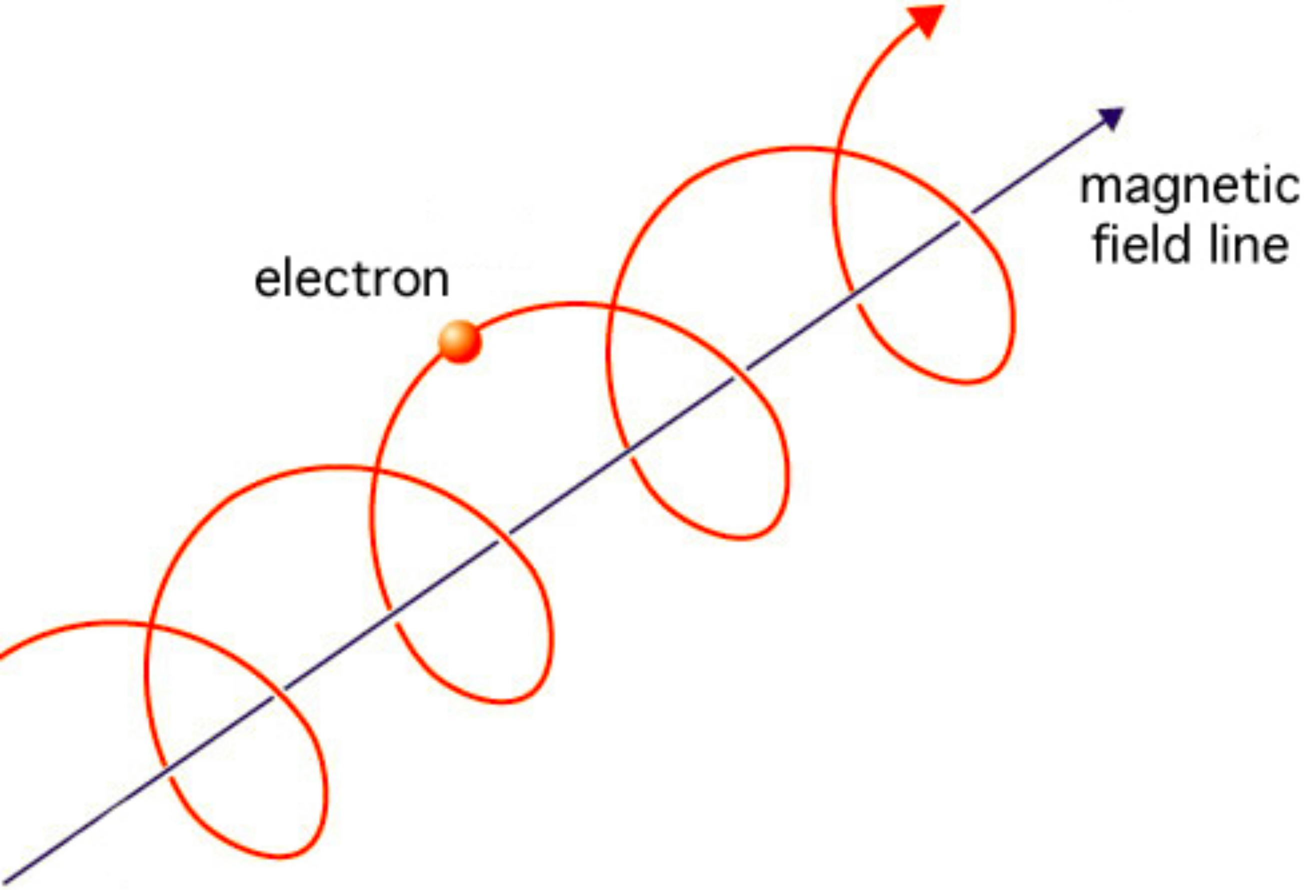
NPAC course on Astroparticles

IV - PLASMA PHYSICS: MagnetoHydroDynamics (MHD)

Outline

- Observational evidences for the presence of magnetic fields: synchrotron radiation
- Plasma physics: basics of MagnetoHydroDynamics (MHD)
- MHD waves: Alfven waves

Motion of a particle in a magnetic field



$$\mu = \cos\vartheta$$

pitch angle = angle between v and B

$$\begin{cases} v_\parallel & = & \mu \; v \\ v_\perp & = & (1-\mu^2)^{1/2} v \end{cases}$$

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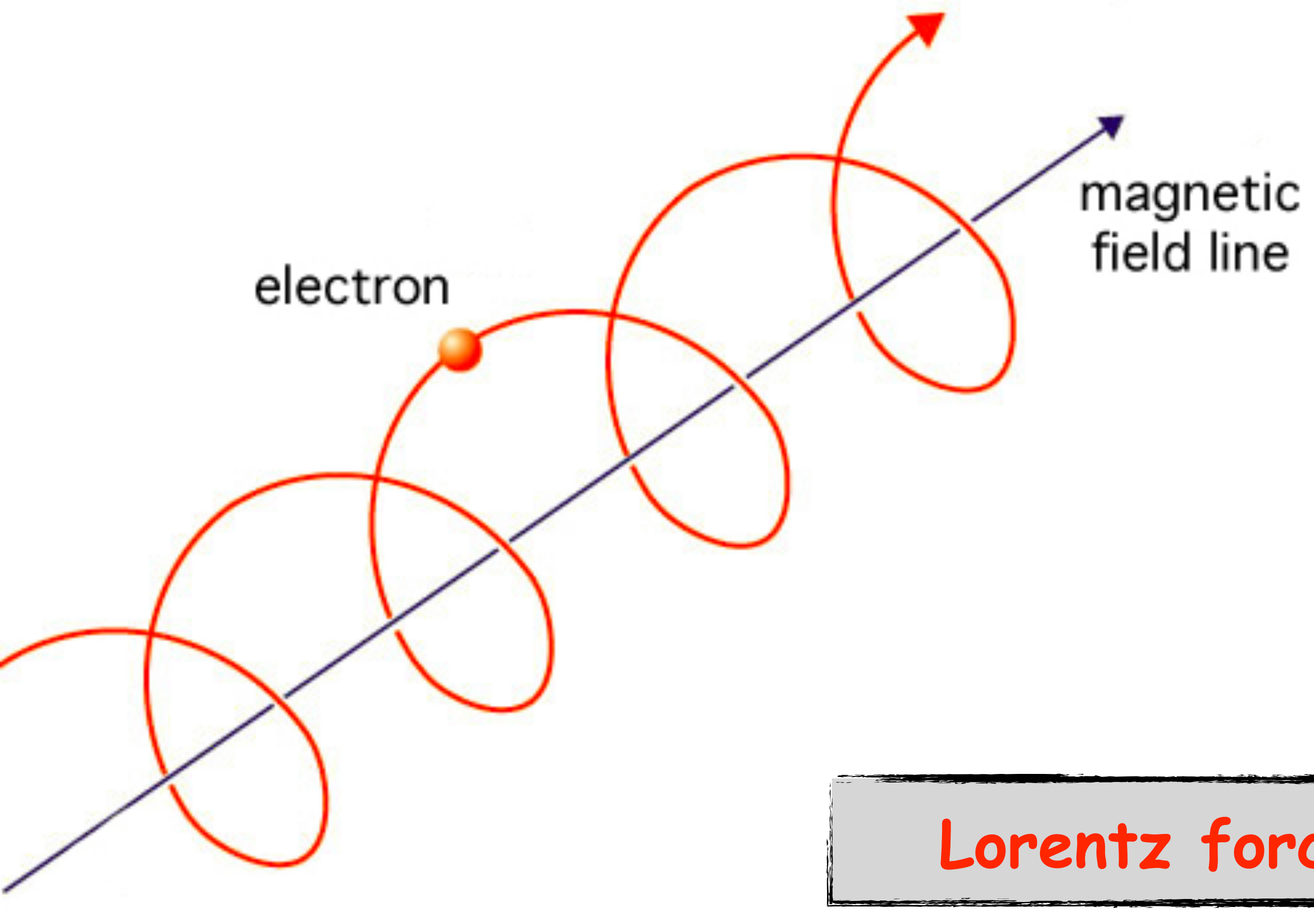

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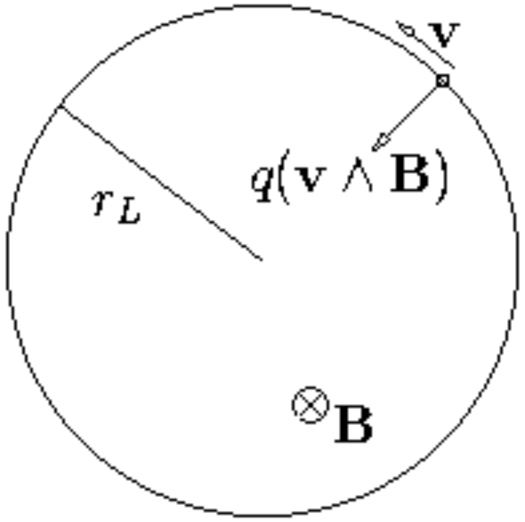
Lorentz force

$$F_L = \frac{q}{c}\vec{v} \times \vec{B}$$

Larmor radius

$$R_L = \frac{p_\perp c}{qB}$$



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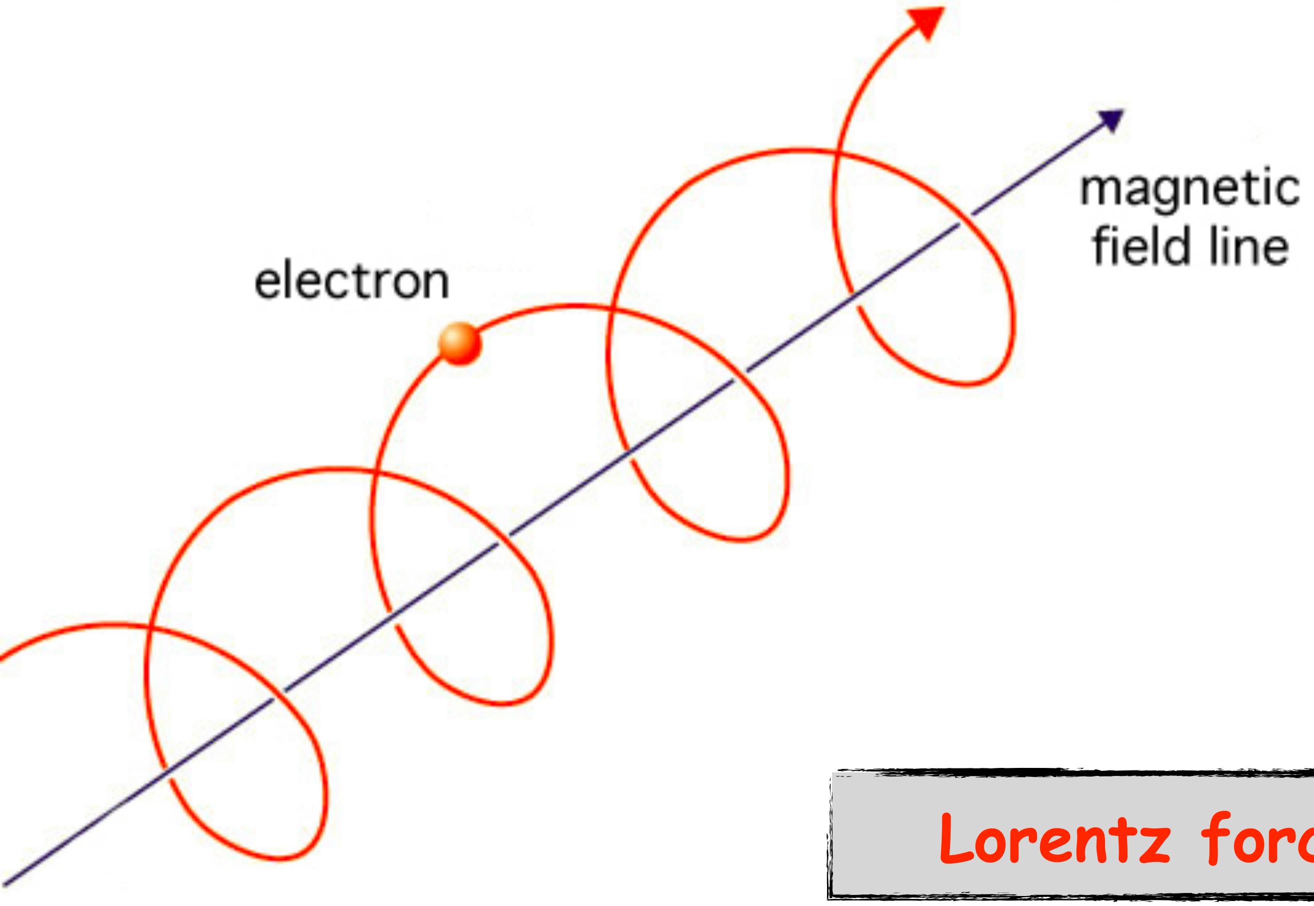

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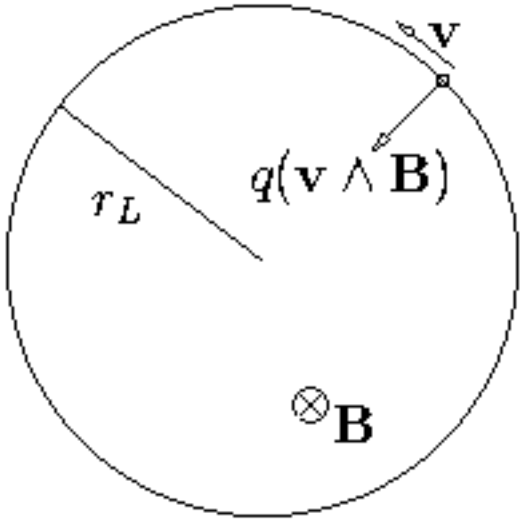
gyration frequency $$\nu_B = \frac{1}{t_g} = \frac{v_{\perp}}{2\pi R_L} = \frac{qB}{2\pi\gamma mc}$$

Lorentz factor



Power emitted by an electron*

non-relativistic $P = \frac{2e^2}{3c^3} a^2 \longrightarrow P = \frac{2e^2}{3c^3} \gamma^4 \left[\gamma^2 a_{\parallel}^2 + a_{\perp}^2 \right]$ **relativistic**

* implicit assumption: the energy of the electron does not change during one gyration around the B-field

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⇒ $$P = \frac{4}{3}\, \sigma_T c\, U_B \gamma^2$$

- Thomson cross section $$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = 6.65 \times 10^{-25} \mathrm{cm}^2$$
- magnetic field energy density $U_B = B^2/8\pi$
- ultra relativistic electrons $\beta \longrightarrow 1$
- isotropic distribution of particles $\langle \sin^2 \vartheta \rangle = 2/3$

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Characteristic frequency

as done for Bremsstrahlung: characteristic time ---> characteristic frequency

gyration frequency?

