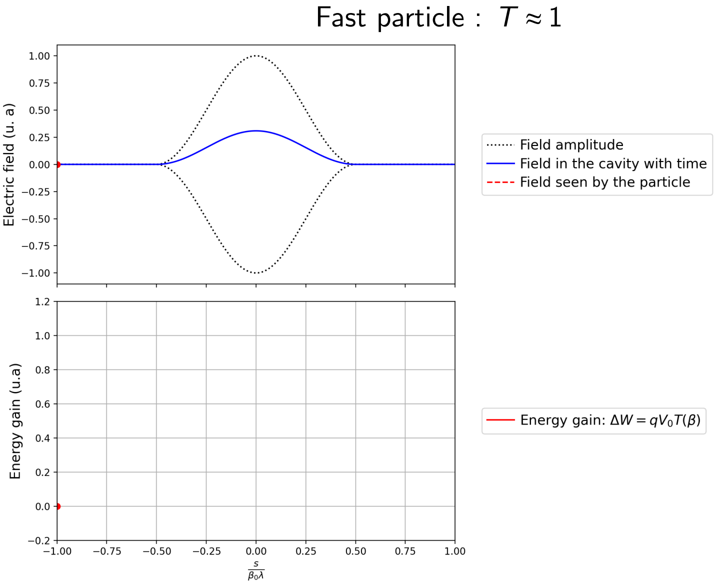
$$\phi_p = \arctan\left(\frac{\int E_z(s) \sin\phi(s) \cdot ds}{\int E_z(s) \cos\phi(s) \cdot ds}\right)$$ Synchronous phase

$$T = \frac{1}{V_0} \int E_z(s) \cos\left(\phi(s) - \phi_p\right)$$ Transit-time factor

$$T = \frac{1}{V_0} \left|\int E_z(s) e^{\imath\phi(s)}\right|$$

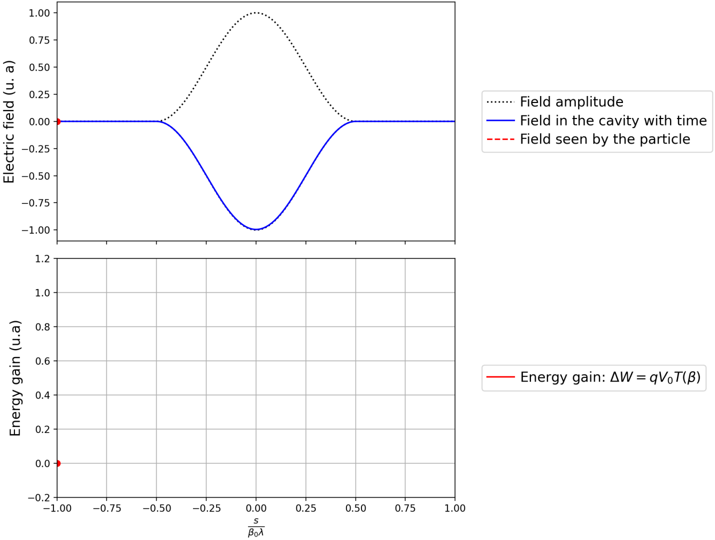
$\boxed{0 < T < 1}$

One-cell cavity ? fast particle



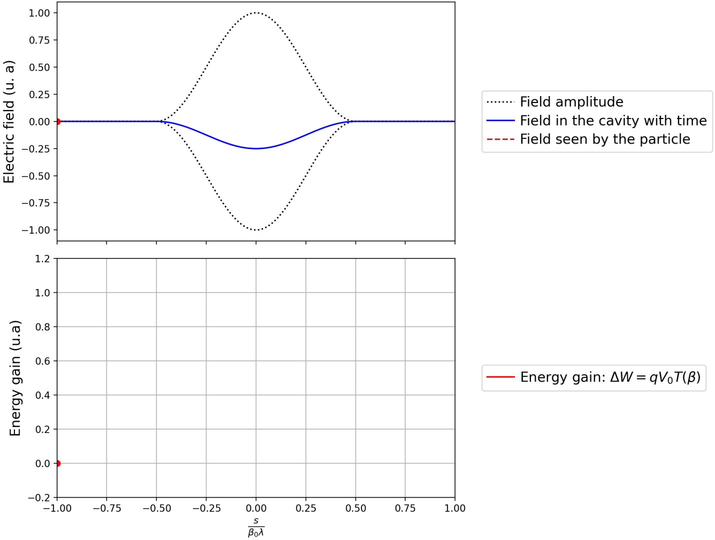
One-cell cavity ? medium particle

Medium particle : $T \approx 0.85$

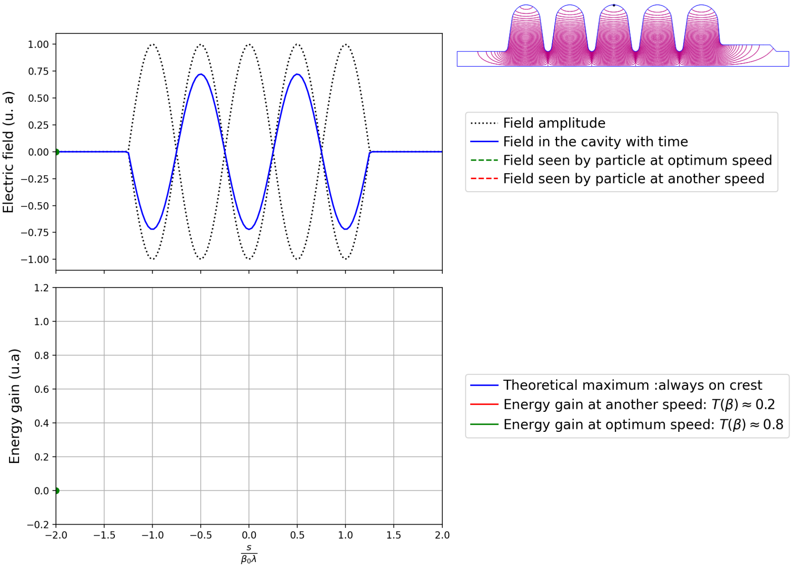


One-cell cavity ? slow particle

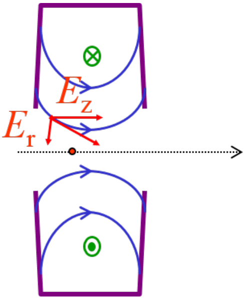
Slow particle : $T \approx 0.3$



Multi-cell cavity



Transverse kick



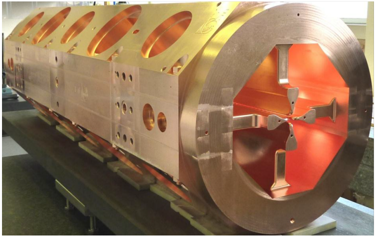
$$F_r = q\left(E_r - v_z \cdot B_\theta\right)$$

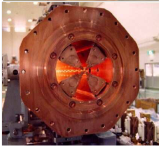
$$F_r = -\frac{q\omega_{\mathrm{RF}} \cdot V_0 T}{2 \cdot \beta c \cdot \gamma^2} \sin\phi_p \cdot r + \mathcal{O}(r^3)$$

- At first order: a linear lens.
- Quickly decreasing with energy.
- Increasing with RF frequency.
- Phase dependent: front and back differently focused. → longitudinal-transverse coupling.
- Max acceleration ($\phi_p = 0$) → no average transverse force.

The RFQ

- The Radio-Frequency Quadrupole (RFQ) is used to bunch continuous beams at low beta ($\beta < 0.1$) and accelerate it to an energy where it can be accelerated by a less expensive structure.
- The transverse focusing is realized with transverse quadrupole geometry.
- The longitudinal field (for bunching and acceleration) is obtained from pole modulation increasing progressively (in amplitude and period).

