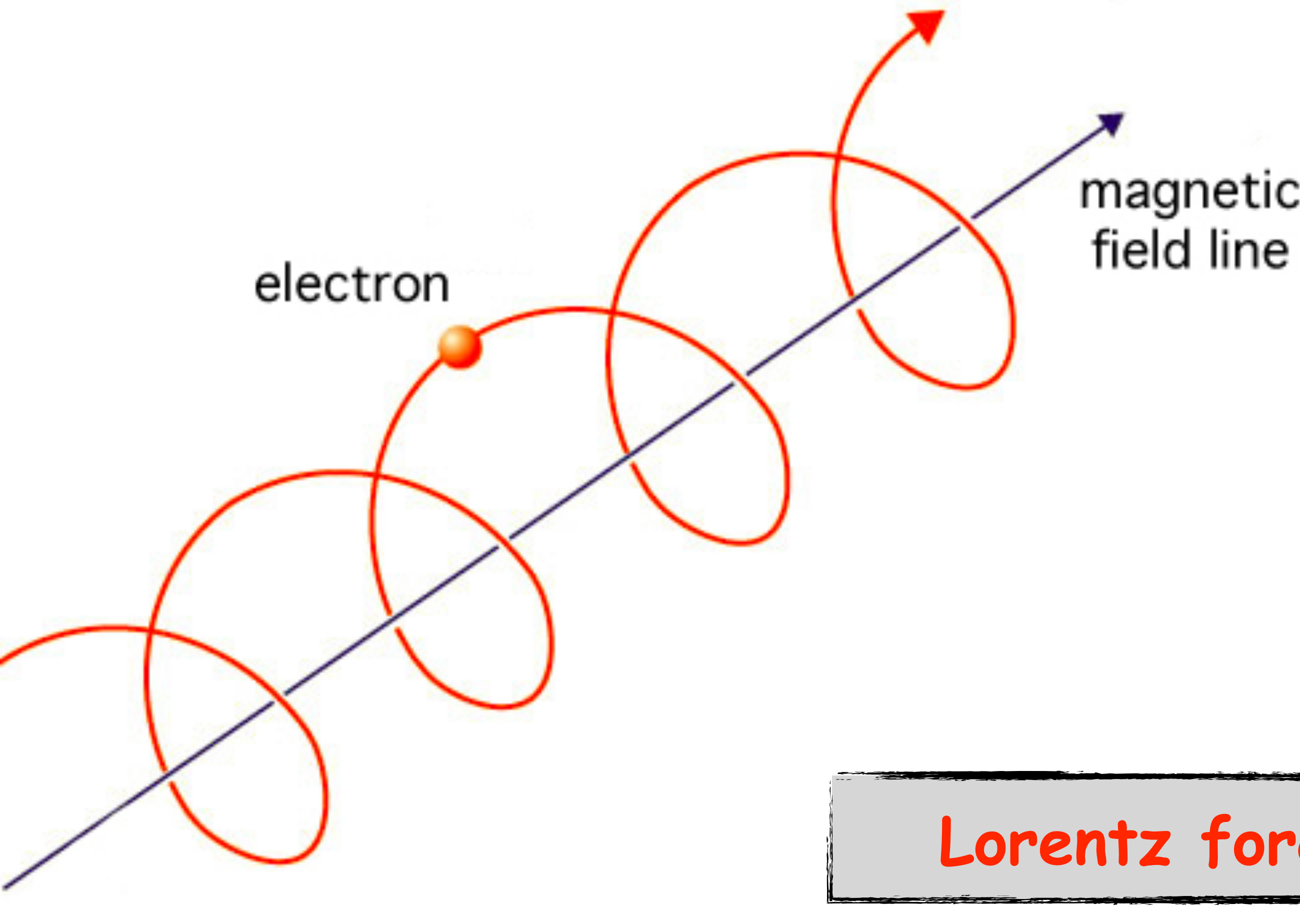
NPAC course on Astroparticles

IV - PLASMA PHYSICS: MagnetoHydroDynamics (MHD)

Outline

- Observational evidences for the presence of magnetic fields: synchrotron radiation
- Plasma physics: basics of MagnetoHydroDynamics (MHD)
- MHD waves: Alfven waves