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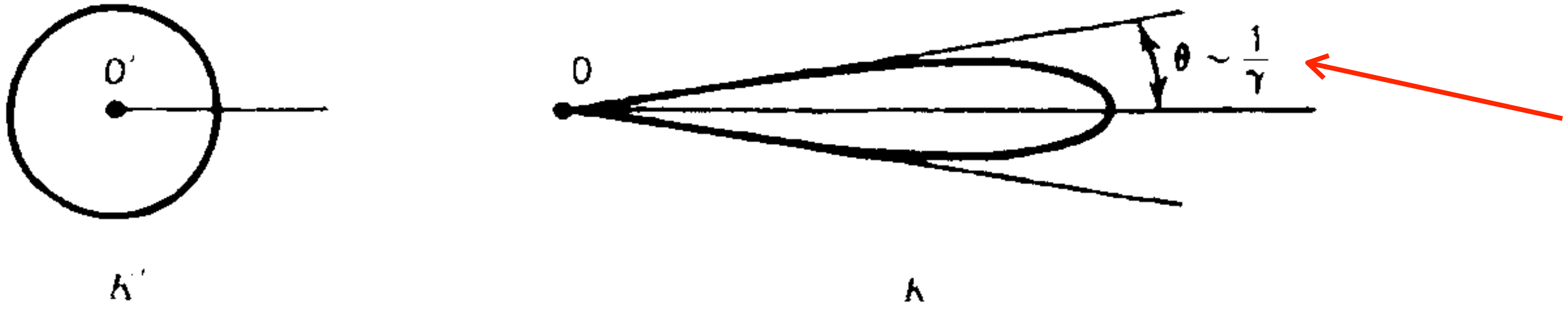
Beaming



$$\vartheta \sim \frac{1}{\gamma}$$

Figure 4.3 ***Relativistic beaming of radiation emitted isotropically in the rest frame K′.***

the radiation emitted by a relativistic particle is concentrated within a cone of opening angle 1/γ entered along the particle velocity

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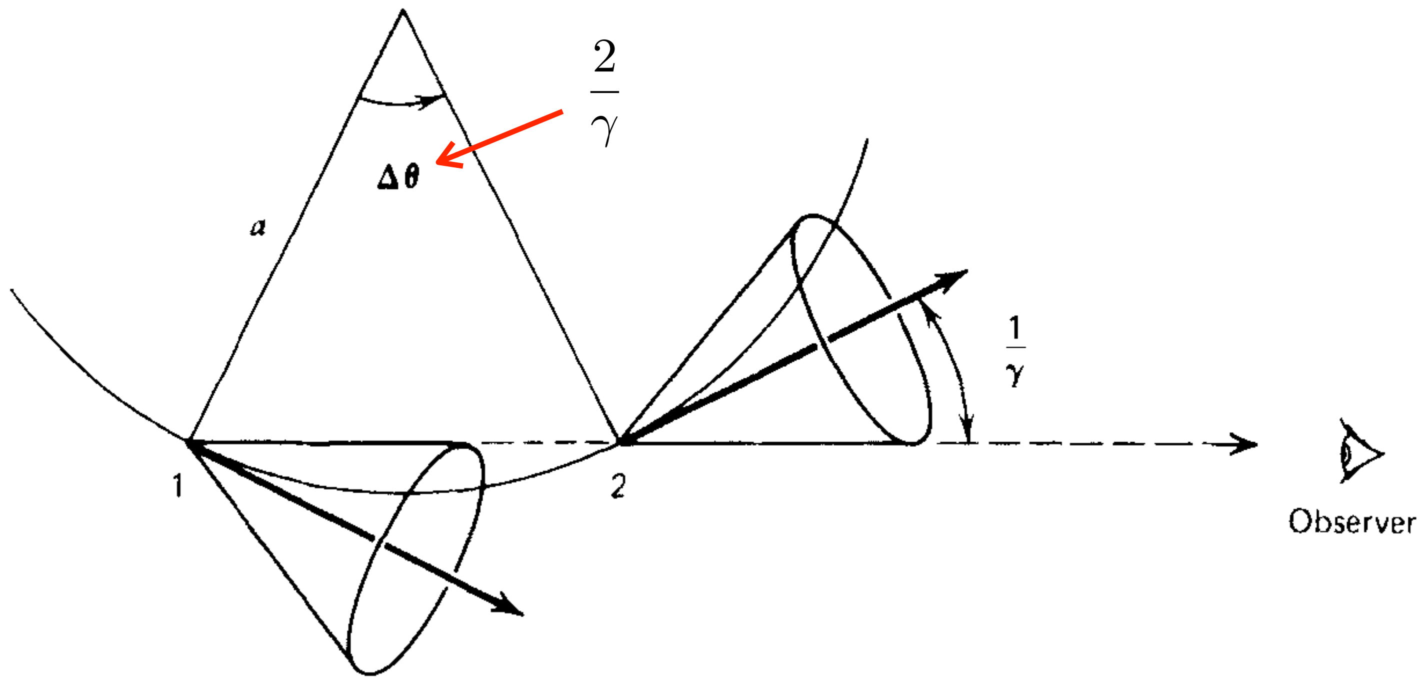

***Figure 6.2** Emission cones at various points of an accelerated particle's trajectory.*

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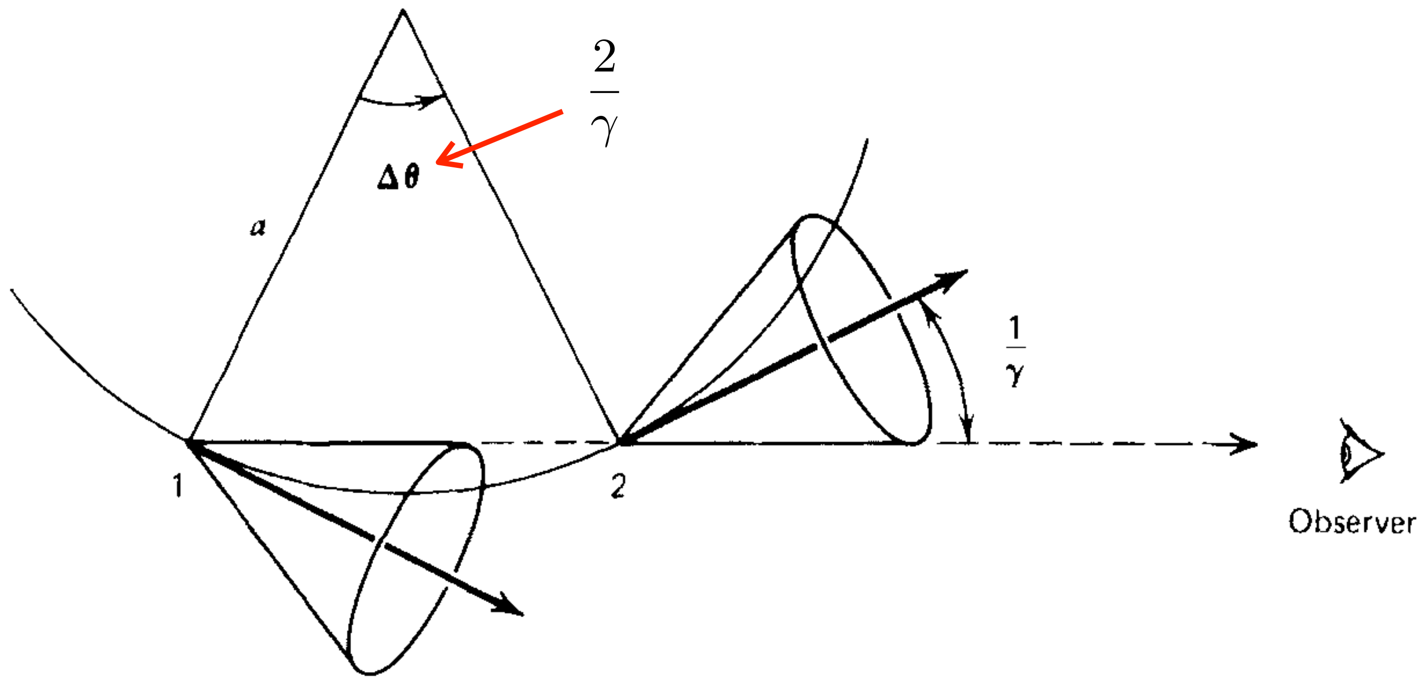

Figure 6.2 Emission cones at various points of an accelerated particle's trajectory.

photons that reach us are emitted in the time interval:

$$\Delta t_e = \frac{R_L \Delta\vartheta}{v_\perp} = \frac{1}{\pi\gamma\nu_B}$$

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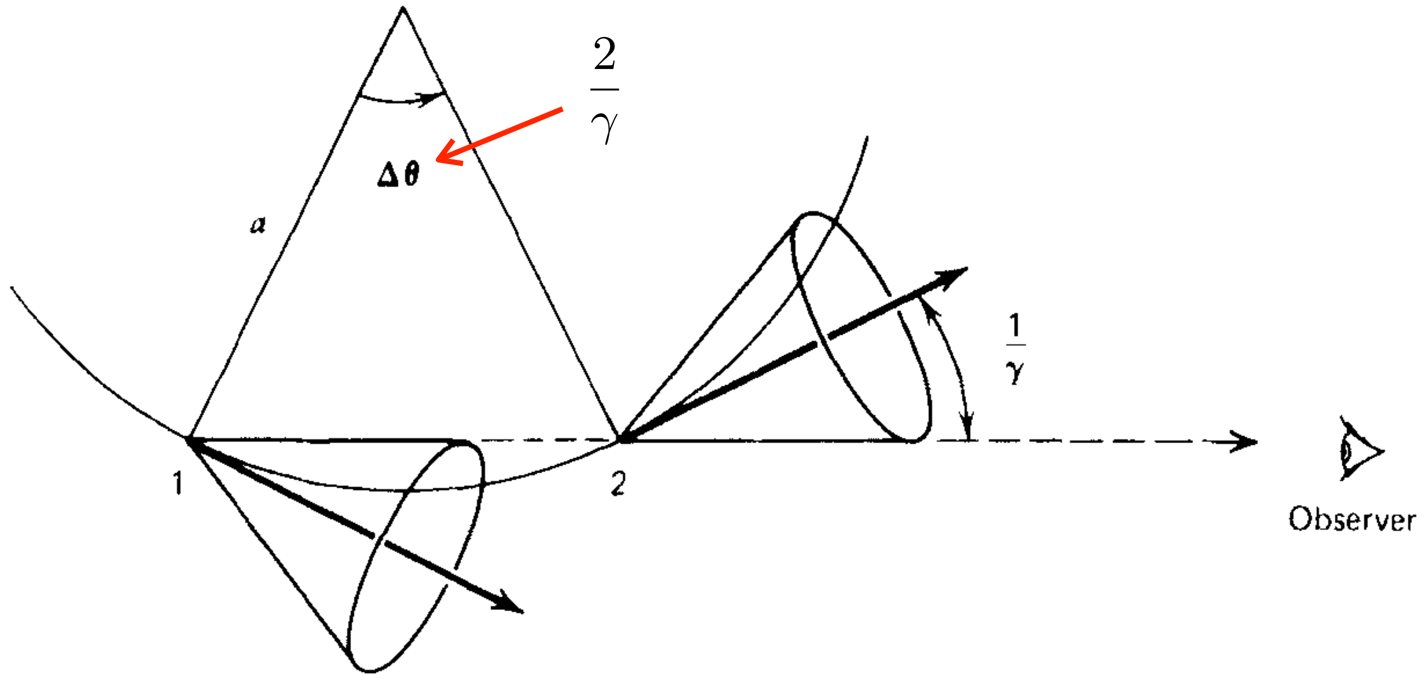

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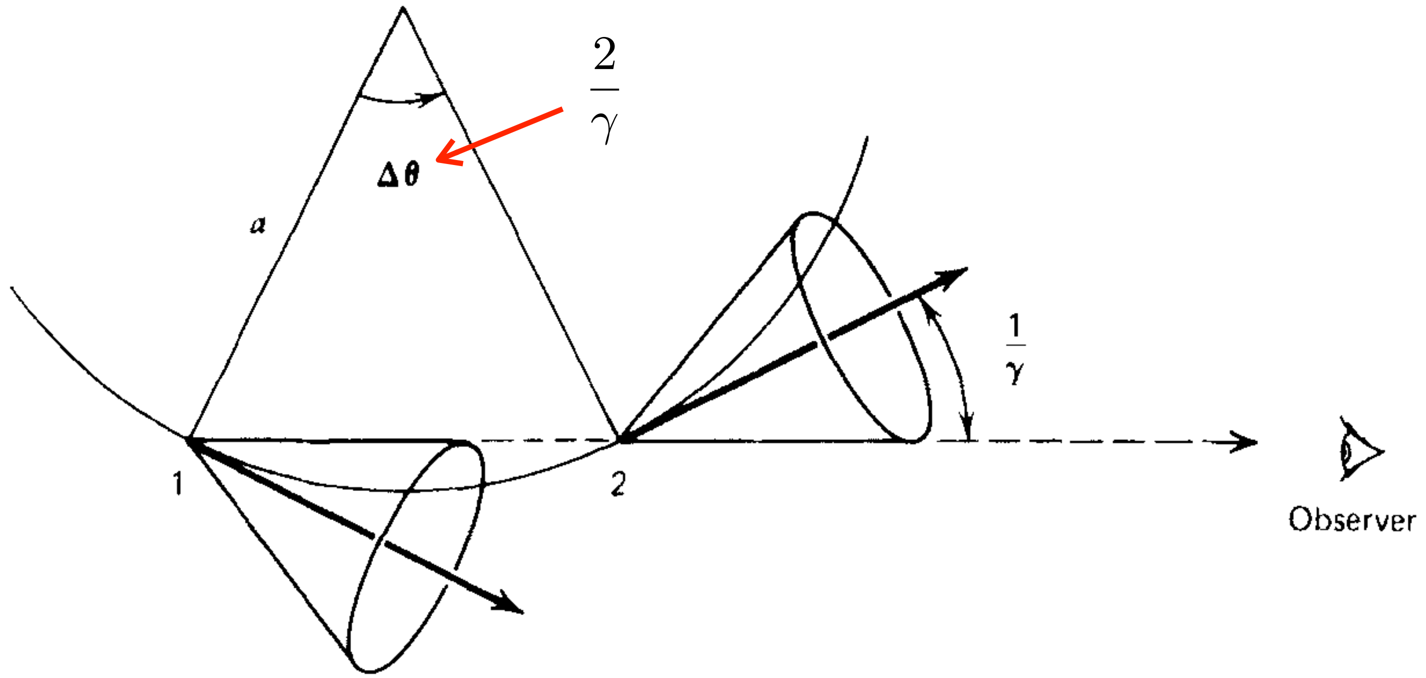

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$$\Delta t_a = \frac{c\Delta t_e - v_\perp \Delta t_e}{c} \approx \Delta t_e\,(1-\beta) = \Delta t_e \frac{1-\beta^2}{1+\beta} \approx \frac{\Delta t_e}{2\gamma^2}$$

Emission from one and many electrons

duration of the received pulse

$$\Delta t_a \approx \frac{\Delta t_e}{2\gamma^2} = \frac{1}{2\pi\gamma^3\nu_B} = \frac{1}{\gamma^2}\frac{mc}{qB}$$