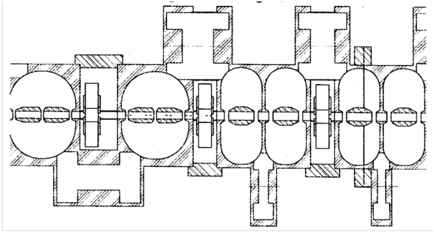




The DTL

- The Drift-Tube Linacs (DTL) is used to accelerate beam with moderate velocity ($0.1 < \beta < 0.4$).
- The phase difference between consecutive gaps is 2π.
- The beam is hidden in drift tube from electric field when decelerated.
- The transverse focusing is made with magnetic quadrupole housed in drift tubes (left) or outside cavities (right).

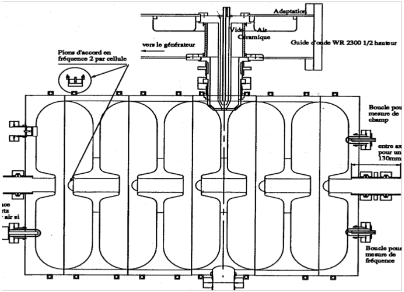
Classical DTL



Separated DTL

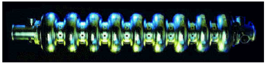
The CCL

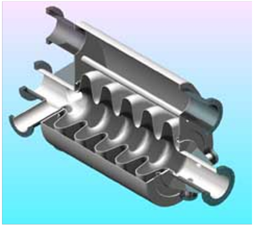
- The Coupled Cavity Linac (CCL) is used to accelerate beam with large velocity ($\beta > 0.4$).
- The phase difference between consecutive gaps is π.
- Accelerating cells are coupled with either inter-cell holes (left) or extern coupling cells (right).
- The beam enters the next cell when its field is positive.
- Transverse focusing with magnetic quadrupole outside cavities.



The superconducting cavities

- The SC cavities can be used at all energy, but are mostly used at high energy.
- Their shape is optimized to minimize the peak fields (magnetic and electric) on the Nb surface.
- The dissipated RF power is small but is made with liquid Helium (low cryogenic efficiency).



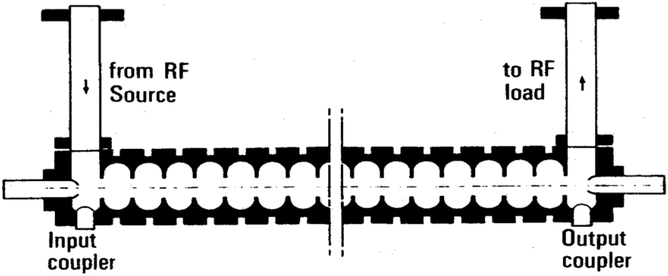


Travelling wave cavity

- Essentially used with relativistic beams ($\beta \approx 1$).
- The longitudinal filed component is :

$$E_z(r,z,t) = \sum_n E_n(r) \cdot e^{\imath(\omega t - k_n z)}$$

- $E_n(r) \cdot e^{\imath(\omega t - k_n z)}$: space harmonic, driven by the cavity periodicity.
- Particles whose velocity v_p is close to the field phase velocity v_ϕ exchange (gain) energy. $v_p \approx v_\phi = \frac{\omega}{k_n}$



RF Accelerator design

3. RF Accelerator design

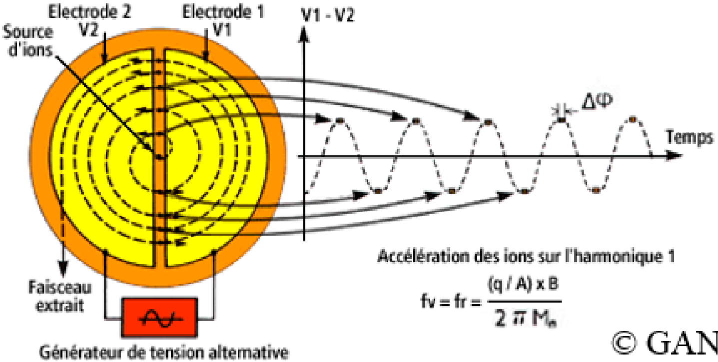
Synchronous particle: definition

- The synchronous particle is an ideal particle travelling on the accelerator reference trajectory (around which all elements are positioned) and whose time arrival is used to synchronize all time-varying elements (mainly cavities).
- The accelerator is designed with this synchronous particle.
- That is a property of the accelerator (representative of the machine).
- The synchronous particle is not linked to the beam !

$B \cdot \rho = \frac{p}{q}$ Magnetic rigidity. ρ curvature radius in B field

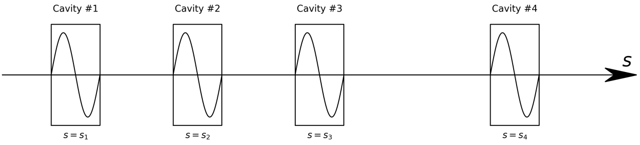
$f_c = \frac{qB}{\gamma m}$ Cyclotron frequency

$\boxed{f_{\mathrm{RF}} = h \cdot f_c}$ Synchronism condition



In a linac

- A linac is made of a set of cavities along a linear path s.
- It is designed with a hypothetical on-axis synchronous particle.



In a linac

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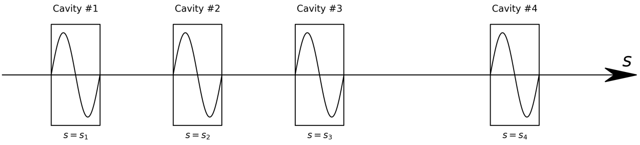
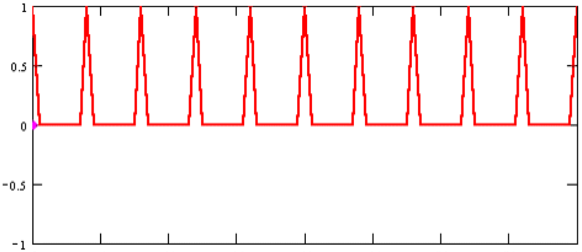
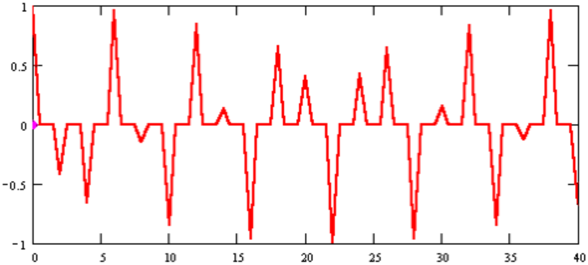


Synchronism conditions $\boxed{\phi_{i+1} = \phi_i + \omega \frac{D_i}{\beta_{s,i} c} + (\phi_{s,i+1} - \phi_{s,i}) + 2\pi n}$

In a linac: examples

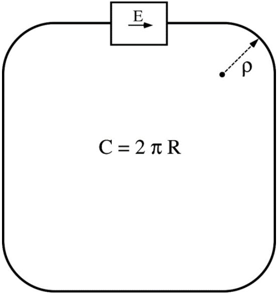


Coupled cavity linac (2π mode)



Independent cavity linac

In a synchrotron



- A synchrotron has h synchronous particles.
- h is the harmonic number. $h \in \mathbb{N}$
- $f_{\text{rev}} = \dfrac{\beta c}{\mathscr{C}}$ the beam revolution frequency
- Synchronism condition:

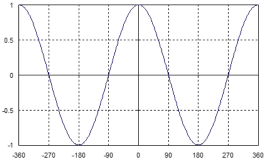
$$\boxed{f_{\text{RF}} = h \cdot f_{\text{rev}}}$$

⇒ Particle acceleration is made by increasing the magnetic field !

⇒ Do you understand the mechanism ?

