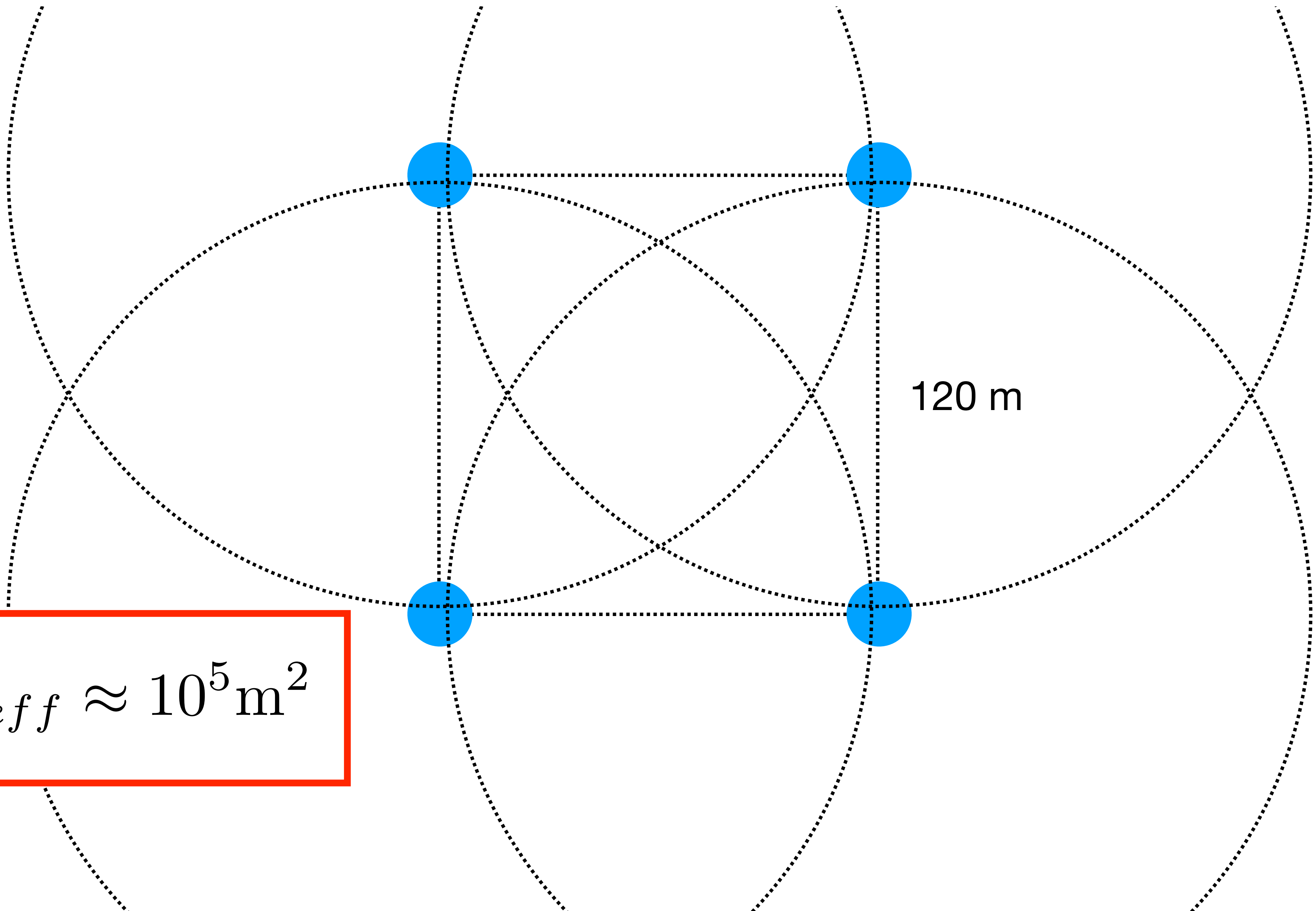
NPAC course on Astroparticles

Exercise #2 — Solution

Effective area

Total effective area

$$0.45 \times 10^5 \mathrm{m}^2 \sim \mathrm{A}^1_{\mathrm{eff}} < A_{eff} < 4 \times A^1_{eff} \sim 1.8 \times 10^5 \mathrm{m}^2$$



$$A_{eff} \approx 10^5 \mathrm{m}^2$$

Effective area for a single telescope $A^1_{eff} = \pi (120\ \mathrm{m})^2 \sim 4.5 \times 10^4 \mathrm{m}^2$