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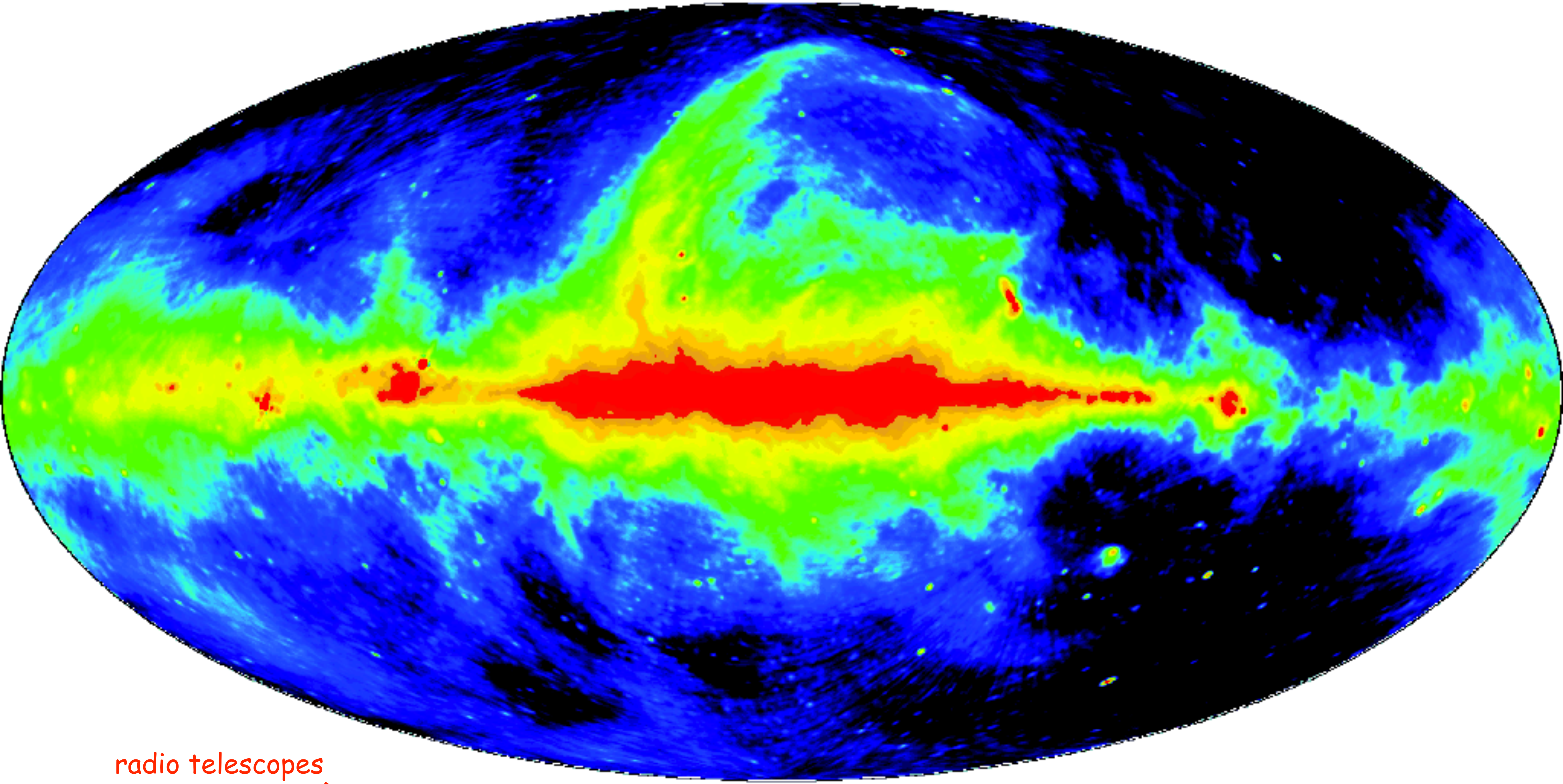
Synchrotron emission from the Milky Way

radio domain → 408 MHz



radio telescopes

Jodrell-Bank 250-ft + Effelsberg 100-m + Parkes 64-m

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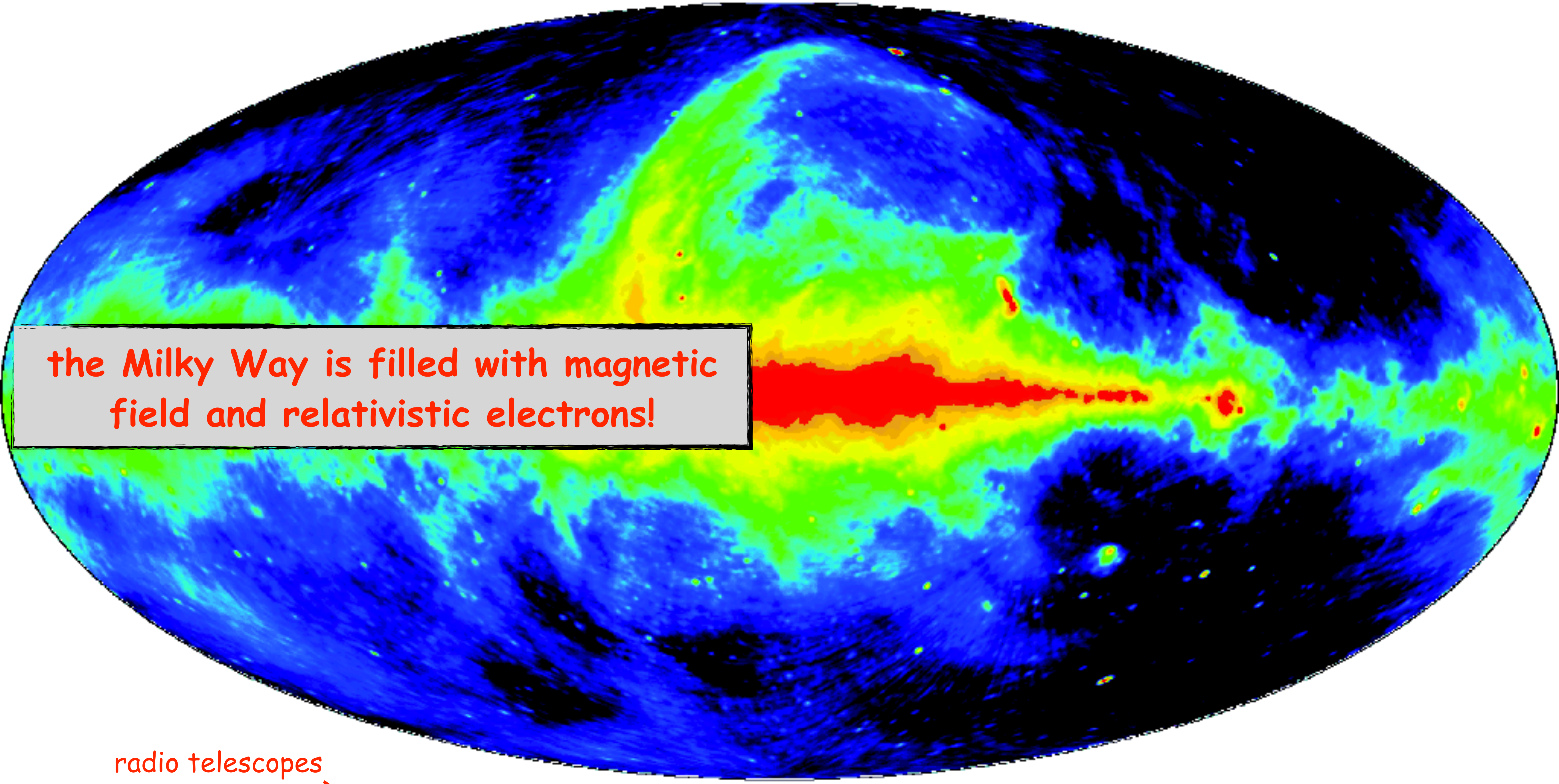

the Milky Way is filled with magnetic field and relativistic electrons!

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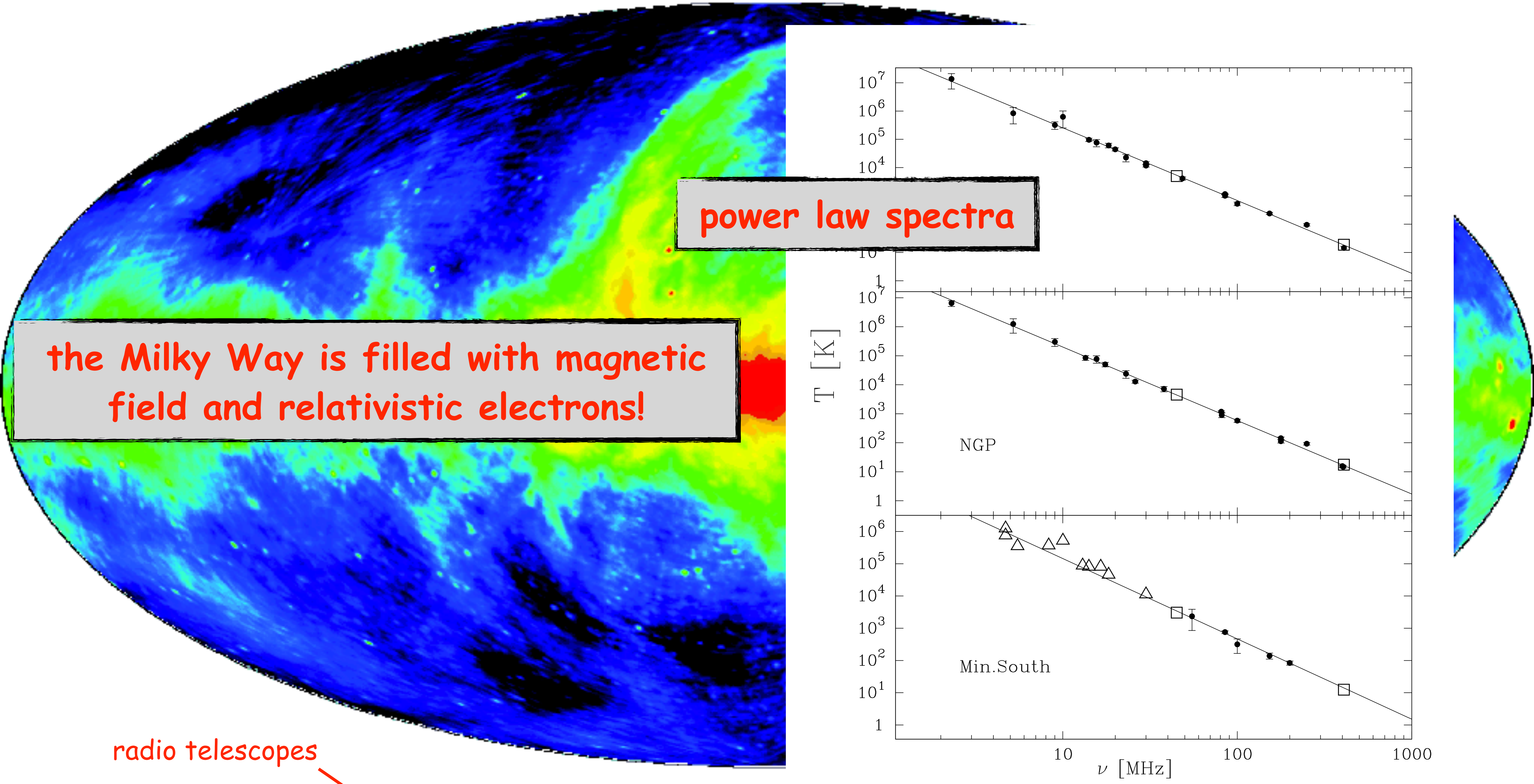

power law spectra

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Synchrotron emission: final considerations

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we can estimate a combination of K and B, but not the two quantities separately!!!

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we can estimate a combination of K and B, but not the two quantities separately!!!

several ways to measure B exist, and they indicate B ~ 3 μG in the Milky Way

$$\nu_s = \gamma^2 \frac{qB}{2\pi mc} \left\{ \begin{array}{l} E_e = 10 \text{ GeV} \longrightarrow \nu_s \sim 3 \text{ GHz} \\ \\ E_e = 100 \text{ TeV} \longrightarrow \nu_s \sim 1 \text{ keV} \end{array} \right.$$

radio

X-rays

Equipartition magnetic field

total energy in a synchrotron emitting source $W_{tot} = W_B + W_{CR}$

$$W_{CR} = W_e + W_p = \left(1 + \frac{W_p}{W_e}\right) W_e = \eta W_e$$

$$\begin{cases} W_B = V \times \dfrac{B^2}{8\pi} \\ W_e = \displaystyle\int_{E_{min}}^{E_{max}} \mathrm{d}E \; E \; K E^{-\delta} = \frac{K}{2-\delta} \left(E_{max}^{2-\delta} - E_{min}^{2-\delta}\right) \end{cases}$$

$$L(\nu) = C(\delta) K B^{\frac{\delta+1}{2}} \nu^{-\frac{\delta-1}{2}}$$

$$\nu_s = A \; E^2 B$$

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Equipartition magnetic field

too much energy!

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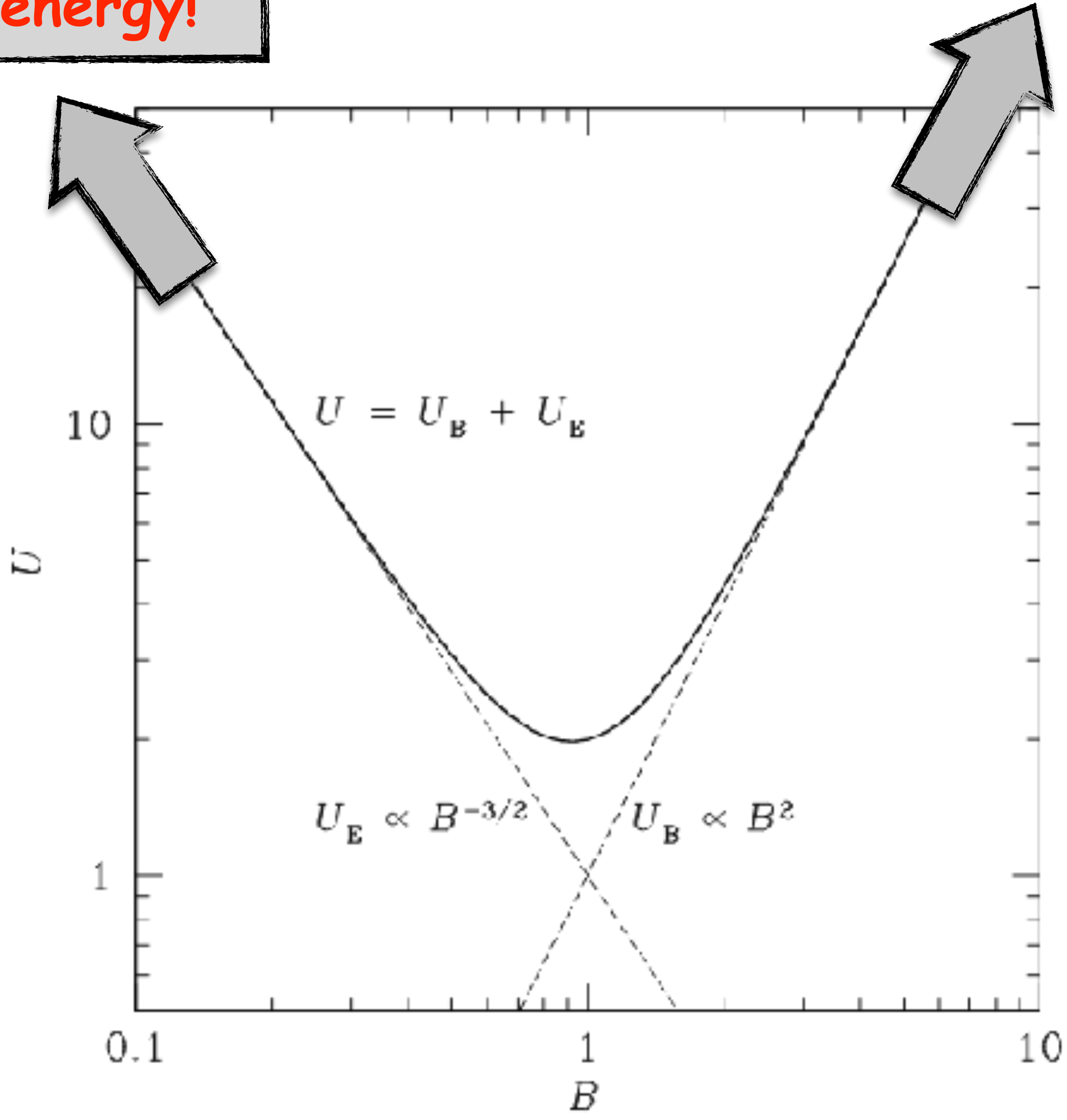