Acceleration condition

- The field should accelerate the particle



- $\int E_z(s)ds = V_0 T \cos\phi \quad \leftarrow$ **Cosine** convention (mostly linac).

$$\begin{array}{c} \Delta W > 0 \\ qV_0 T \cos\phi_p > 0 \end{array} \quad \Rightarrow \quad \begin{array}{ll} qV_0 T > 0: & -90^\circ < \phi < 90^\circ \\ qV_0 T < 0: & 90^\circ < \phi < 270^\circ \end{array}$$

- $\int E_z(s)ds = V_0 T \sin\phi \quad \leftarrow$ **Sine** convention (mostly synchrotron).

$$\begin{array}{c} \Delta W > 0 \\ qV_0 T \sin\phi_p > 0 \end{array} \quad \Rightarrow \quad \begin{array}{ll} qV_0 T > 0: & 0^\circ < \phi < 180^\circ \\ qV_0 T < 0: & -180^\circ < \phi < 0^\circ \end{array}$$

Stability condition

Energy gain should allow late particles to catch up early ones.

- In a linac,
  - Higher energy (and velocity) particles catch up lower energy particles.
  - Electric field in cavities should then be growing when synchronous particle cross it.
  - Latest particles gain more energy than earliest particles.
- In a synchrotron,
  - Higher energy means higher velocity but also higher magnetic rigidity leading to higher curvature radius in dipole magnets and then longer trajectory over one turn.
  - A higher energy particle goes faster but on a longer path.
  - Knowing if it will gain or lose time is a balance.
  - The parameter that tells if the velocity change or path change dominates is called the slipping factor η.

η parameter

$\eta = \frac{1}{\delta}\frac{df_{\text{rev}}}{f_{\text{rev}}}$ is the relative variation of revolution frequency with respect to the relative momentum $\delta = \frac{\Delta p}{p}$

$\eta > 0$ A higher energy particle turns faster: linacs and low energy synchrotrons.

$\eta < 0$ A lower energy particle turns faster : high energy synchrotrons.

$f_{\text{rev}} = \frac{\beta c}{\mathscr{C}}$ $\Rightarrow \frac{df_{\text{rev}}}{f_{\text{rev}}} = \frac{d\beta}{\beta} - \frac{d\mathscr{C}}{\mathscr{C}}$

$\Rightarrow \boxed{\eta = \frac{1}{\gamma^2} - \frac{1}{\delta}\frac{d\mathscr{C}}{\mathscr{C}} = \gamma^{-2} - \alpha}$

$\alpha = \frac{1}{\delta}\frac{d\mathscr{C}}{\mathscr{C}}$ is the momentum compaction.

$\gamma_t = \frac{1}{\sqrt{\alpha}}$ is the synchrotron transition energy.

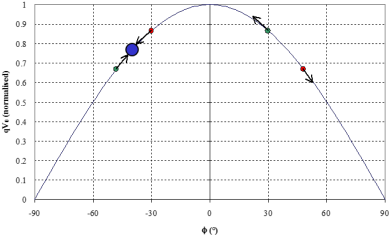
Stability condition

Cavities should allow latest particles to recover earliest ones.

$\eta > 0$

linac: $\eta = \dfrac{1}{\gamma^2}$

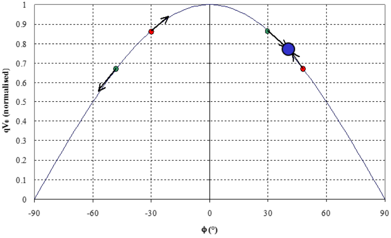
LE synchrotron: $\dfrac{1}{\gamma^2} > \alpha$



Stable with a rising field

$\eta < 0$

HE synchrotron: $\dfrac{1}{\gamma^2} < \alpha$



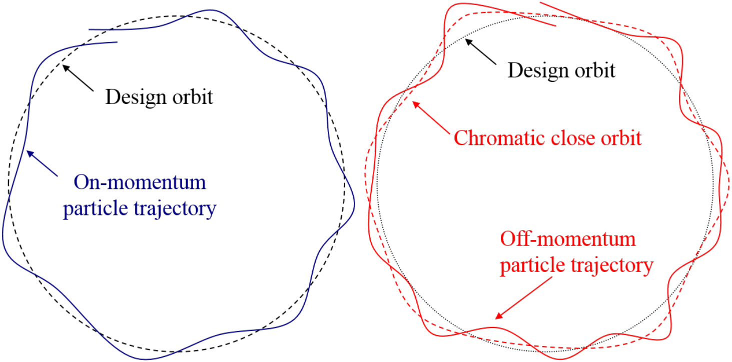
Stable with a falling field

Periodic dispersion function

Off-momentum particles are not oscillating around the design orbit, but around a chromatic closed orbit, whose distance from the design orbit depends linearly on δ.

$$x_\delta(s) = D_p(s)\delta$$

D_p is the periodic dispersion function.



Momentum compaction α

$$\alpha = \frac{1}{\delta}\frac{d\mathscr{C}}{\mathscr{C}} = \frac{1}{\delta}\frac{dR}{R}$$

The momentum compaction is the relative variation of path length with respect to the relative momentum

- Generally $\alpha > 0$: longer path for higher energy particles.
- The momentum compaction can be calculated from the periodic dispersion function D_p.

$$D_p = \frac{\partial x}{\partial \delta}$$

Variation of the transverse position with relative momentum.

$$\begin{aligned} \mathscr{C} &= 2\pi\rho + n \cdot L = 2\pi R \\ d\mathscr{C} &= 2\pi\left(\rho + \langle dx\rangle_{\text{dipoles}}\right) - 2\pi\rho \\ &= 2\pi\left\langle D_p\right\rangle_{\text{dipoles}}\delta \end{aligned}$$

$$\boxed{\alpha = \frac{\left\langle D_p\right\rangle_{\textbf{dipoles}}}{R}}$$

Acceleration summary

- Only electric field gives kinetic energy.
- Energy gain = integral of electric longitudinal component.
- Many technologies to give energy.
- Mostly used: RF cavity.
- Characterized by: modes, shunt impedance, transit time factor.
- Accelerator tuned with a synchronous particle (phase, energy).
- Different synchronism conditions in linacs, synchrotrons and cyclotrons.
- Synchronous phase :
  - Acceleration condition, field should be accelerating,
  - Stability condition, field should be :
    - increasing in linac and synchrotron at low energy → velocity driven,
    - decreasing synchrotron at high energy → trajectory driven.

Longitudinal dynamics

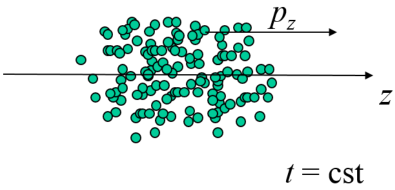
4. Longitudinal dynamics

Longitudinal phase spaces

Independent variable: time t

Longitudinal position:
δz (m)

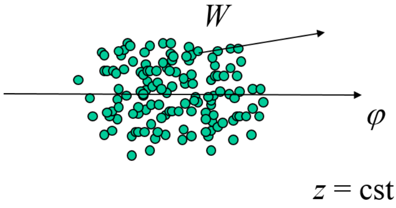
Longitudinal momentum:
δp_z (eV/c)



Independent variable: position s

Phase or time:
$\delta\varphi = \omega \cdot \delta t$

Energy
δW (eV)



Synchronous particle reference

Particle's longitudinal coordinates are represented with respect to those of the synchronous particle.

$$\begin{aligned} \phi = \varphi - \varphi_s & \quad \text{is the particle relative phase} \\ \delta E = E - E_s & \quad \text{is the particle relative energy} \end{aligned}$$

Phase evolution in a drift

$$\boxed{\frac{d\varphi}{ds} = \frac{2\pi f_{\mathbf{RF}}}{v_z} = \frac{2\pi}{\beta_z \lambda_{\mathbf{RF}}}}$$

- f_{RF} is the RF frequency
- $v_z = \beta_z \cdot c$ is the longitudinal component of the particle velocity,
- c is the physics constant corresponding to the speed of light in vacuum,
- λ_{RF} is the RF wavelength in vacuum.

In the frame attached to the synchronous particle:

Assuming: $\dfrac{\beta - \beta_s}{\beta} \ll 1 \qquad \dfrac{\delta E}{E_s} \ll 1$

We obtain: $\boxed{\dfrac{d\phi}{ds} = -\dfrac{2\pi}{\lambda_{\mathbf{RF}}} \cdot \dfrac{\delta E}{\beta_s^3 \gamma_s^3 mc^2}}$

Q : What is the particle motion in longitudinal phase-space ?