Flux sensitivity

Flux of the Crab Nebula $\Phi_\gamma(>1\ \mathrm{TeV}) \sim 2\times 10^{-11}\mathrm{photons\ cm^{-2}\ s^{-1}}$

Photon detection rate $R_s = \Phi_\gamma(>1\ \mathrm{TeV})A_{eff} = \boxed{2\times 10^{-2} s^{-1}}$

$1/R_s = 50\mathrm{s}$ about a photon per minute

Cosmic ray intensity

$$j_{CR}(E) = A\left(\frac{E}{E_0}\right)^{-2.7} = 1.3\left(\frac{E}{E_0}\right)^{-2.7}\ \mathrm{GeV^{-1}cm^{-2}s^{-1}sr^{-1}}$$

Background rate

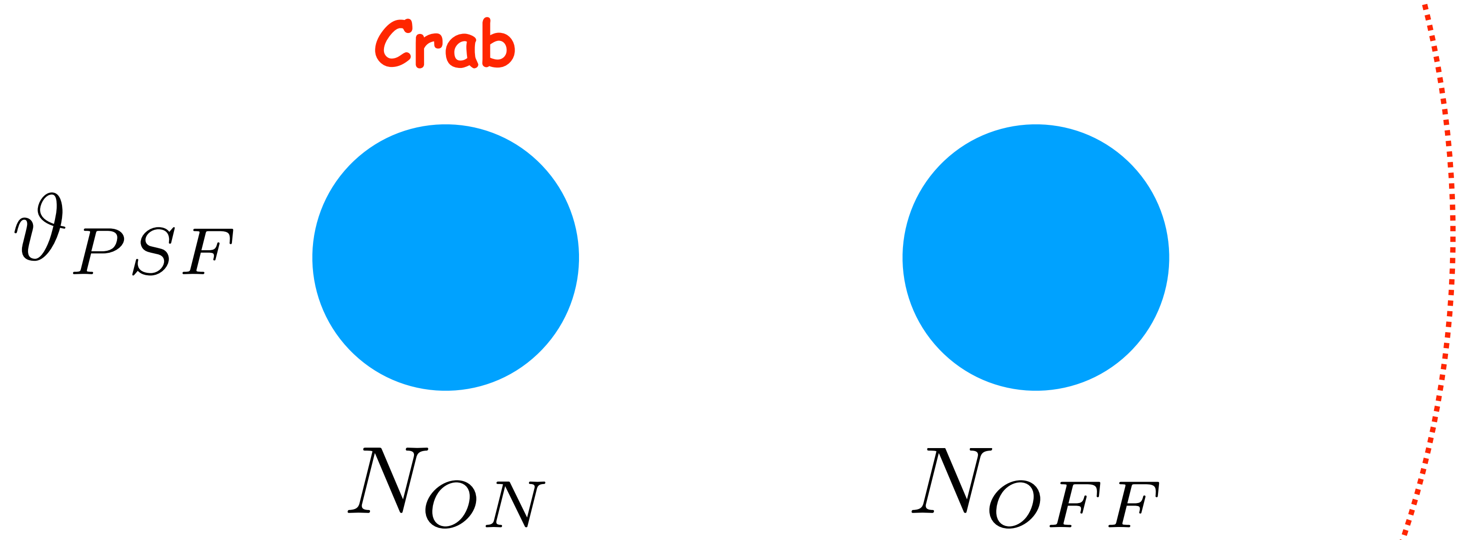
$$\phi_B(>\mathrm{TeV}) = \int_{\mathrm{TeV}} \mathrm{d}E\ j_{CR}(E) = A\int_{\mathrm{TeV}} \mathrm{d}E\left(\frac{E}{E_0}\right)^{-2.7} = AE_0\int_{1000}\mathrm{d}x\ x^{-2.7}$$

$$= \frac{AE_0}{1.7}(1000)^{-1.7} \sim 6.1\times 10^{-6}\mathrm{cm^{-2}s^{-1}sr^{-1}}$$

Angular resolution (point spread function)

$$R_B = \Phi_B A_{eff} = \phi_B \pi\left(\frac{\vartheta_{PSF}}{2}\right)^2 A_{eff} = \boxed{1.5\times 10^{-2}\mathrm{s}^{-1}}$$

How long it takes to detect the Crab?