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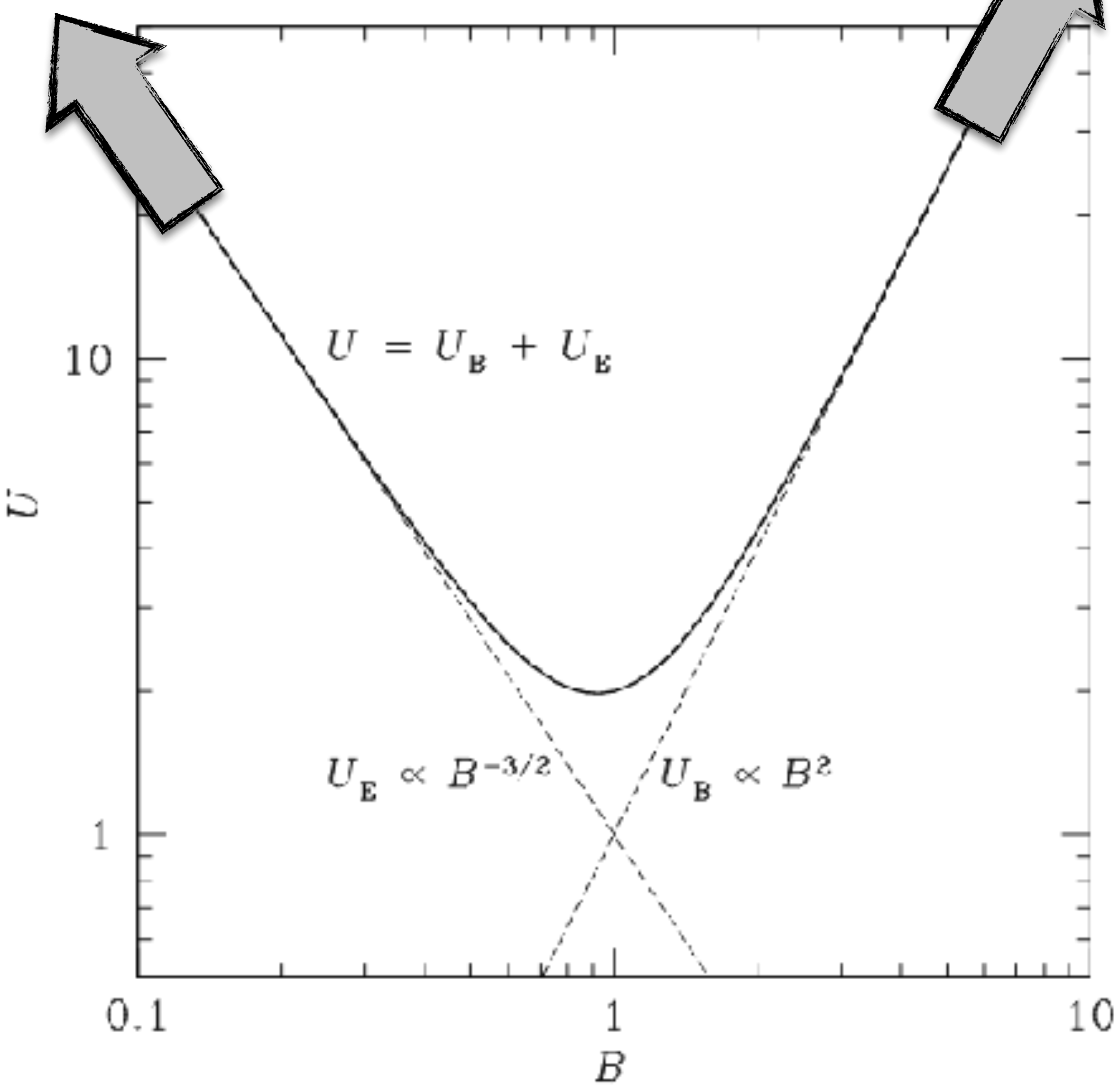

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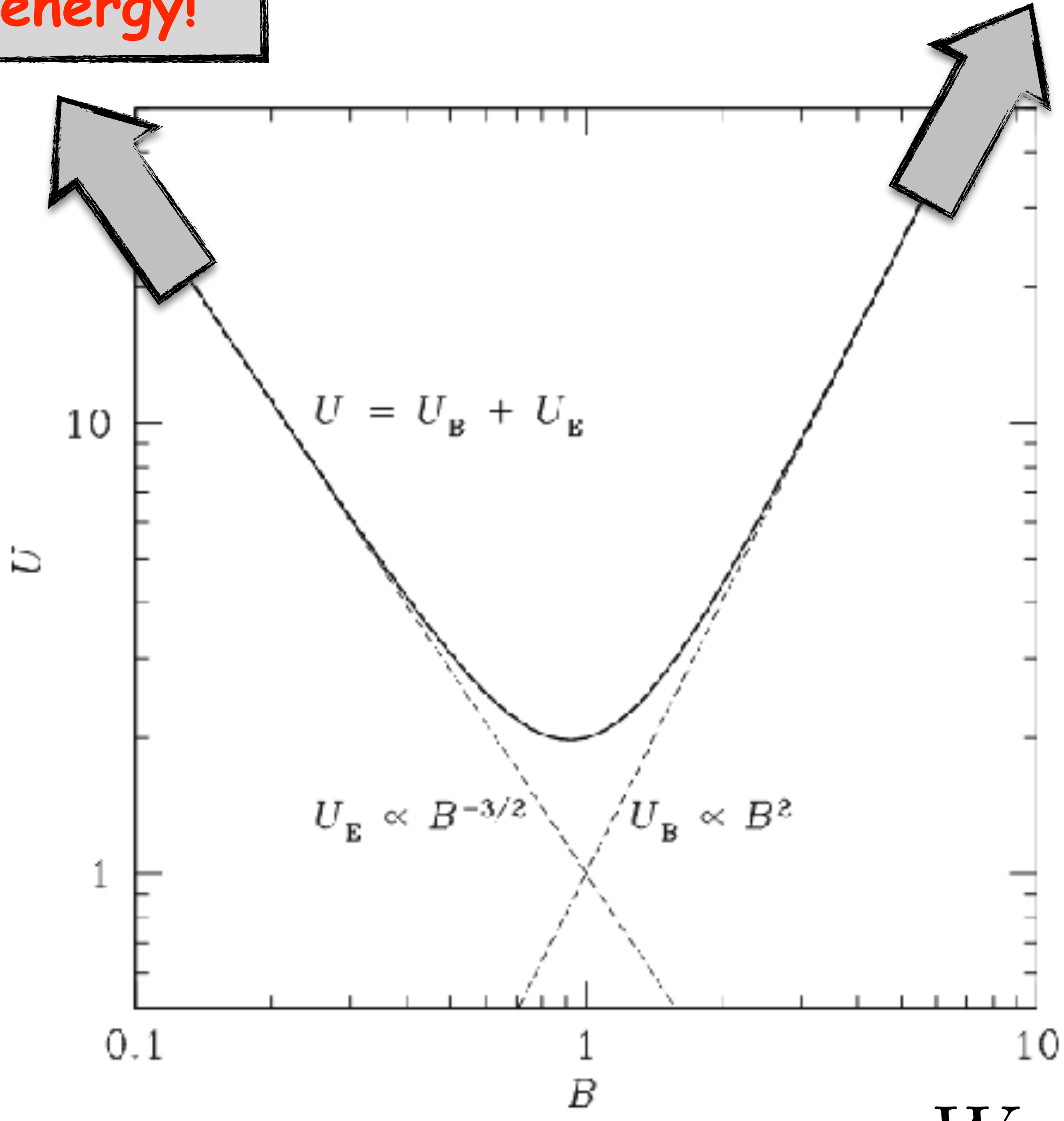

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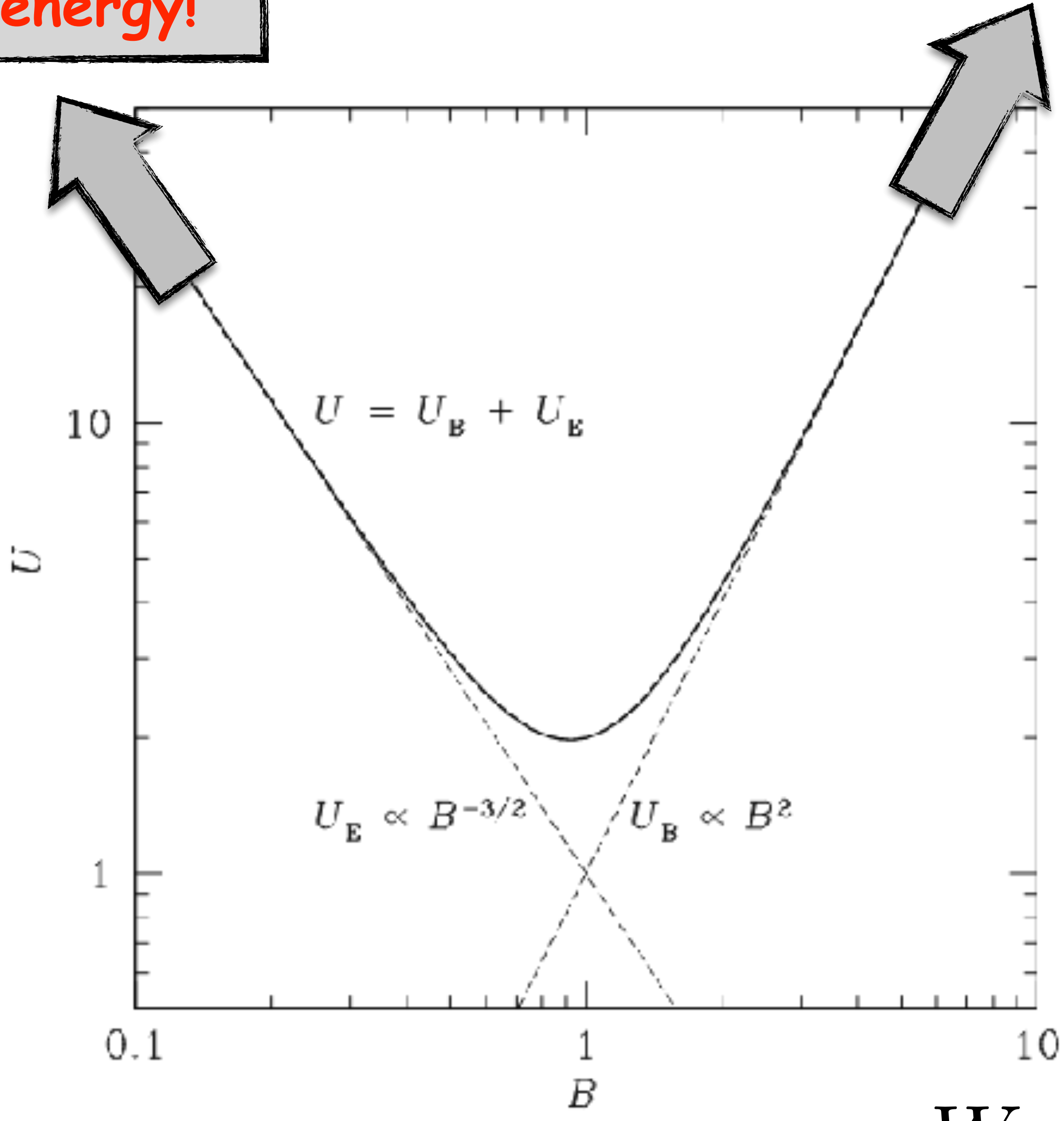

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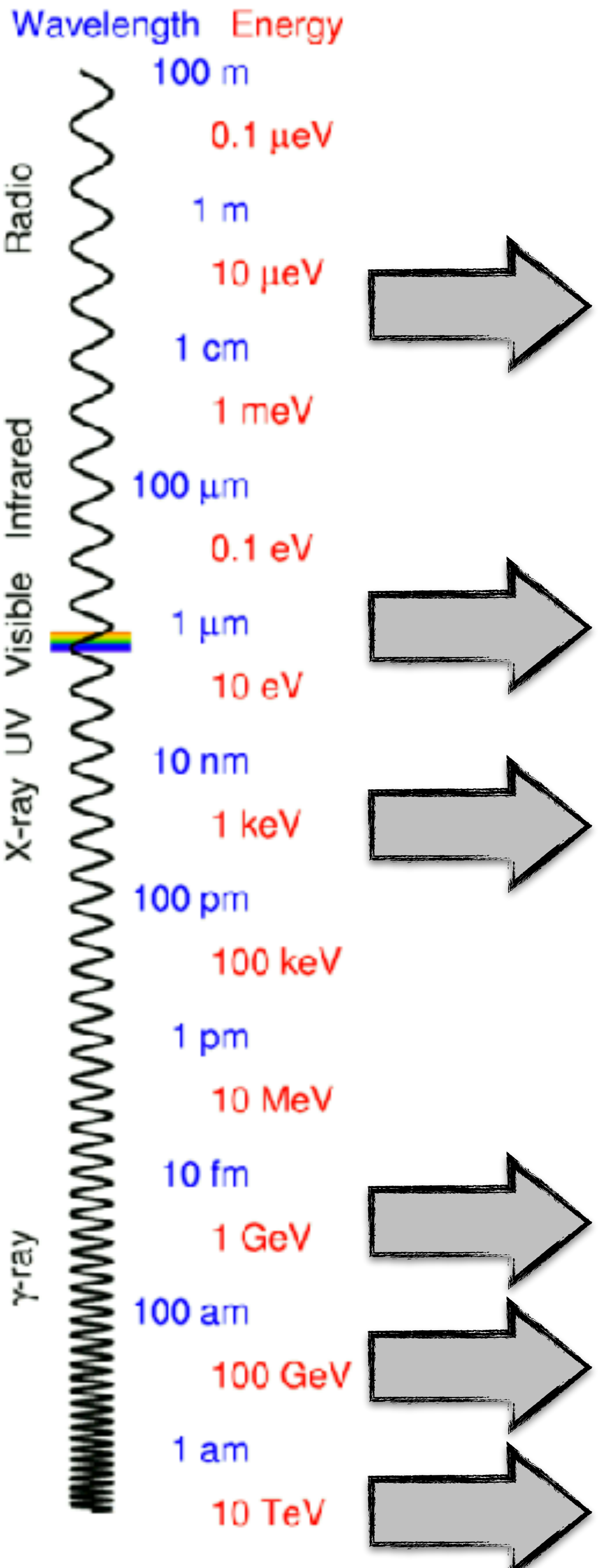
EQUIPARTITION

in the absence of other estimates, the assumption of equipartition is used to estimate a reference value for the magnetic field

The electromagnetic spectrum



21 cm line -> atomic neutral Hydrogen (HI)
synchrotron emission of multi-GeV electrons in a 3 μG field

visible light -> stars

thermal emission from hot plasmas (millions of K)
synchrotron emission of 100 TeV electrons in a 3 μG field

interactions of cosmic ray protons with matter

Definition of plasma

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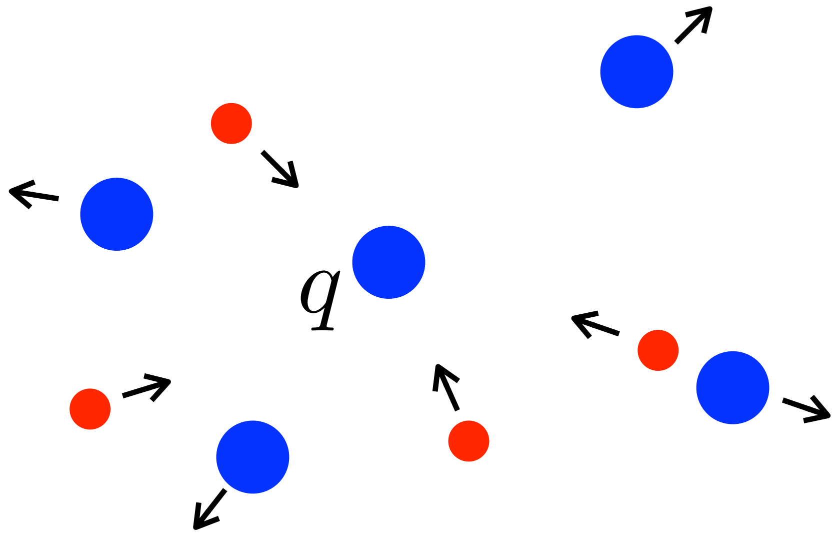