Energy evolution in a cavity

$$\boxed{\Delta E = q \cdot V_0 \cdot T \sin\varphi}$$

- T is the transit time factor,
- φ is the synchronous phase of the particle trough the cavity.

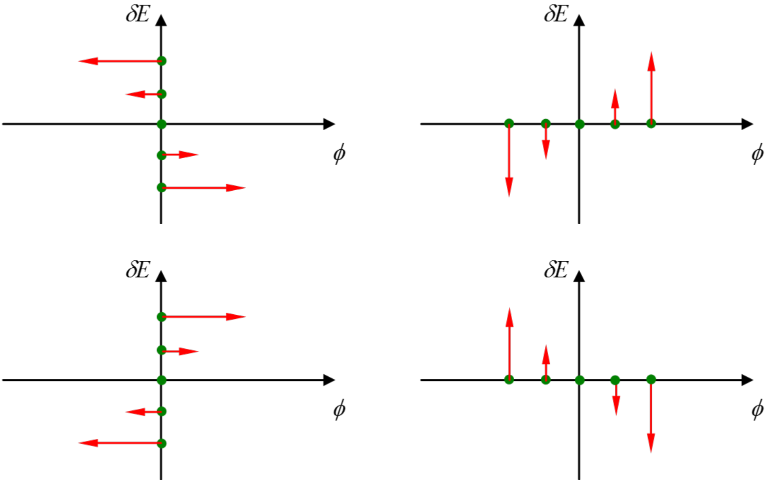
In the frame attached to the synchronous particle:

$$\boxed{\Delta\delta E = qV_0 T \left(\cos\varphi_s \cdot \sin\phi - \sin\varphi_s \left(1 - \cos\phi\right)\right)}$$

Q : What is the particle motion in longitudinal phase-space ?

In a drift

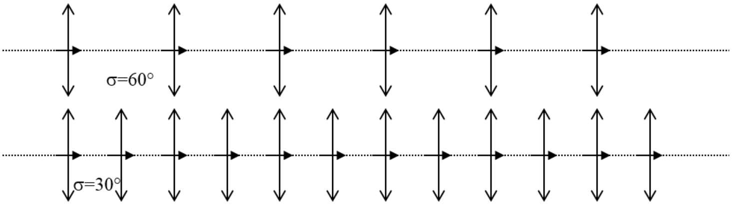
In a thin cavity



Q : What is the sign of η and $qV_0 \sin \varphi_s$?

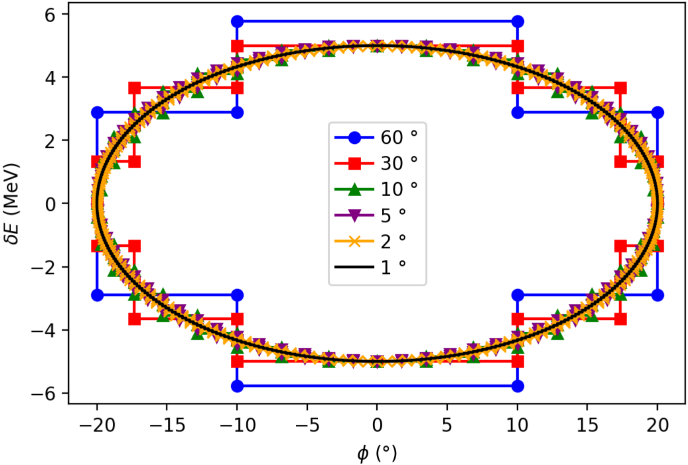
Equivalent channel

- In a real accelerator, cavities are generally regularly distributed.
- These cavities are tuned to keep the particle oscillating around the synchronous particle (stability condition).
- This set of cavities can be considered as a periodic focusing channel, and, like in transverse dynamics, one can define a longitudinal phase advance per lattice (period) σ as the fraction of oscillation (called synchrotron motion) of a particle over one lattice of length L.
- Two focusing channels are said equivalent if they have the same phase advance per unit length ($\frac{\sigma}{L}$).



Phase-space trajectories

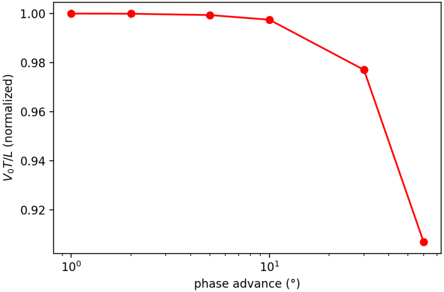
Smaller phase advance	$\Rightarrow$	More regular trajectory
More regular trajectory	$\Rightarrow$	Model of the continuous focusing channel more relevant.



Continuous focussing channel

- In a continuous focusing channel, the beam is considered in a continuous field E_0.
- The motion equations are then simple (at least to write) with a confinement force not depending on time.

$$\begin{cases} \dfrac{d\phi}{ds} = -\dfrac{2\pi}{\lambda_{\mathrm{RF}}} \cdot \dfrac{\delta E}{\beta_s^3 \gamma_s^3 mc^2} \\ \dfrac{d\delta E}{ds} = qE_0 T \left(\sin\left(\phi + \varphi_s\right) - \sin\varphi_s \right) \end{cases}$$



Average field in equivalent channels slightly depends on phase advance per lattice.

Synchrotron oscillation

- Preceding equations have been established considering a straight drift-space between cavities (linac case).
- In a synchrotron, drift spaces contain dipolar magnets, which complicated (a little) the preceding equations:
- One has to replace: $\dfrac{1}{\gamma_s^2}$ by η. Reminder: $\eta = \dfrac{1}{\gamma_s^2} - \alpha$.
- One gets:

$$\begin{cases} \dfrac{d\phi}{ds} = -\dfrac{2\pi \cdot \eta}{\lambda_{\mathrm{RF}}} \cdot \dfrac{\delta E}{\beta_s^3 \gamma_s m c^2} \\ \dfrac{d\delta E}{ds} = -qE_0 T \left(\sin\varphi_s \left(1 - \cos\phi\right) - \cos\varphi_s \sin\phi \right) \end{cases}$$

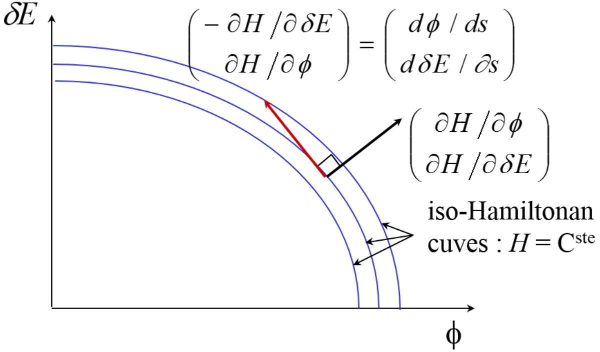
$$\Rightarrow \quad \frac{d^2\phi}{ds^2} = \frac{2\pi \cdot \eta}{\lambda_{\mathrm{RF}}} \frac{qE_0 T}{\beta_s^3 \gamma_s m c^2} \left(\sin\varphi_s \left(1 - \cos\phi\right) - \cos\varphi_s \sin\phi \right)$$

- This is a non-linear oscillator equation describing the synchrotron oscillation.