We extract the background from a region within the telescope field of view. For simplicity let's consider the 2 regions to be identical. We observed the Crab for a time T.

Signal —> $S = R_s \times T$

Background —> $B = R_B \times T$

Background rejection rate (shape of the shower) —> $\eta_B \approx 10^{-2}$

$$N_{ON} = S + \eta_B B$$

$$N_{OFF} = \eta_B B$$

$$\sigma = \frac{N_{ON} - N_{OFF}}{\sqrt{N_{ON} + N_{OFF}}} = \frac{S}{\sqrt{S + 2\eta_B B}} = \frac{R_s T}{\sqrt{R_s T + 2\eta_B R_B T}}$$

$$= \frac{R_s}{\sqrt{R_s + 2\eta_B R_B}} T^{1/2} \longrightarrow T = \frac{R_s + 2\eta_B R_B}{R_s^2} \sigma^2$$

For the Crab Nebula —> $R_s \gg \eta_B R_B$

The source is detected when $N_\gamma > 5$

There is of course some level of arbitrariness in the choice of the number 5. This can be rephrased as: in order to detect a source we need to detect "enough" photons.

For the Crab we have: $N_\gamma = R_s \times T = 5 \longrightarrow \boxed{T = 4 \text{ minutes}}$

Minimum detectable flux in a long observation

Typically we consider T = 50 h (keep in mind that in a year there are about 1000-1500 h of available observing time). Let's use the result obtained 2 slides ago:

$$\sigma = \frac{R_s}{\sqrt{R_s + 2\eta_B R_B}} T^{1/2}$$

$$\sigma = 5$$
$$T = 50 \text{ h}$$

$$T R_s^2 - \sigma^2 R_s - 2\sigma^2 \eta_B R_B = 0$$

$$R_s^{min} = \frac{\sigma^2 \pm \sqrt{\sigma^4 + 8\sigma^2 \eta_B R_B T}}{2T} \quad \sim 3 \times 10^{-4} \text{s}^{-1}$$

$$\frac{R_s^{min}}{R_s^{Crab}} = \frac{3 \times 10^{-4}}{2 \times 10^{-2}} \approx 1\%$$

The sensitivity of current Cherenkov telescopes is at the percent level of the Crab

$$\Phi_\gamma^{min}(> 1 \text{ TeV}) = 2 \times 10^{-11} \frac{3 \times 10^{-4}}{2 \times 10^{-2}} \approx 10^{-13} \text{cm}^{-2}\text{s}^{-1}$$

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Exercise #3— Solution

The spectrum of cosmic rays injected in the Milky Way is (see Exercise 1)

$$Q_{CR}(E) = B \left(\frac{E}{E_0} \right)^{-2.4} \qquad B \sim 2.5 \times 10^{34} \mathrm{eV}^{-1} \mathrm{s}^{-1}$$

To compute the gamma-ray luminosity of the Milky Way we start from the equation derived in the class:

$$Q_\gamma(E_\gamma) E_\gamma^2 = \frac{\eta_\pi}{3} Q_p(E_p) E_p^2$$

And we rewrite it as:

$$Q_\gamma(E_\gamma) = \frac{\eta_\pi}{3} Q_p(E_p) \frac{E_p^2}{E_\gamma^2} = \frac{100}{3} \eta_\pi Q_p(E_p)$$

$$Q_\gamma(E_\gamma) = \frac{100}{3}\eta_\pi Q_p(E_p) \qquad \eta_\pi = 1 - \exp\left(-\frac{\tau_{res}}{\tau_{pp}}\right)$$

Residence time in the disk

$$\tau_{res} \sim 3\left(\frac{E_p}{10\ \mathrm{GeV}}\right)^{-0.3}\ \mathrm{Myr} \quad \sim 3\times 10^{0.3}\left(\frac{E_p}{E_0}\right)^{-0.3}\ \mathrm{Myr}$$