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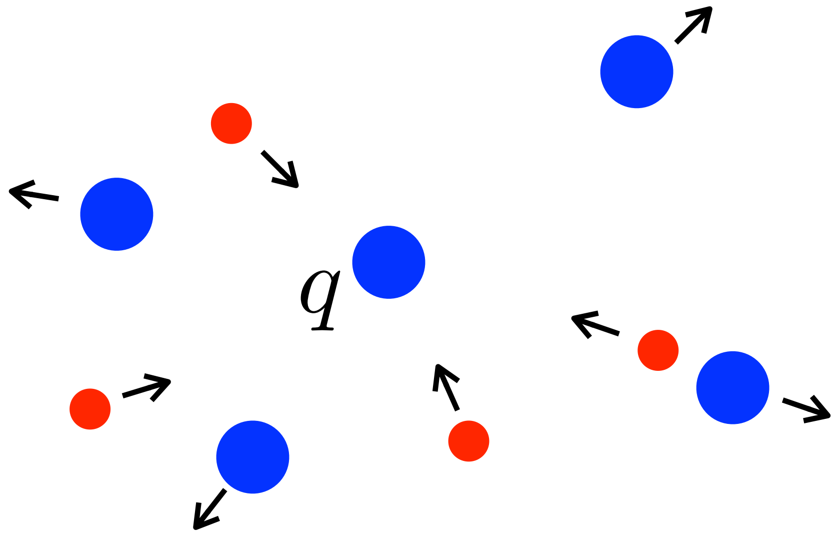

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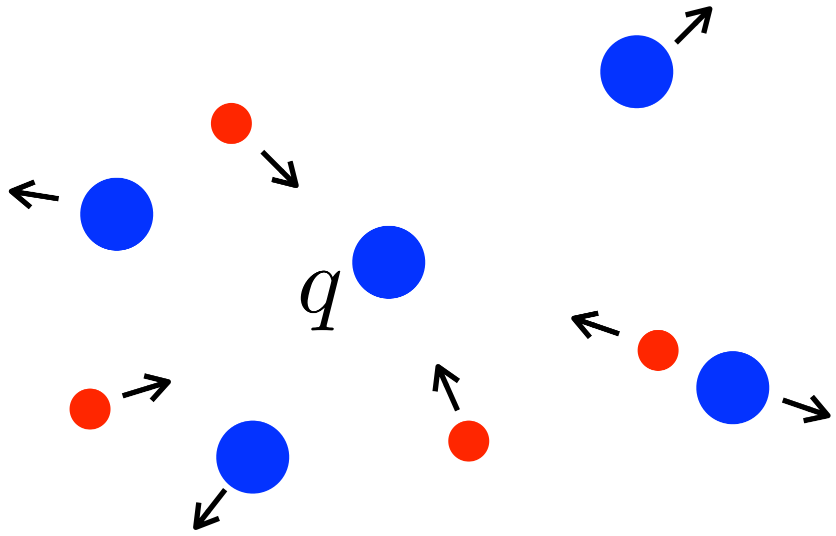

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$$\xrightarrow[q\phi \ll kT]{} q\ \delta(\vec{r}) + \frac{2n_e q^2}{kT}\phi$$

● protons

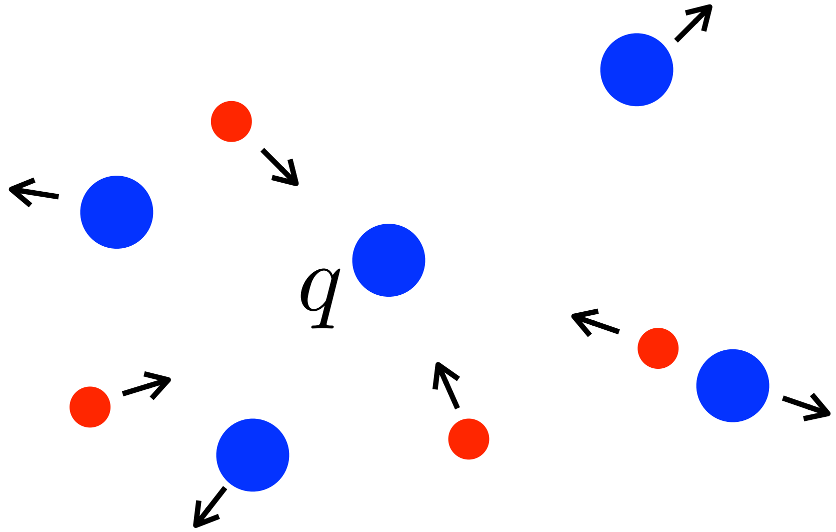
● electrons

$\}$ thermal equilibrium $T_e = T_p = T$ -> Boltzmann distribution

Definition of plasma

Wikipedia: “Plasma is an electrically neutral medium of unbound positive and negative particles (i.e. the overall charge of a plasma is roughly zero)”.

fully ionised plasma global neutrality -> $N_p = N_e$ local neutrality -> $n_p = n_e$



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Debye length

$L_D^{-2}/4\pi$

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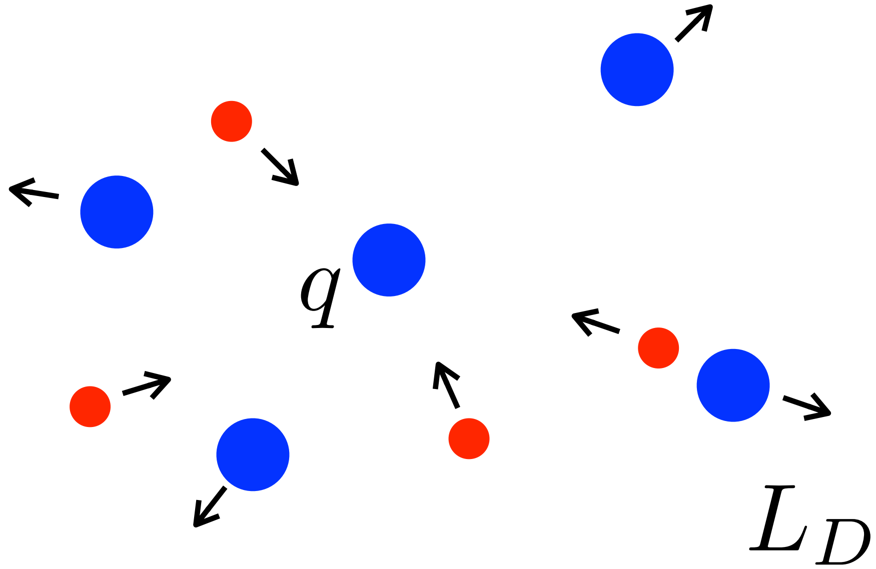

$$\nabla^2\phi = -4\pi q\delta(\vec{r}) + \frac{\phi}{L_D^2} \rightarrow \phi = \frac{q}{r}e^{-\frac{r}{L_D}}$$

Debye shielding

$$L_D \sim 5\times10^2\left(\frac{T}{10^4\ \mathrm{K}}\right)^{1/2}\left(\frac{n}{\mathrm{cm}^{-3}}\right)^{-1/2}\ \mathrm{cm}$$

● protons
● electrons
} thermal equilibrium $T_e = T_p = T$ -> Boltzmann distribution

Magnetohydrodynamics (MHD)

dynamics of electrically conducting fluids in the presence of magnetic fields

plasma motion —> electric fields —> currents —> magnetic fields —> …

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Maxwell equations

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characteristic length scale

characteristic time scale

$$\nabla \vec{B} = 0$$

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displacement current

electric currents —> only source of B-field

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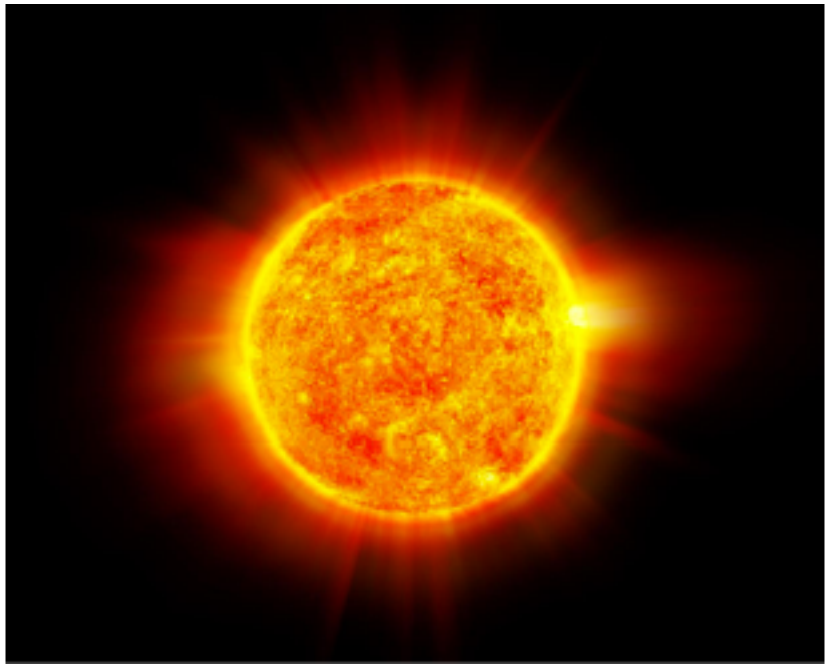
the Sun



- B fields up to 10^3 G!
- generated in a convective region of L ~ 2 x 10^{10} cm
- average electron density n_e ~ 10^{23} cm^{-3}

$$v_{ei} \approx 10^{-12}\mathrm{cm/s}$$

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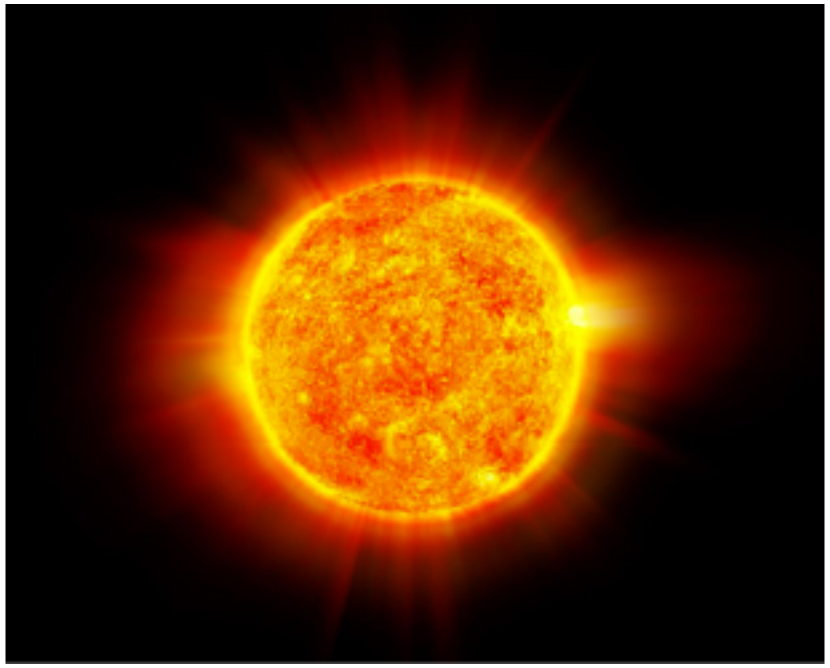

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for any practical purpose we can consider a 1-component plasma
electrons and ions are fully coupled

MHD equation for the magnetic field

Ohm's law: relates the electric current j to the other variables of the problem

$$\vec{j}' = \overset{\substack{\text{electric conductivity}\\ \downarrow}}{\sigma}\vec{E}' = \vec{j}$$

primed quantities -> rest frame where the plasma is at rest

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Lorentz transformation for u/c << 1

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⇒

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{B}\right) - \nabla \times \left(\frac{c^2}{4\pi\sigma} \nabla \times \vec{B}\right)$$

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using the vectorial identity $\nabla \times \left(\nabla \times \vec{B}\right) = \nabla \left(\nabla \vec{B}\right) - \nabla^2 \vec{B}$ and $\nabla \vec{B} = 0$

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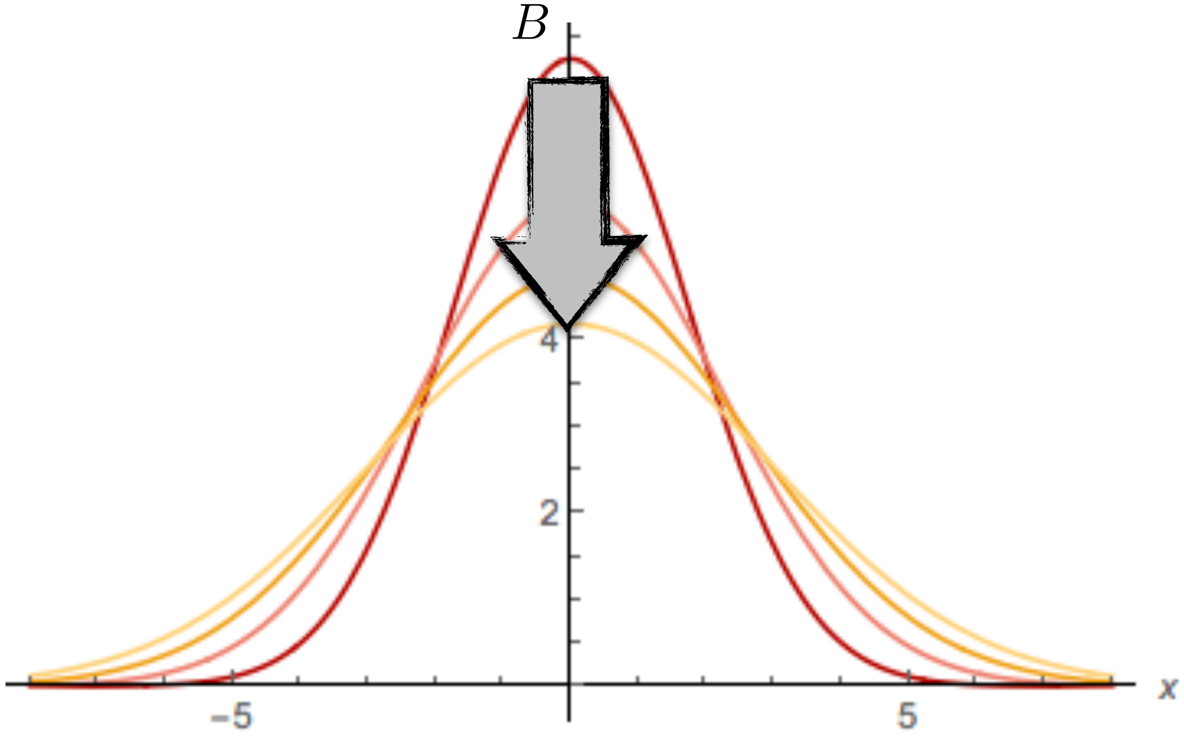

magnetic diffusion

$$T_d \approx \frac{L^2}{\eta} \qquad \eta = \frac{c^2}{4\pi\sigma}$$

magnetic conductivity

Mass, momentum, and energy

mass

$$\frac{\partial \varrho}{\partial t} + \nabla \left(\varrho \vec{u} \right) = 0$$

Mass, momentum, and energy

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momentum

Lorentz force ↙

$$\varrho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla P + \frac{1}{c} \vec{j} \times \vec{B}$$

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energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(P \varrho^{-\gamma} \right) = 0$$

adiabatic condition

MHD equations

$$\frac{\partial \varrho}{\partial t} + \nabla\left(\varrho \vec{u}\right) = 0$$

$$\varrho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = \nabla P + \frac{1}{4\pi} \left(\nabla \times \vec{B} \right) \times \vec{B}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(P \varrho^{-\gamma} \right) = 0$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{B} \right) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}$$

8 equations for 8 variables: $\varrho \quad P \quad \vec{u} \quad \vec{B}$

we got rid of $\vec{E}$ and $\vec{j}$!