The particle motion can be described using a function of phase and energy: the motion Hamiltonian $H(\phi, \delta E; s)$

Hamiltonian definition:

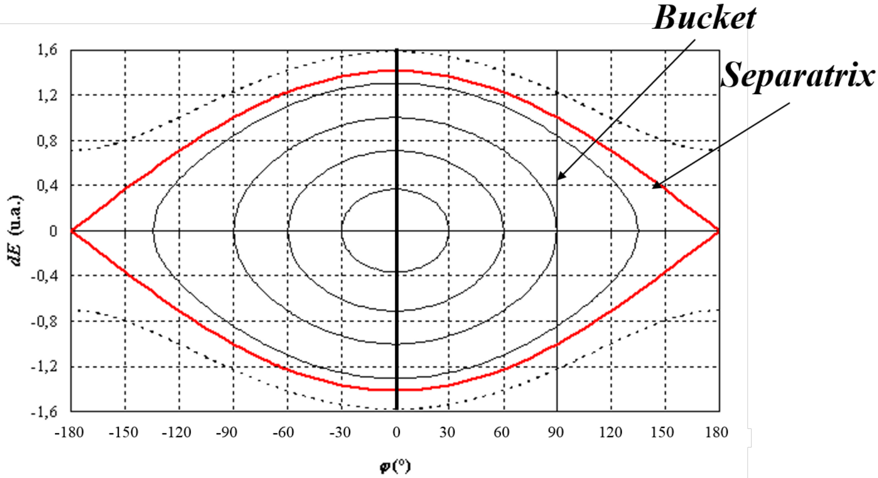
$$\begin{cases} \dfrac{d\phi}{ds} = -\dfrac{\partial H}{\partial \delta E} \\ \dfrac{d\delta E}{ds} = \dfrac{\partial H}{\partial \phi} \end{cases}$$



$$H(\phi, \delta E; s) = \frac{\pi \cdot \eta}{\lambda_{\mathbf{RF}}} \frac{\delta E^2}{\beta_s^3 \gamma_s m c^2} - qE_0 T \left(\sin\varphi_s (\phi - \sin\phi) - \cos\varphi_s (1 - \cos\phi)\right)$$

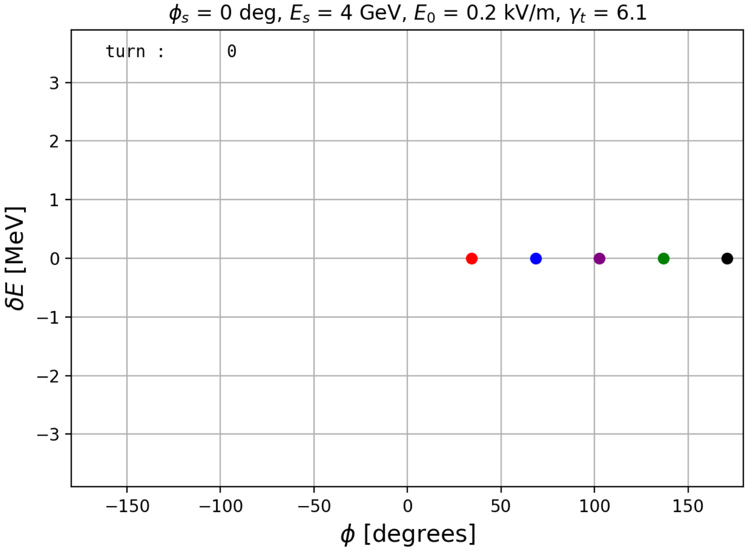
Phase-space trajectory $\varphi_s = 0^\circ$ or 180°

When $\varphi_s = 0^\circ$ or 180°, the synchronous particle is not accelerated.

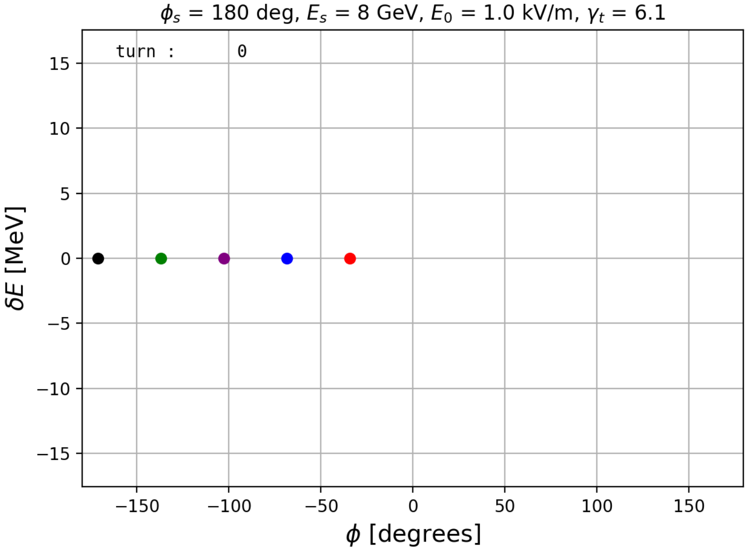


Q : In which sense are the particles turning ?

Animation $\varphi_s = 0°$: below the transition $\gamma < \gamma_t$

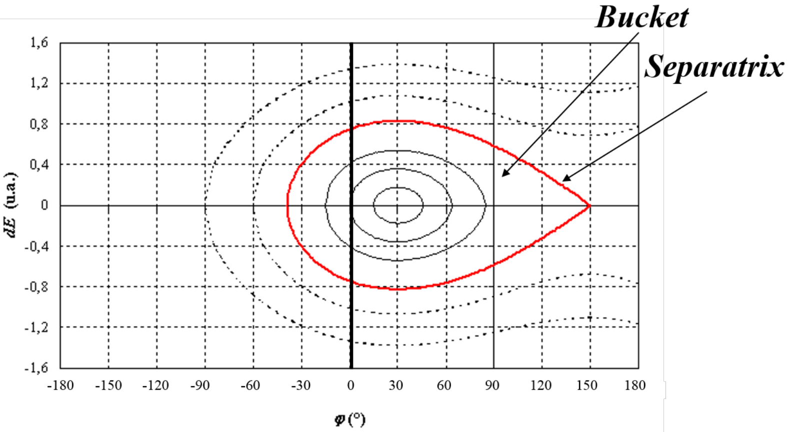


Animation $\varphi_s = 180°$: above the transition $\gamma > \gamma_t$



Phase-space trajectory $\varphi_s = 30^\circ$ or 150°

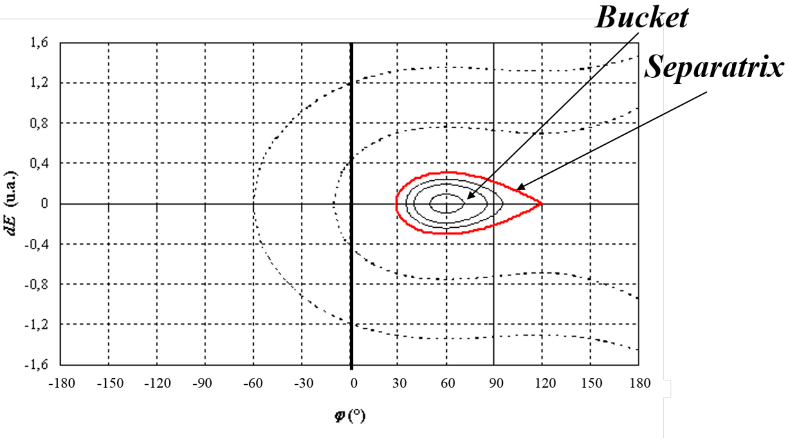
When $\varphi_s = 30^\circ$ or 150°, synchronous particle gets 50% of the possible energy gain ($\sin 30^\circ$).



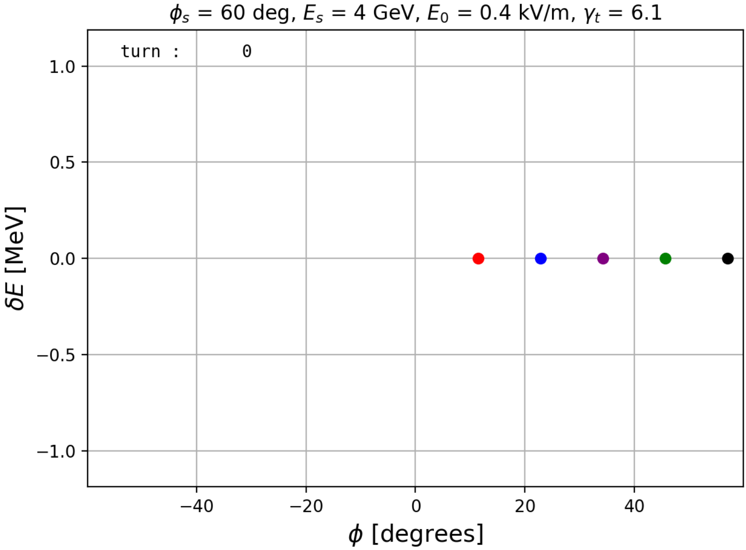
Q : In which sense are the particles turning ?