Energy loss time (p-p interactions)

$$\tau_{pp} \sim (n_{gas}\sigma_{pp}ck)^{-1} \ \sim 60\ \mathrm{Myr}$$

$$\tau_{res} \ll \tau_{pp} \longrightarrow \eta_\pi \sim \frac{\tau_{res}}{\tau_{pp}}$$

$$Q_\gamma(E_\gamma) = \frac{100}{3}\frac{\tau_{res}}{\tau_{pp}}Q_p(E_p) = 10^{2.3}\frac{B}{\tau_{pp}}\left(\frac{E_p}{E_0}\right)^{-2.7}$$

$$L_\gamma^{MW} = \int_{1\ \mathrm{TeV}}^{\infty} \mathrm{d}E_\gamma Q_\gamma(E_\gamma) \ = 10^{2.3} \frac{B}{\tau_{pp}} \int_{1\ \mathrm{TeV}}^{\infty} \mathrm{d}E_\gamma \left(\frac{E_p}{E_0}\right)^{-2.7}$$

$$= 10^{2.3} \frac{B}{\tau_{pp}} \int_{10\ \mathrm{TeV}}^{\infty} \frac{\mathrm{d}E_p}{10} \left(\frac{E_p}{E_0}\right)^{-2.7} = 10^{1.3} \frac{B}{\tau_{pp}} E_0 \int_{10^4}^{\infty} \mathrm{d}x\ x^{-2.7}$$

$$= 10^{1.3} \frac{B}{\tau_{pp}} E_0 \frac{(10^4)^{-1.7}}{1.7} \ \sim \boxed{8 \times 10^{35} \mathrm{s}^{-1}}$$

Gamma-ray flux $$F_\gamma = \frac{L_\gamma}{4\pi d^2} \equiv \Phi_\gamma^{min}$$

$$\rightarrow d_{max} = \left(\frac{L_\gamma}{4\pi\Phi_\gamma^{min}}\right)^{1/2} \sim 250 \text{ kpc}$$

Andromeda is at ~ 1 Mpc —> normal galaxies cannot be observed with Cherenkov telescopes of current generation
(in fact, only active galaxies are observed in TeV gamma rays)

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Exercise #4— Solution

Supernova remnants with dimensional analysis only

Physical constants aree **dimensional** quantities

E.g. G, c, h…

We can use them to define scales of physical problems

Example: self gravitating fluid

A fluid (monoatomic gas) is defined by the following physical quantities:
pressure, density, velocity, temperature

$$P \quad \varrho \quad u \quad T$$

And we should specify an equation of state for the gas

$$P = c_s^2 \varrho \qquad\qquad c_s = \sqrt{\gamma \frac{kT}{m}}$$

And gravity

$$G$$

—> consider a subsonic flow: $u \ll c_s$

Question: can we combine these quantities to form a length scale?

$$\varrho \qquad c_s \qquad G$$

$$[\varrho] = ML^{-3} \quad [c_s] = LT^{-1} \quad [G] = M^{-1}L^3T^{-2}$$

$$\lambda \sim G^\alpha c_s^\beta \varrho^\gamma$$

$$L \sim (M^{-1}L^3T^{-2})^\alpha (LT^{-1})^\beta (ML^{-3})^\gamma$$

$$\begin{cases} 1 = 3\alpha + \beta - 3\gamma \;\longrightarrow 1 = 3\alpha - 2\alpha - 3\alpha \rightarrow \alpha = -\dfrac{1}{2} \\ 0 = -\alpha + \gamma \;\; \rightarrow \alpha = \gamma \quad \longrightarrow \gamma = -1/2 \\ 0 = -2\alpha - \beta \;\; \rightarrow \beta = -2\alpha \quad \longrightarrow \beta = 1 \end{cases}$$

Jeans mass

$$\lambda_J \sim \frac{c_s}{\sqrt{G\varrho}}$$

What does it mean?
Consider a spherical mass (M) of radius R

Gravitational energy –>

$$E_g \sim \frac{GM^2}{R} \sim G\varrho^2 R^5$$

Thermal energy –>

$$E_{th} \sim \epsilon V \sim PV \sim \varrho c_s^2 R^3$$

$$E_{th} \sim E_g \longrightarrow R \sim \frac{c_s}{\sqrt{G\varrho}}$$

$R > \lambda_J \longrightarrow$ Gravitational collapse

$R < \lambda_J \longrightarrow$ Stable system (gravity is not important)

Let's proceed further

$$\varrho \qquad \not{c_s} \qquad G$$

$$R > \lambda_J$$ Gravity dominates, pressure forces are negligible

We can build a time scale: $$t_{ff} \sim (G\varrho)^{-1/2}$$ **Free fall time**

NOTE: Without G it would have been impossible to build a spatial and temporal scale

Is gravity relevant in supernova remnants?