Ideal MHD equations

$$\frac{\partial \varrho}{\partial t} + \nabla \left(\varrho \vec{u} \right) = 0$$

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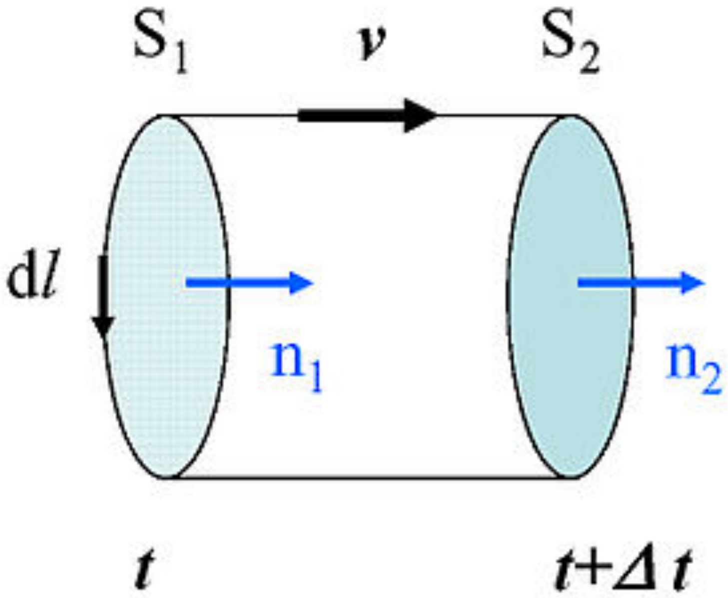
under most astrophysical conditions $T_d \approx \dfrac{L^2}{\eta} \longrightarrow \infty$

Magnetic flux freezing



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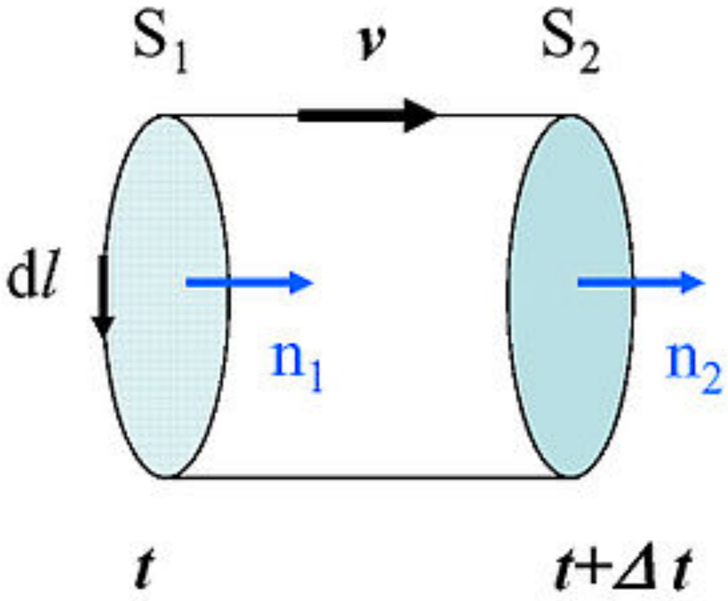

$$\Phi_1 = \int_{S_1} \vec{B}(\vec{x},t) \cdot \mathrm{d}\vec{S}_1$$

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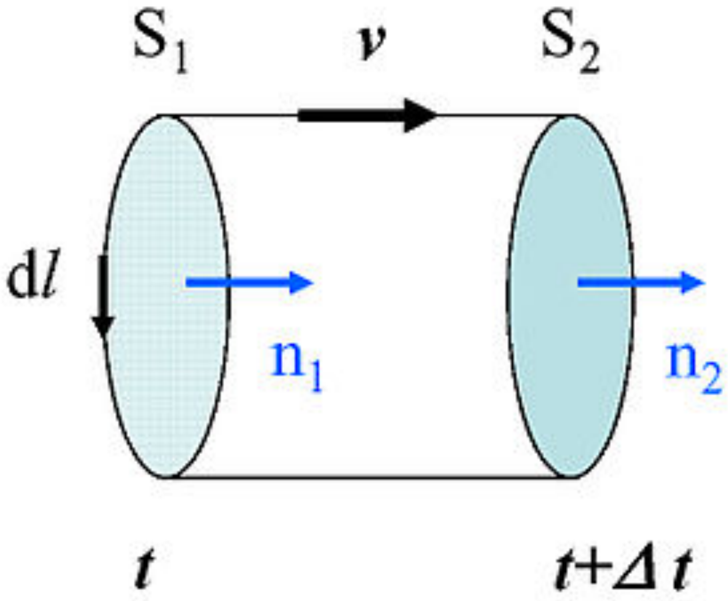

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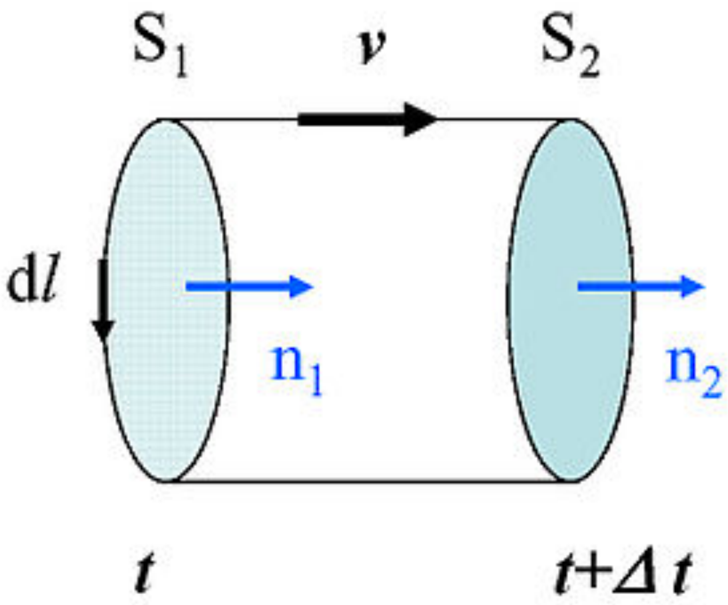

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$$\approx \int_{S_2} \vec{B}(\vec{x} + \vec{v}\Delta t, t) \cdot \mathrm{d}\vec{S}_2 + \Delta t \int_{S_2} \frac{\partial}{\partial t} \vec{B}(\vec{x} + \vec{v}\Delta t, t) \cdot \mathrm{d}\vec{S}_2$$

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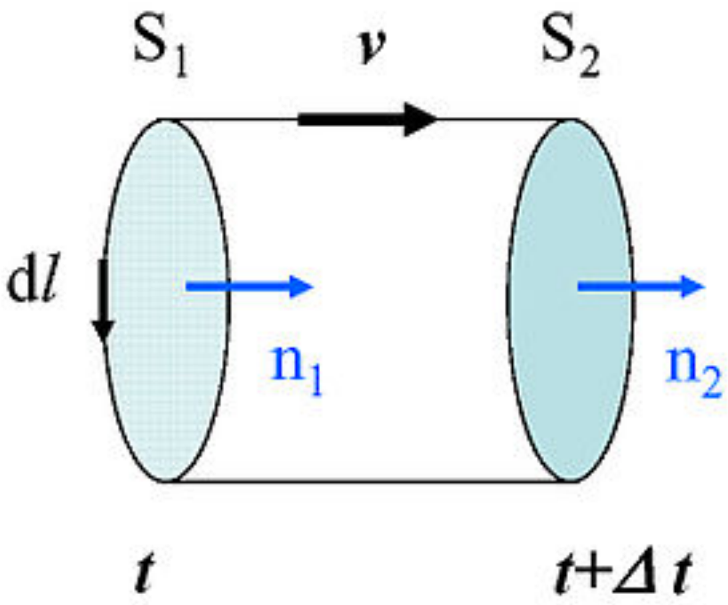

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S₂ differs from S₁ by terms of order Δt

↓

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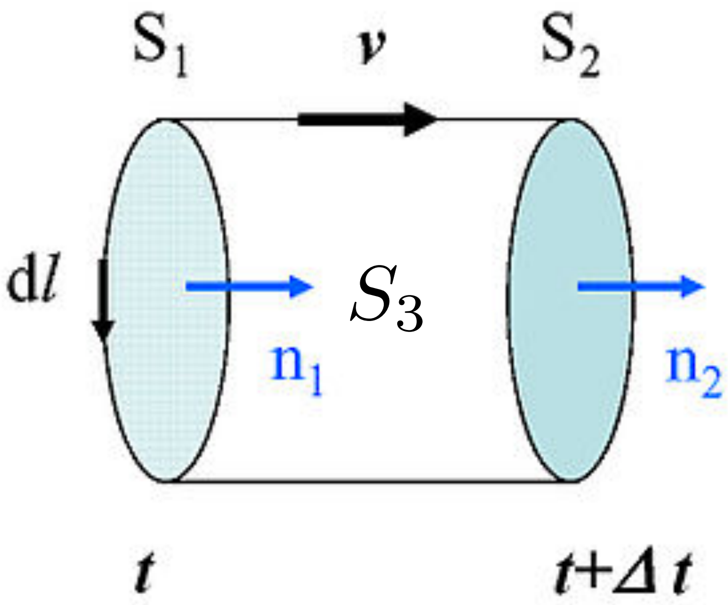

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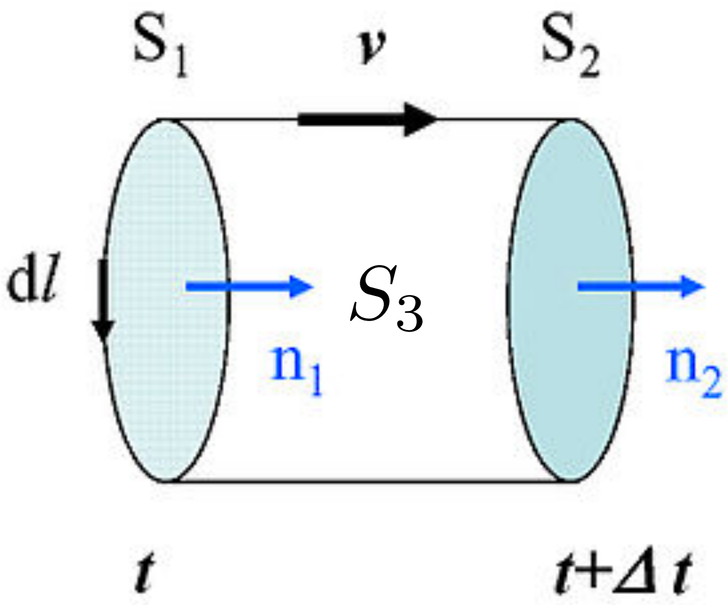

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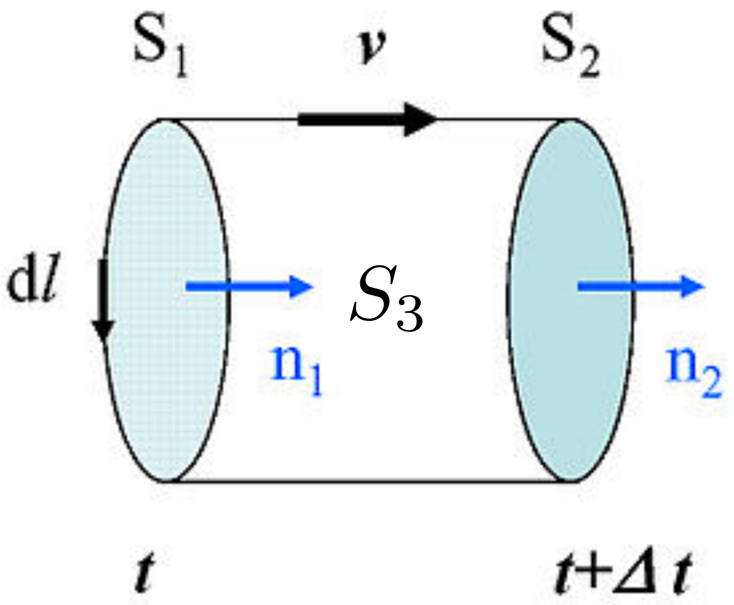
Gauss theorem

$$\oint \vec{B}\cdot \mathrm{d}\vec{S}_{tot} = \int \nabla\cdot\vec{B}\ \mathrm{d}V = 0$$

Magnetic flux freezing



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Gauss theorem

$$-\int_{S_1} \vec{B}\cdot \mathrm{d}\vec{S}_1 + \int_{S_2} \vec{B}\cdot \mathrm{d}\vec{S}_2 + \int_{S_3} \vec{B}\cdot \mathrm{d}\vec{S}_3 = \oint \vec{B}\cdot \mathrm{d}\vec{S}_{tot} = \int \nabla\cdot\vec{B}\,\mathrm{d}V = 0$$

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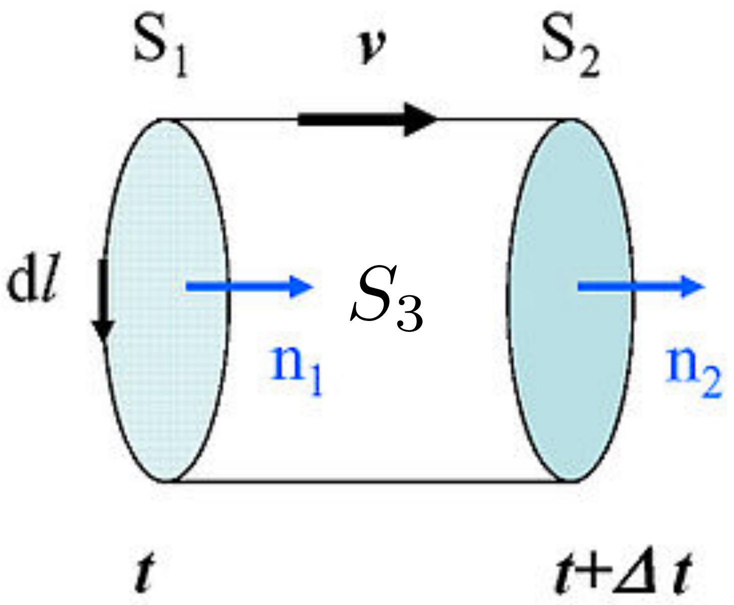


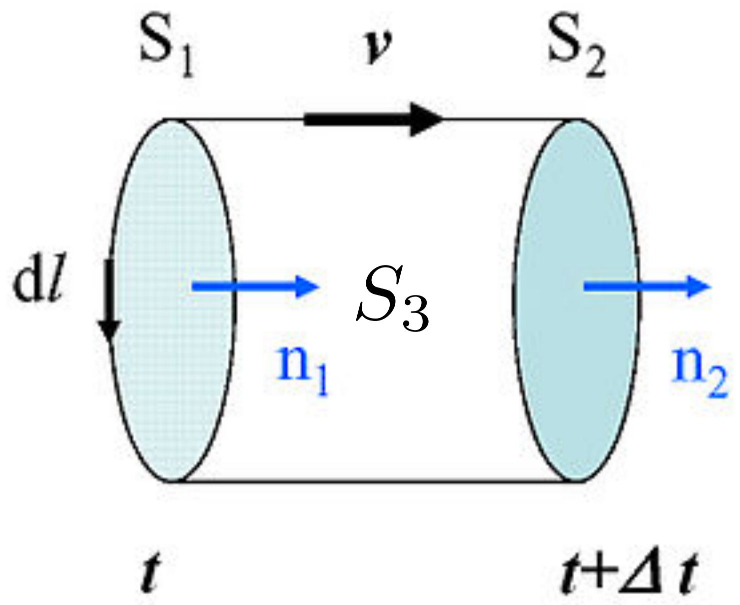
$$\Phi_1 = \boxed{\int_{S_1} \vec{B}(\vec{x},t) \cdot \mathrm{d}\vec{S}_1}$$

$$\Phi_2 \approx \boxed{\int_{S_2} \vec{B}(\vec{x}+\vec{v}\Delta t,t) \cdot \mathrm{d}\vec{S}_2} + \Delta t \int_{S_1} \frac{\partial}{\partial t}\vec{B}(\vec{x}+\vec{v}\Delta t,t) \cdot \mathrm{d}\vec{S}_1$$

Gauss theorem

$$-\boxed{\int_{S_1} \vec{B} \cdot \mathrm{d}\vec{S}_1} + \boxed{\int_{S_2} \vec{B} \cdot \mathrm{d}\vec{S}_2} + \int_{S_3} \vec{B} \cdot \mathrm{d}\vec{S}_3 = \oint \vec{B} \cdot \mathrm{d}\vec{S}_{tot} = \int \nabla \cdot \vec{B}\, \mathrm{d}V = 0$$