Phase-space trajectory $\varphi_s = 60^\circ$ or 120°

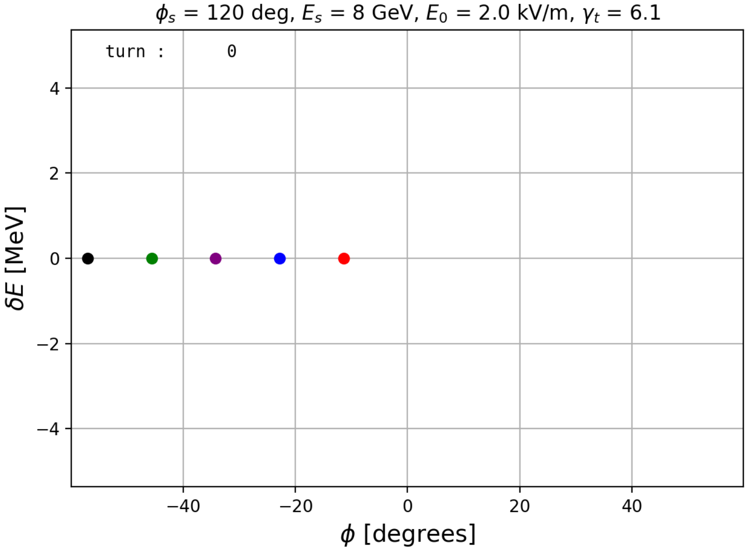
When $\varphi_s = 60^\circ$ or 120°, synchronous particle gets 87% of the possible energy gain ($\sin 60^\circ$).



Animation $\varphi_s = 60°$: below the transition $\gamma < \gamma_t$



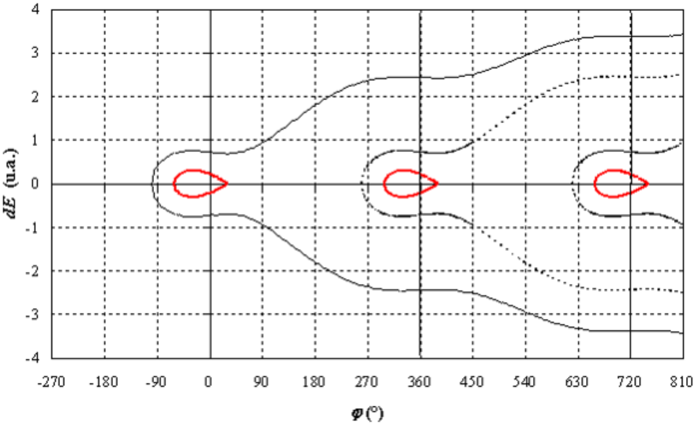
Animation $\varphi_s = 120°$: above the transition $\gamma > \gamma_t$



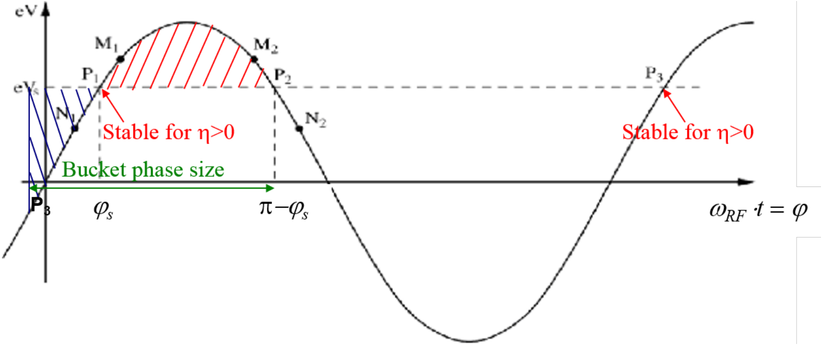
Remarks

- The Bucket is the phase-surface where particles are accelerated. They oscillate around synchronous particle and get the same average energy gain.
- The separatrix is the bucket frontier.
- The closer the synchronous phase from the crest, the higher the acceleration but the lower the bucket size.
- At injection in a synchrotron, the synchronous phase is $\varphi_s = 0^\circ$ or 180°.

Unhooked particles



- The unhooked particles are not accelerated in average.
- They get late ($\eta > 0$) or early ($\eta < 0$) on synchronous particles.



First phase stability limit, P_2, such as: $\phi(P_2) = \pi - 2\varphi_s$

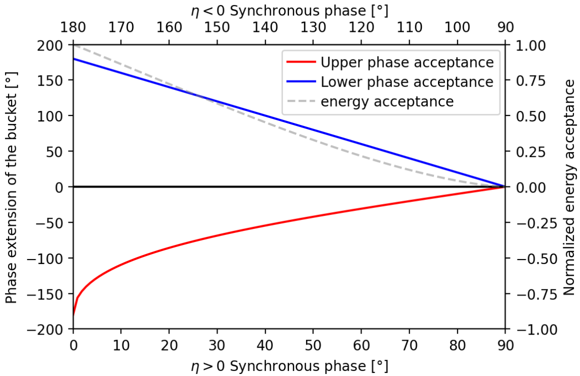
Second phase stability limit, P_3, such as: $H(\phi(P_3),0) = H(\phi(P_2),0)$

Energy stability limit, $\delta E_{\max}$, such as: $H(0,\delta E_{\max}) = H(\phi(P_2),0)$

Bucket phase size

The second phase stability limit, P_3 , is such as:

$$H(\phi(P_3),0) = H(\pi - 2\varphi_s, 0)$$
$$H(\pi - 2\varphi_s, 0) = 2qE_0T \cdot \left(\cos\varphi_s - \left(\frac{\pi}{2} - \varphi_s\right)\sin\varphi_s\right)$$

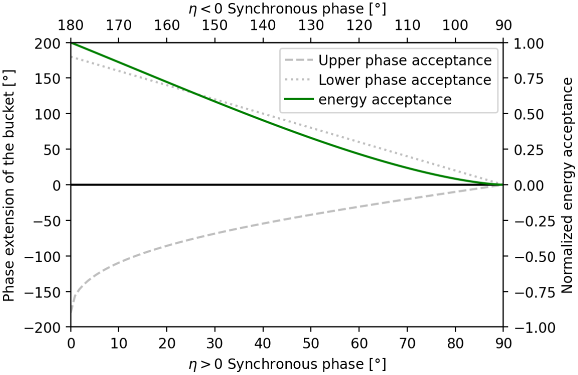


Bucket energy size

The energy stability limit, δE_{max}, is such as:

$$H(0, \delta E_{\text{max}}) = 2qE_0 T \cdot \left(\cos\varphi_s - \left(\varphi_s - \frac{\pi}{2}\right)\sin\varphi_s\right)$$

$$\delta E_{\text{max}} = \sqrt{2qE_0 T \cdot \left(\cos\varphi_s - \left(\frac{\pi}{2} - \varphi_s\right)\sin\varphi_s\right) \cdot \frac{\beta_s^3 \gamma_s mc^2 \lambda_{\text{RF}}}{\pi\eta}}$$



Linearisation at low amplitude

- For $\phi \ll 1$, the Hamiltonian can be approximated by:

$$H(\phi, \delta E) = \frac{\pi\eta}{\lambda_{\mathrm{RF}}} \cdot \frac{\delta E^2}{\beta_s^3 \gamma_s m c^2} + q E_0 T \cos\varphi_s \cdot \frac{\phi^2}{2}$$

- This is an ellipse equation in phase-space.
- Particle motion is an harmonic oscillator:

$$\frac{d^2\phi}{ds^2} = -\frac{2\pi\eta}{\lambda_{\mathrm{RF}}} \frac{q E_0 T}{\beta_s^3 \gamma_s m c^2} \cos\varphi_s \cdot \phi = -\frac{\Omega_{s,0}^2}{\beta_s^2 c^2} \cdot \phi$$

- The synchrotron wave number is:

$$Q_{s,0} = \frac{\Omega_{s,0}}{2\pi f_{\mathrm{rev}}} = \mathscr{C} \sqrt{\frac{\eta}{2\pi\lambda_{\mathrm{RF}}} \cdot \frac{q E_0 T}{\beta_s^3 \gamma_s m c^2} \cdot \cos\varphi_s}$$

Wave number spread (equations)

$$Q_s = \frac{\mathscr{C}}{\mathscr{S}} = \frac{\mathscr{C}}{2\int_{\phi_{\min}}^{\phi_{\max}} \frac{d\phi}{d\phi/ds}}$$

$\mathscr{C}$ trajectory path length
$\mathscr{S}$ trajectory length over one synchrotron oscillation.