Free expansion phase: gravity does not count (otherwise there would be no explosion!)

Sedov phase: expanding shock in a pressureless gas (P=0)

E ~~u~~ ϱ G

however, we expect u to change with time, so it is not a good quantity... can we substitute it with another one?

The explosion energy E is a constant during the Sedov phase

We now combine the physical quantities above to obtain a length scale

$$\lambda \sim \left(\frac{E}{G\varrho} \right)^{1/5} \approx 0.5 \text{ kpc}$$

Much larger than the typical SNR radius —> we can neglect gravity!

Scale free problems

As gravity is negligible, the expansion of an energy conserving SNR must be determined by the two following physical quantities only

$$E \qquad \varrho$$

It is impossible to build a length (or a time) scale. So?

The solution must be scale free.

Examples of non-scale free functions...

$$\exp\left(\frac{t}{\tau}\right) \qquad \cos\left(\omega t\right) \quad \ldots$$

Power laws are scale free!

$$R_s \propto t^{\alpha}$$

Sedov (energy conserving) phase

Explosion of energy E in a pressureless uniform gas

$$E \qquad \varrho$$

$$R_s \sim E^\beta \varrho^\gamma t^\alpha$$

$$L = (ML^2T^{-2})^\beta (ML^{-3})^\gamma T^\alpha$$

$$\begin{cases} 1 = 2\beta - 3\gamma \quad \longrightarrow \gamma = -1/5 \\ 0 = \beta + \gamma \ \rightarrow \beta = -\gamma \ \rightarrow \beta = 1/5 \\ 0 = -2\beta + \alpha \quad \longrightarrow \alpha = 2/5 \end{cases}$$

$$R_s \sim \left(\frac{E}{\varrho}\right)^{1/5} t^{2/5}$$

Thin shell approximation

$$R_s \sim 1.12 \left(\frac{E}{\varrho}\right)^{1/5} t^{2/5}$$

Snowplough (radiative) phase

The shell cools but the interior conserves energy

Adiabatic invariant: $K = PV^{5/3} \sim PR^5$

$$[K] = (ML^{-1}T^{-2})(L)^5 = ML^4T^{-2}$$

$$R_s = K^{\beta} \varrho^{\gamma} t^{\alpha}$$

$$L = (ML^4T^{-2})^{\beta}(ML^{-3})^{\gamma}T^{\alpha}$$

$$\begin{cases} 1 = 4\beta - 3\gamma \\ 0 = \beta + \gamma \\ 0 = -2\beta + \alpha \end{cases} \longrightarrow \qquad R_s \sim \left(\frac{K}{\varrho}\right)^{1/7} t^{2/7}$$

Momentum conserving (radiative) phase

$$[\mu] = MLT^{-1} \qquad \mu \qquad \varrho$$

$$R_s = \mu^\beta \varrho^\gamma t^\alpha$$

$$L = (MLT^{-1})^\beta (ML^{-3})^\gamma T^\alpha$$

$$\begin{cases} 1 = \beta - 3\gamma \\ 0 = \beta + \gamma \\ 0 = -\beta + \alpha \end{cases} \longrightarrow \quad R_s \sim \left(\frac{\mu}{\varrho}\right)^{1/4} t^{1/4}$$

Supernova remnant in a stellar wind (Sedov phase)

The density is not uniform but scales as a power law (scale free!)

$$\varrho = BR^{-2}$$

$$R_s \sim \left(\frac{E}{\varrho}\right)^{1/5} t^{2/5} = \left(\frac{E}{BR_s^{-2}}\right)^{1/5} t^{2/5} = \left(\frac{E}{B}\right)^{1/5} t^{2/5} R_s^{2/5}$$

$$R_s^{3/5} = \left(\frac{E}{B}\right)^{1/5} t^{2/5} \longrightarrow R_s = \left(\frac{E}{B}\right)^{1/3} t^{2/3}$$

$2/3 > 2/5$ Larger expansion rate