Magnetic flux freezing



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curl theorem

$$= -\Delta t \left[\int_{S_1} \left(\frac{\partial \vec{B}}{\partial t} - \nabla \times \left(\vec{v} \times \vec{B}\right) \right) \cdot \mathrm{d}\vec{S}_1 \right]$$

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ideal MHD

Magnetic flux freezing

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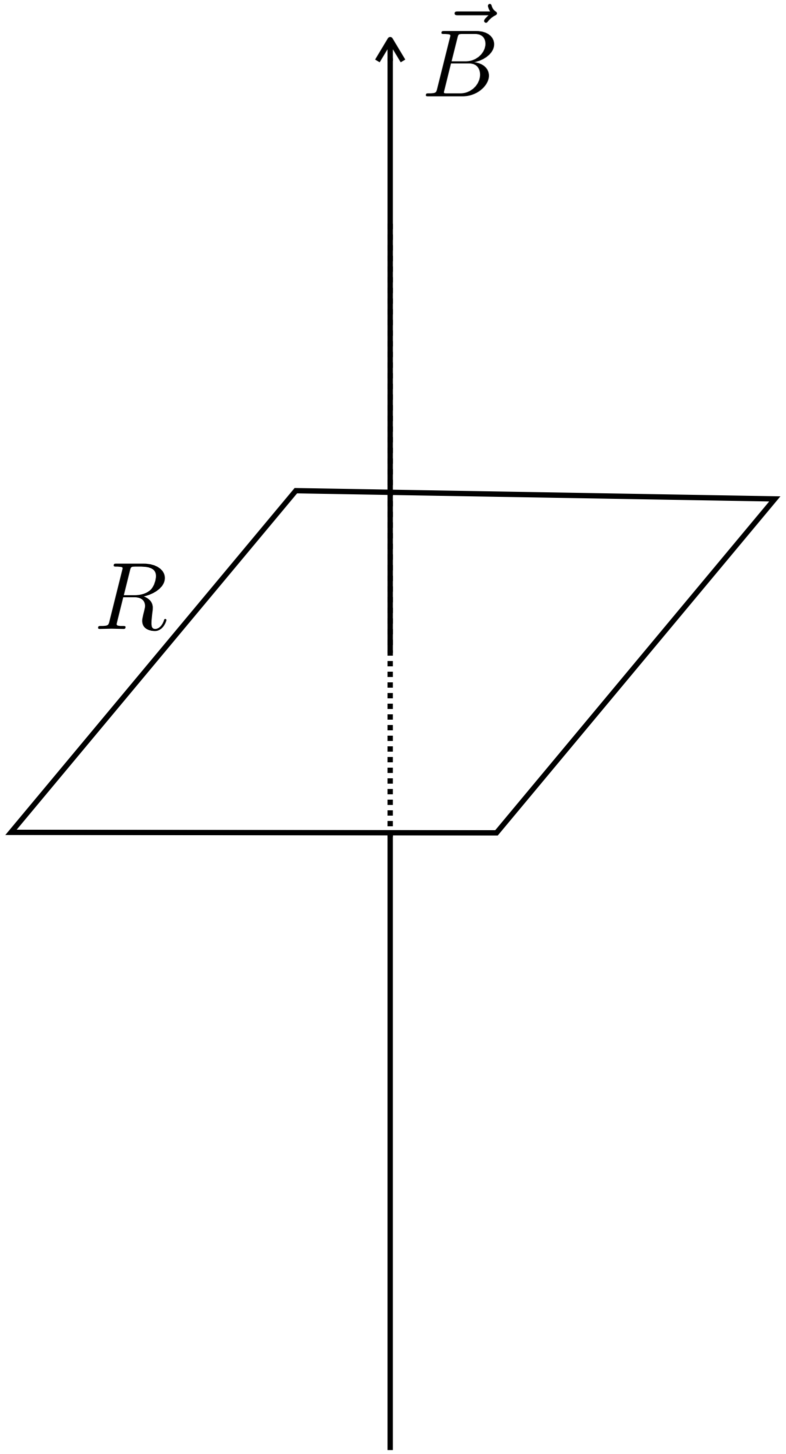
ideal MHD

the magnetic flux is constant across a surface moving with the plasma!

Magnetic flux freezing —> B-field line move with the plasma

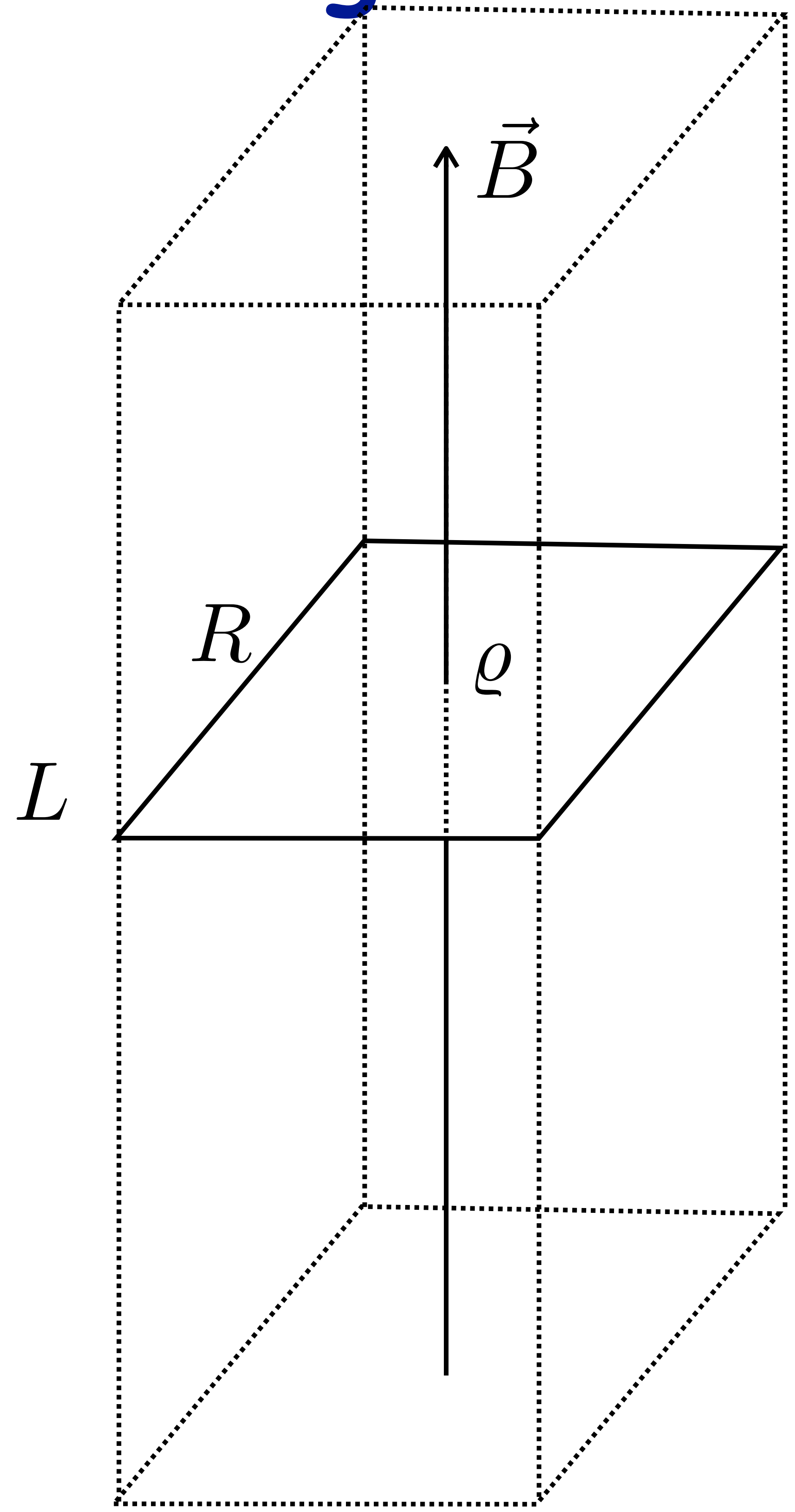
Magnetic flux freezing: examples



magnetic flux $\Phi = BR^2$

gas density ϱ

Magnetic flux freezing: examples



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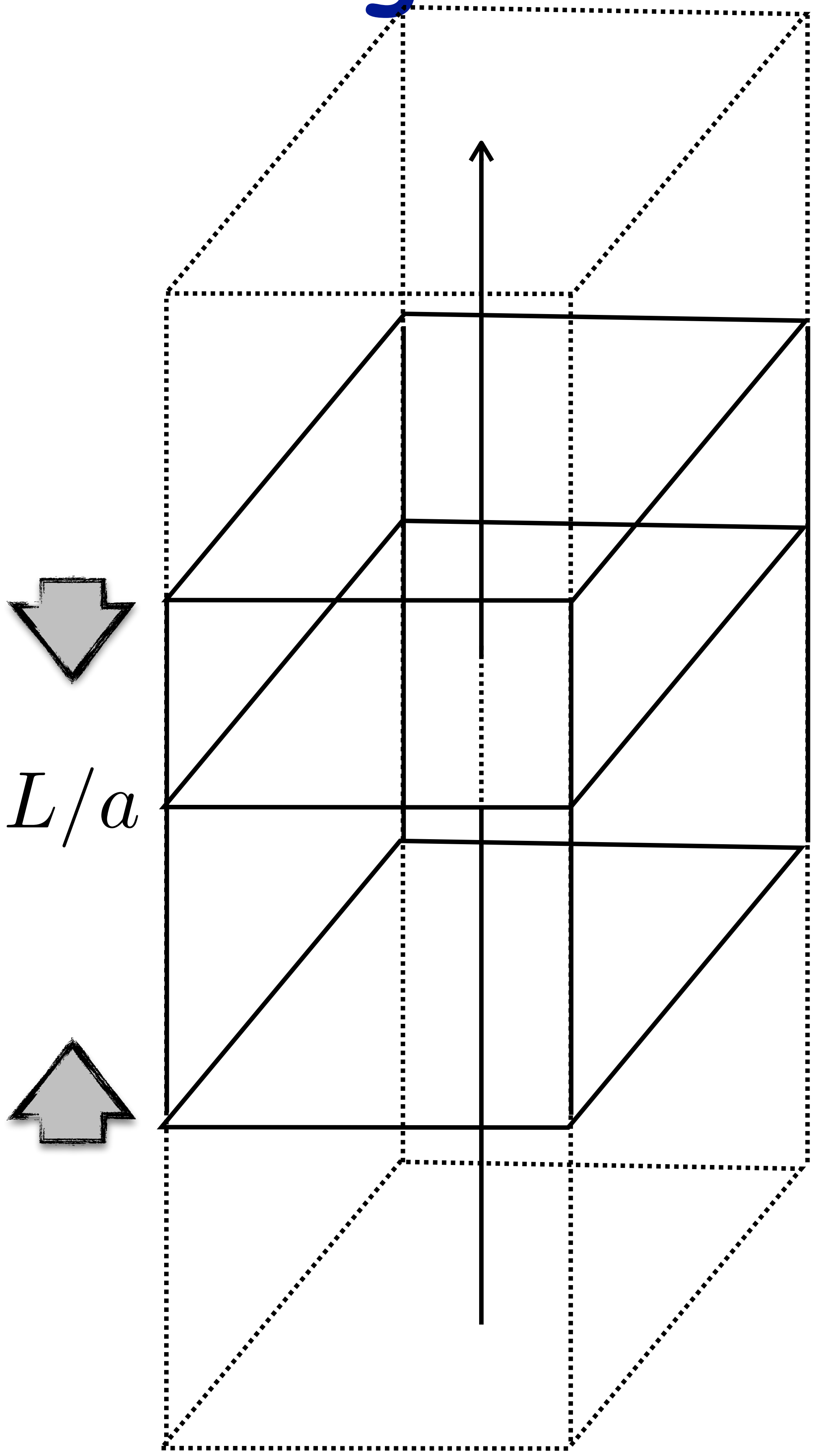
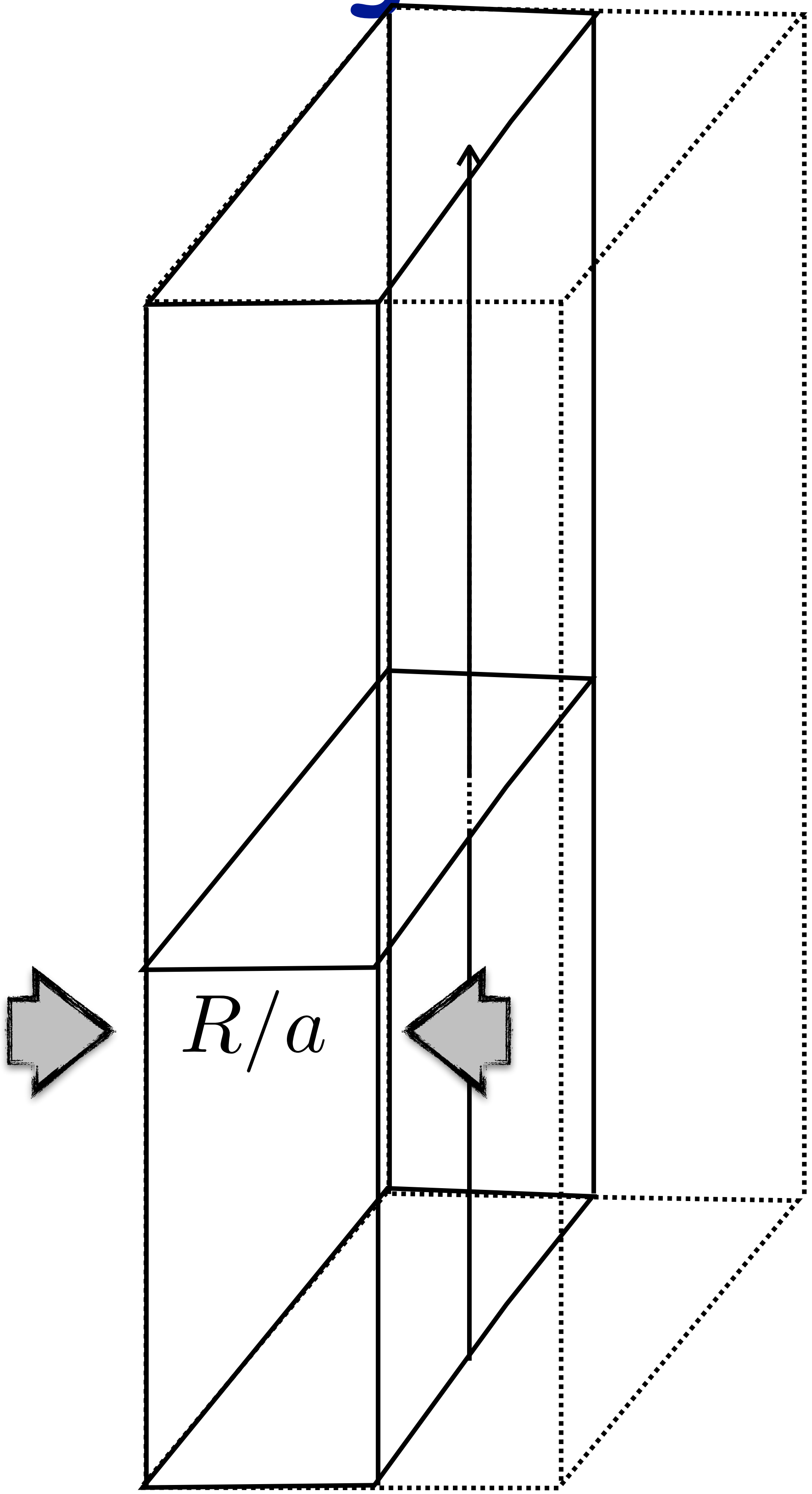