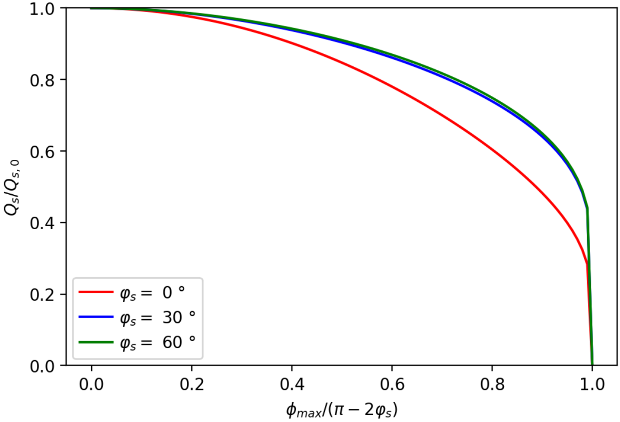
with

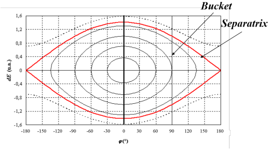
$$\frac{d\phi}{ds} = -\frac{2\pi\eta}{\lambda_{\mathrm{RF}}} \cdot \frac{\delta E}{\beta_s^3 \gamma_s mc^2}$$

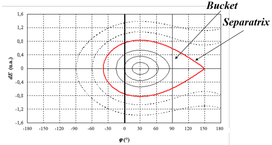
$$\delta E = \sqrt{\frac{\beta_s^3\gamma_s mc^2 \lambda_{\mathrm{RF}}}{\pi\eta}\left(H_0 + qE_0 T\left(\sin\varphi_s(\phi - \sin\phi) - \cos\varphi_s(1-\cos\phi)\right)\right)}$$

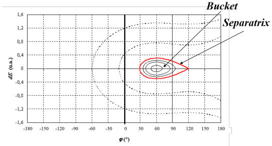
$$\boxed{\frac{Q_s}{Q_{s,0}} = \frac{\sqrt{2}\pi}{\int_{\phi_{\min}}^{\phi_{\max}} \frac{d\phi}{\sqrt{\left|(\cos\phi - \cos\phi_{\max}) + \tan\varphi_s((\phi-\phi_{\max}) - (\sin\phi - \sin\phi_{\max}))\right|}}}}$$

Wave number spread (plots)

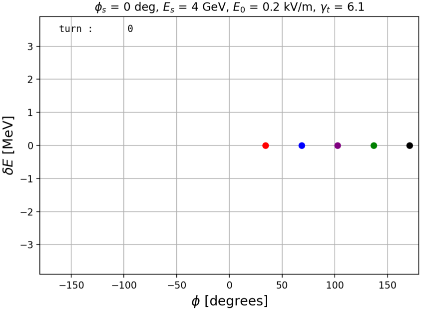


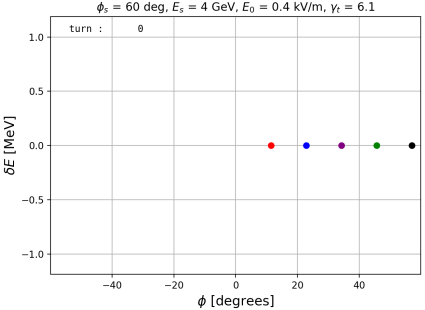






Wave number spread (animation)





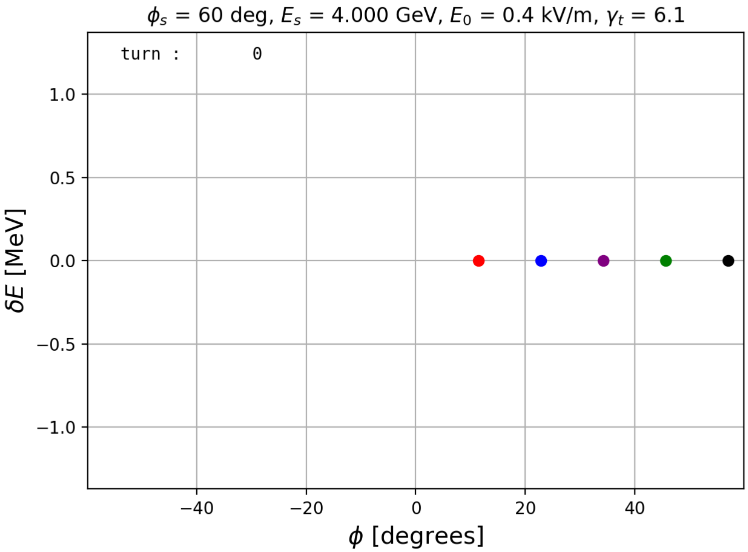
Adiabatic damping

$$\begin{cases} \dfrac{d\phi}{ds} = -\dfrac{2\pi\cdot\eta}{\lambda_{\mathrm{RF}}}\cdot\dfrac{\delta E}{\beta_s^3\gamma_s mc^2} \\ \dfrac{d\delta E}{ds} = qE_0 T\left(\cos\varphi_s \sin\phi - \sin\varphi_s\left(1-\cos\phi\right)\right) \end{cases}$$

$$\frac{d}{ds}\left(\frac{\beta_s^3\gamma_s}{\eta}\cdot\frac{d\phi}{ds}\right) = -\frac{2\pi}{\lambda_{\mathrm{RF}} mc^2} qE_0 T\left(\sin\varphi_s\left(\cos\phi-1\right)+\cos\varphi_s\sin\phi\right)$$

$$\frac{d^2\phi}{ds^2} + \underbrace{\frac{\frac{d}{ds}\left(\frac{\beta_s^3\gamma_s}{\eta}\right)}{\left(\frac{\beta_s^3\gamma_s}{\eta}\right)}}_{\text{Adiabatic matching}}\cdot\frac{d\phi}{ds} = -\frac{2\pi\cdot\eta\cdot qE_0 T}{\lambda_{\mathrm{RF}}\beta_s^3\gamma_s mc^2}\left(\sin\varphi_s\left(\cos\phi-1\right)+\cos\varphi_s\sin\phi\right)$$

Animation



ϕ_s = 60 deg, E_s = 4.000 GeV, E_0 = 0.4 kV/m, γ_t = 6.1
turn : 0
1.0
0.5
0.0
−0.5
−1.0
δE [MeV]
−40
−20
0
20
40
ϕ [degrees]

Periodic, linear force

- In the highest simplification level, the external force along direction w (x, y or φ) can be considered periodic, linear, uncoupled and undamped over one period :

$$\text{Hill's equation}: \quad \frac{d^2 w}{ds^2} + k_w(s) \cdot w = 0 \qquad k_w(s+\mathscr{S}) = k_w(s)$$