magnetic flux $\Phi = BR^2 \longrightarrow BR^2$

gas density $\varrho \longrightarrow a\varrho$

B-field $B \longrightarrow B$

Magnetic flux freezing: examples



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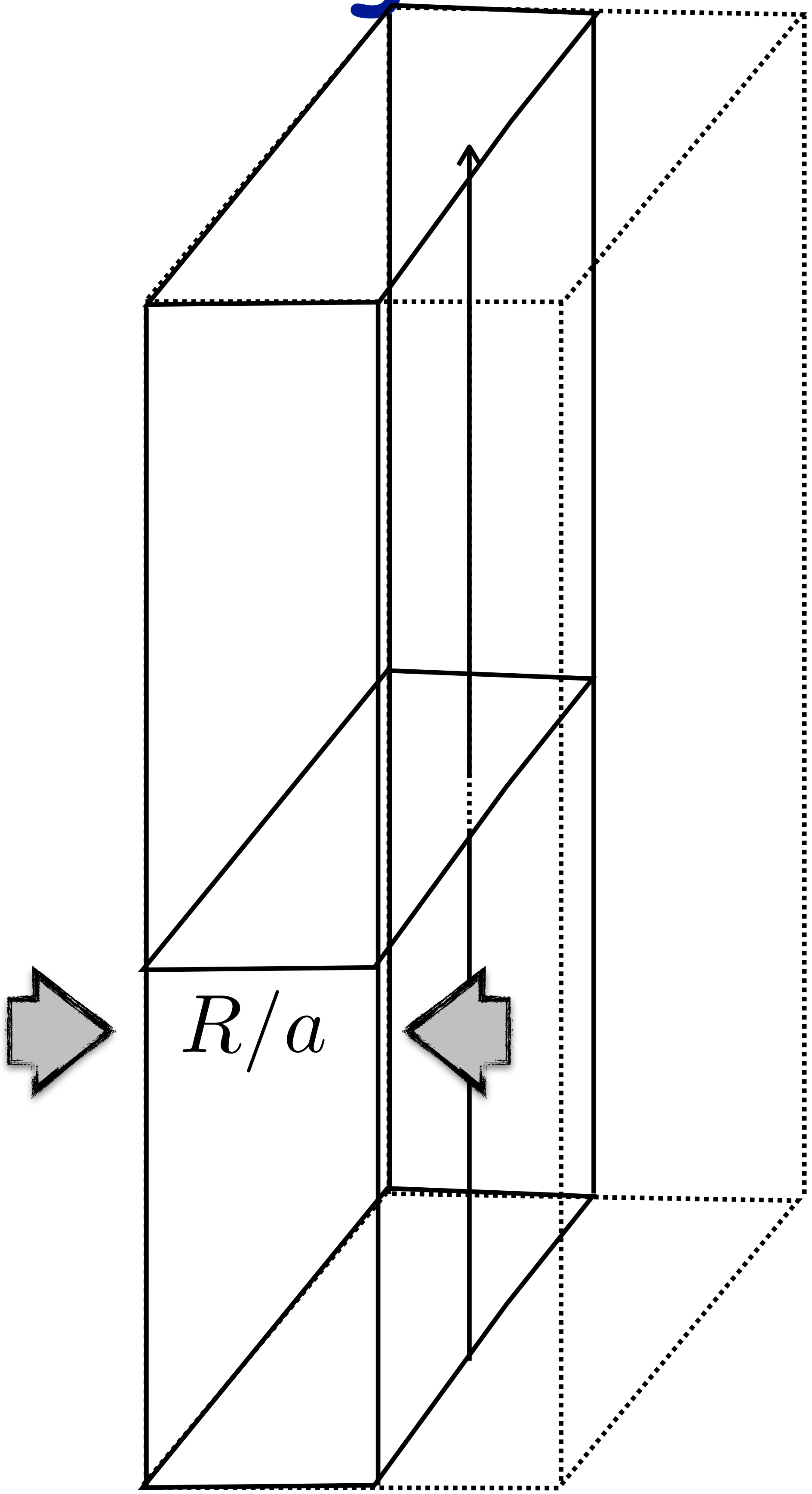

magnetic flux $\Phi = BR^2 \longrightarrow BR^2 a^{-1}$

gas density $\varrho \longrightarrow a\varrho$

B-field $B \longrightarrow aB$

in general: $\dfrac{B}{\varrho^n} = const$

- n = 0 —> parallel contraction/expansion
- n = 1 —> perpendicular contraction/expansion
- n = 2/3 —> isotropic contraction or expansion

Magnetic pressure and tension

Lorentz force

$$F_L = \frac{1}{c}\,\vec{j} \times \vec{B}$$

Magnetic pressure and tension

Lorentz force

$$F_L = \frac{1}{c}\,\vec{j} \times \vec{B} \;= \frac{1}{4\pi}\left(\nabla \times \vec{B}\right) \times \vec{B}$$

Ampere law

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}$$

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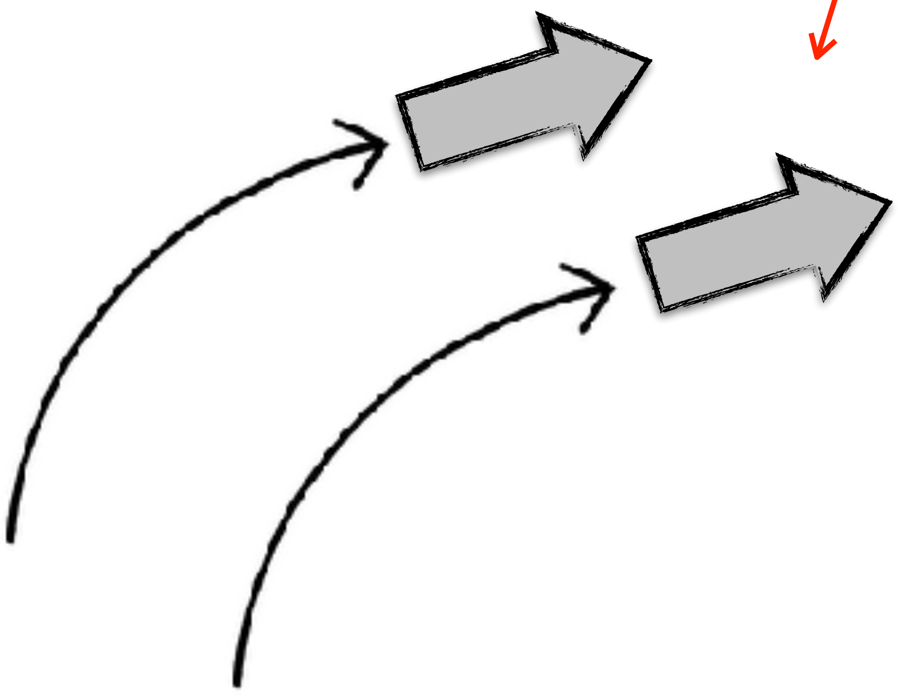
Ampere law

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}$$

$$F_L = \frac{1}{4\pi}\underset{\downarrow}{\overset{\text{magnetic tension}\ \searrow}{\left(\vec{B}\cdot\nabla\right)\vec{B}}} - \nabla\left(\frac{\vec{B}^2}{8\pi}\right)$$



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magnetic tension

gradient of magnetic pressure

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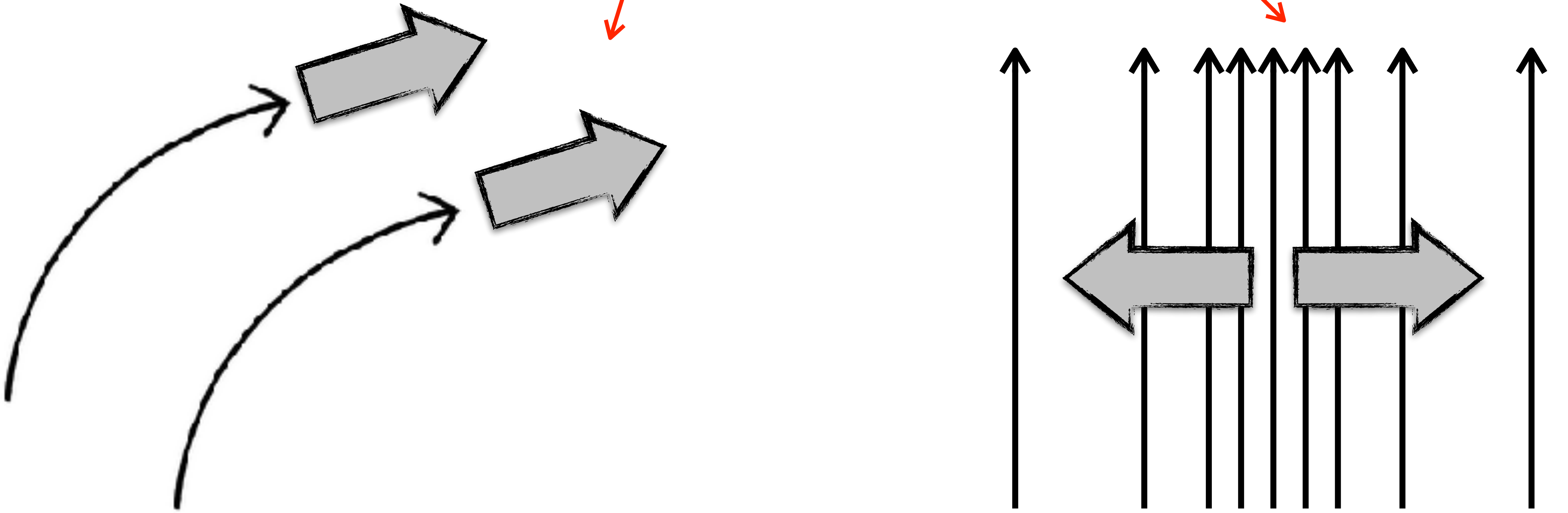
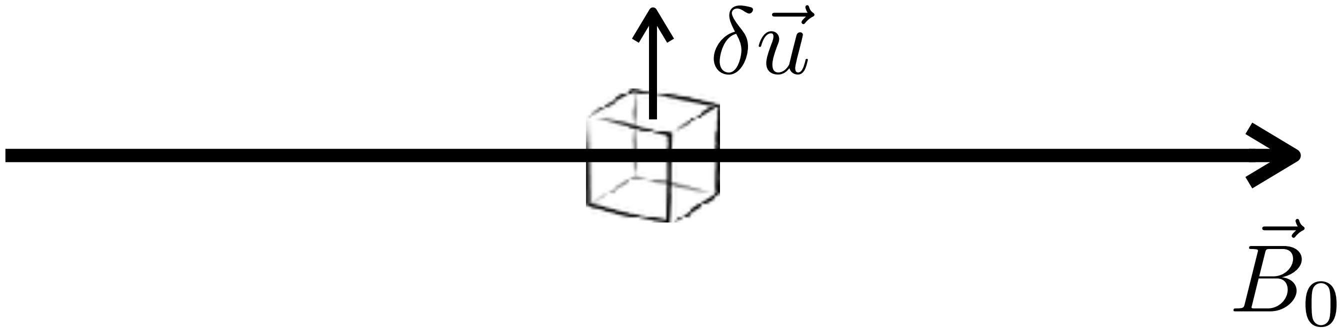


Alfven waves

$\vec{u} = 0$



- displace a fluid element orthogonally to B_0

kinetic energy

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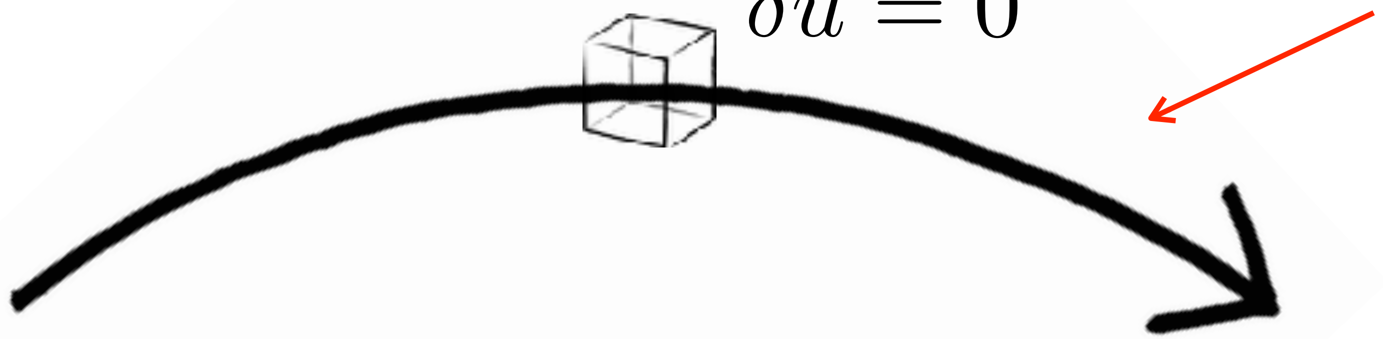


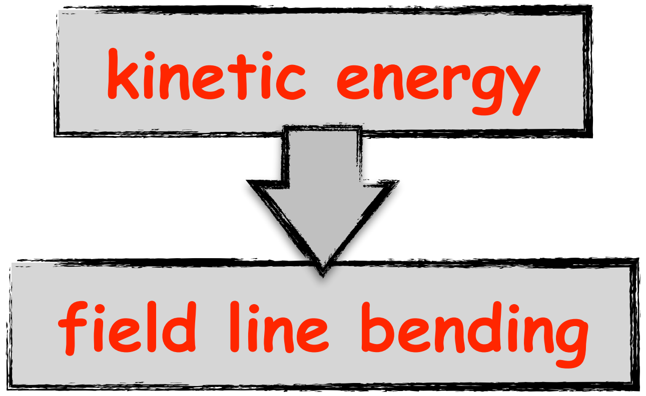
$\delta\vec{u} = 0$

$\vec{B}_0 + \delta\vec{B}$

$\delta\vec{B} \perp \vec{B}_0$

- displace a fluid element orthogonally to B0
- the field is bent by plasma motion (freezing)

  -> work is needed to bend B

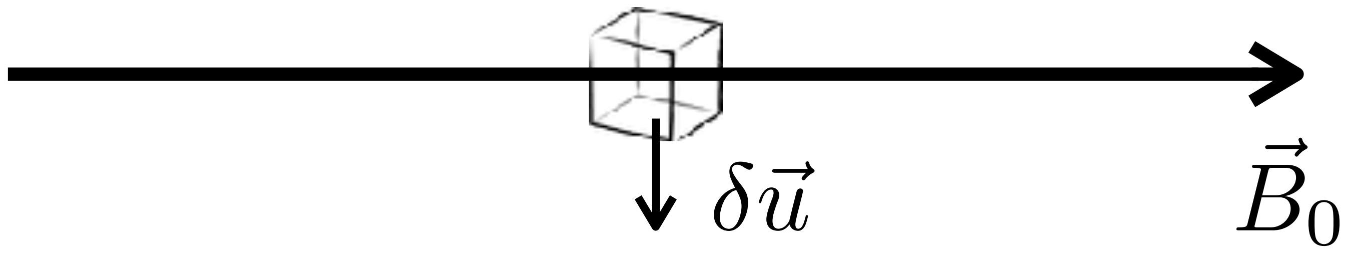


kinetic energy

field line bending

Alfven waves

$\vec{u} = 0$



- displace a fluid element orthogonally to B_0
- the field is bent by plasma motion (freezing)

  -> work is needed to bend B
- magnetic tension will push the plasma back

  into motion

kinetic energy

field line bending

kinetic energy

WAVE MOTION

Alfven speed

Alfven waves propagate along magnetic field lines like waves on a string

$$c_W^2 = \frac{tension}{linear\ density} \longrightarrow v_A^2 = \frac{B^2/4\pi L}{\varrho_i/L}$$

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in the ISM:

$$v_A = \frac{B}{\sqrt{4\pi\varrho_i}} = 20\left(\frac{B}{3\ \mu\mathrm{G}}\right)\left(\frac{n_i}{0.1\ \mathrm{cm}^{-3}}\right)^{-1/2}\mathrm{km/s}$$

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$v_A \ll v_{SNR}$ SNR shocks are super-Alfvenic

$v_A \ll c$ cosmic rays “see” Alfven waves at rest

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component of wave vector k along B_0

dispersion relation $$\omega = k_{\|} v_A$$