- Giving : $\quad w(s) = \sqrt{2J_w \beta_{wm}(s)} \cdot \cos\left(\psi_w(s) + \psi_{w0}\right)$ with:

$$\text{with : } \mu_0 \text{ and } J_w \text{ constant,}$$

$$\psi_w(s) = \psi_{w0} + \int_{s_0}^{s} \frac{ds}{\beta_{wm}(s)},$$

$$\text{and } \beta_{wm}(s) = \beta_{wm}(s+\mathscr{S})$$

Periodic, linear force (2)

See also:
transverse dynamics

- In the (w, w') phase-space, the particle is moving on an ellipse of equation :

$$\gamma_{wm}(s) \cdot w^2 + 2\alpha_{wm}(s) \cdot w \cdot w' + \beta_{wm}(s) \cdot w'^2 = 2J_w$$

Courant and Snyder parameters: $\alpha_{wm}(s) = -\frac{1}{2}\frac{d\beta_{wm}(s)}{ds}$

$$\gamma_{wm}(s) = \frac{1 + \alpha_{wm}^2(s)}{\beta_{wm}(s)}$$

- The phase advance of the particle in the lattice is:

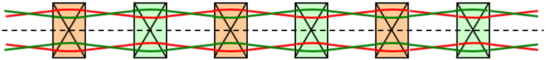
$$\mu = \psi(s + \mathscr{S}) - \psi(s)$$

Periodic, linear force (animations)

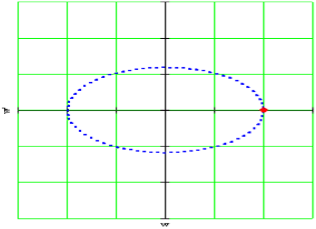
See also:
transverse dynamics

Particle

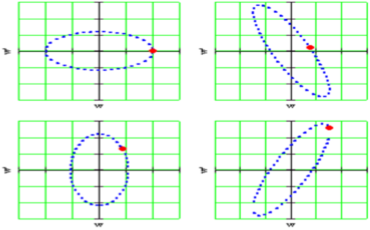
Particle ellipse

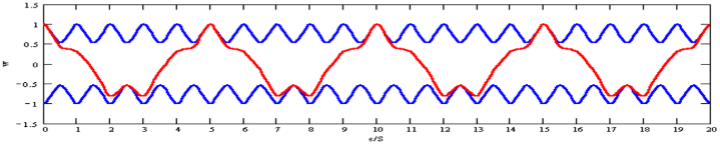


Phase-space trajectory



Phase-space periodic looks





Particle trajectory

Particle ellipse maximum size

RMS dimensions

The rms dimensions of the beam are defined statistically as followed :

RMS size: $\sigma_w = \sqrt{\left\langle (w - \langle w \rangle)^2 \right\rangle}$

RMS divergence: $\sigma_{w'} = \sqrt{\left\langle (w' - \langle w' \rangle)^2 \right\rangle}$

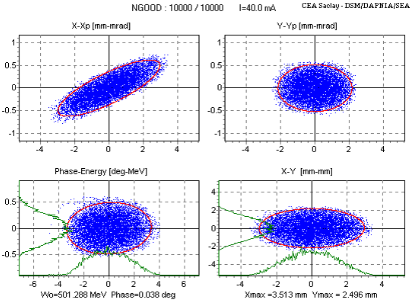
RMS emittance: $\epsilon_w = \sqrt{\sigma_w \cdot \sigma_{w'} - \left\langle (w - \langle w \rangle)(w' - \langle w' \rangle) \right\rangle^2}$

The beam Twiss parameters are then :

$$\beta_w = \frac{\sigma_w^2}{\epsilon_w} \quad \gamma_w = \frac{\sigma_{w'}^2}{\epsilon_w}$$

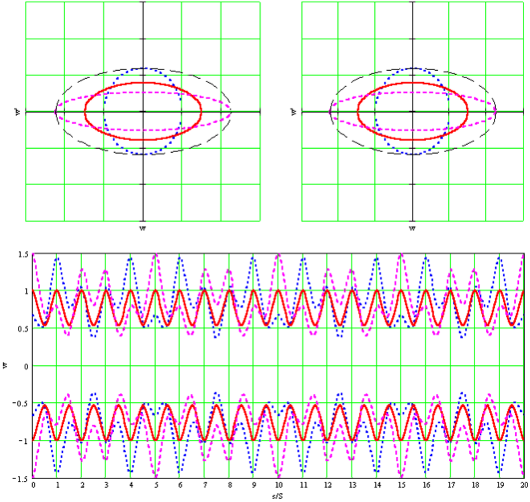
$$\alpha_w = \frac{\left\langle (w - \langle w \rangle)(w' - \langle w' \rangle) \right\rangle}{\epsilon_w}$$

— $\gamma_{wm} \cdot w^2 + 2\alpha_{wm} \cdot w \cdot w' + \beta_{wm} \cdot w'^2 = 5\epsilon_w$



50% mismatched beam

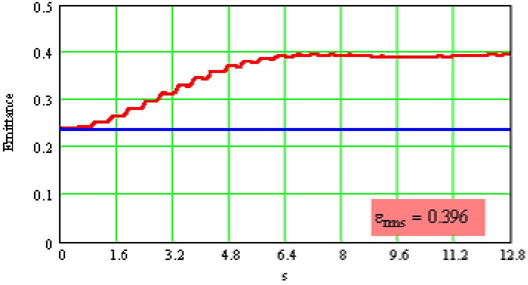
Phase-space trajectory — Phase-space periodic looks

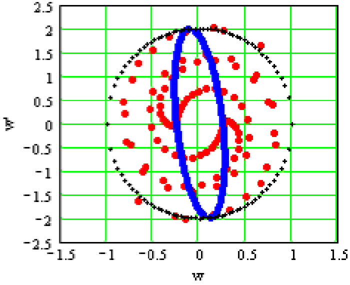


$$\alpha_w = \alpha_{wm}$$
$$\beta_w = \beta_{wm}$$
$$\gamma_w = \gamma_{wm}$$

- Matched beam
- Bigger input beam
- Smaller input beam
- Mismatched beams (Phase-space scan)

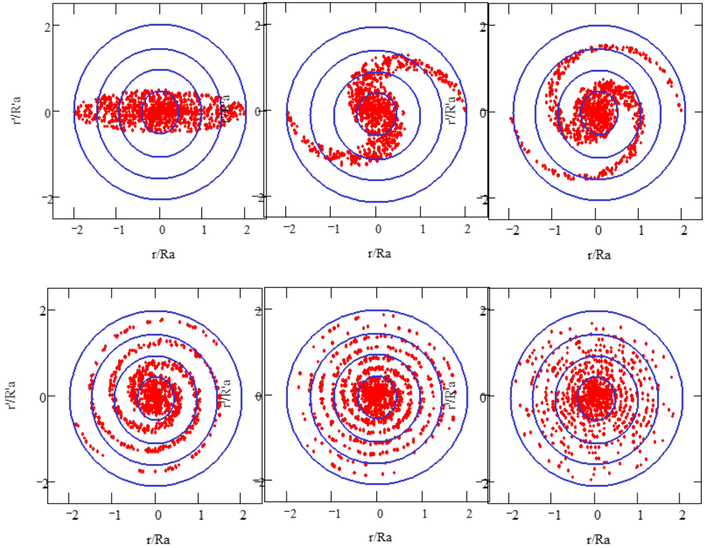
- Mismatching $\Rightarrow$ Emittance growth and Halo formation through :
  - non linear forces (external or space-charge),
  - resonance of some particle motion with core oscillation (space-charge).





(Mis-)matched beam (illustration)

See also:
transverse dynamics



Summary – Longitudinal dynamics

- Particle longitudinal motion described in (Phase; Energy) phase-space.
- With respect to the synchronous particle (representing accelerator)
- Particle oscillation around synchronous particle (synchrotron oscillation)
- Periodic focusing → continuous focusing
- Motion ⊥ Hamiltonian gradient
- Stable region: bucket inside separatrix
- Adiabatic damping in phase when acceleration
- Non-linear motion
- Filamentation for mismatched beam → emittance growth