Motion of a particle in a magnetic field



$$\mu = \cos\vartheta$$

pitch angle = angle between v and B

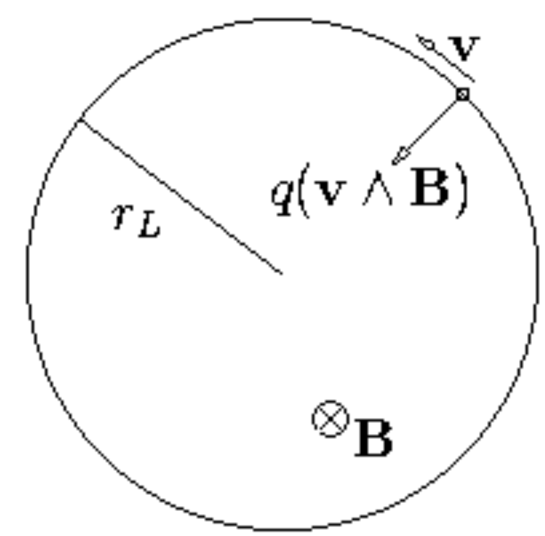
$$\begin{cases} v_{\parallel} = \mu\, v \\ v_{\perp} = (1-\mu^2)^{1/2} v \end{cases}$$

Lorentz force $$F_L = \frac{q}{c}\vec{v}\times\vec{B}$$

Larmor radius $$R_L = \frac{p_{\perp}c}{qB}$$

gyration frequency $$\nu_B = \frac{1}{t_g} = \frac{v_{\perp}}{2\pi R_L} = \frac{qB}{2\pi\gamma mc}$$

Lorentz factor



Power emitted by an electron*

non-relativistic $$P = \frac{2e^2}{3c^3} a^2 \longrightarrow P = \frac{2e^2}{3c^3}\gamma^4 \left[\cancel{\gamma^2 a_\parallel^2} + a_\perp^2\right]$$ **relativistic**

Lorentz force is orthogonal to v

$$F_L = F_{L,\perp} = \frac{e v_\perp B}{c} \equiv \gamma m \frac{\mathrm{d}v_\perp}{\mathrm{d}t} \longrightarrow a_\perp = \frac{e v_\perp B}{\gamma m c}$$

$$P = \frac{4}{3}\,\sigma_T c\, U_B \gamma^2$$

- Thomson cross section $\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 = 6.65 \times 10^{-25}\mathrm{cm}^2$
- magnetic field energy density $U_B = B^2/8\pi$
- ultra relativistic electrons $\beta \longrightarrow 1$
- isotropic distribution of particles $\langle \sin^2\vartheta \rangle = 2/3$

* implicit assumption: the energy of the electron does not change during one gyration around the B-field

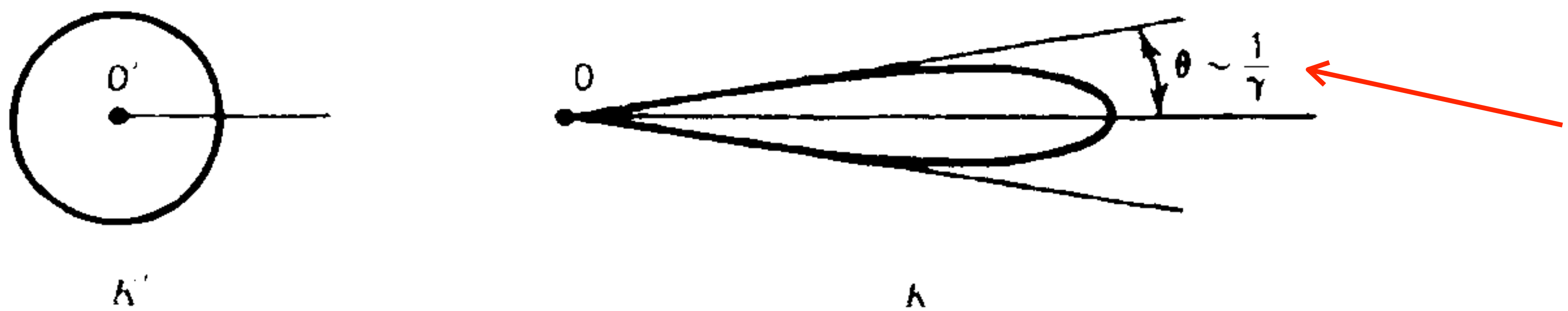
Characteristic frequency

as done for Bremsstrahlung: characteristic time ——> characteristic frequency

gyration frequency?

$$\nu_B = \frac{1}{t_g} = \frac{v_\perp}{2\pi R_L} = \frac{qB}{2\pi\gamma m c}$$

Beaming



$$\vartheta \sim \frac{1}{\gamma}$$

***Figure 4.3** Relativistic beaming of radiation emitted isotropically in the rest frame K'.*

the radiation emitted by a relativistic particle is concentrated within a cone of opening angle 1/γ entered along the particle velocity

Characteristic frequency

as done for Bremsstrahlung: characteristic time ——> characteristic frequency

gyration frequency?

$$\nu_B = \frac{1}{t_g} = \frac{v_\perp}{2\pi R_L} = \frac{qB}{2\pi\gamma m c}$$

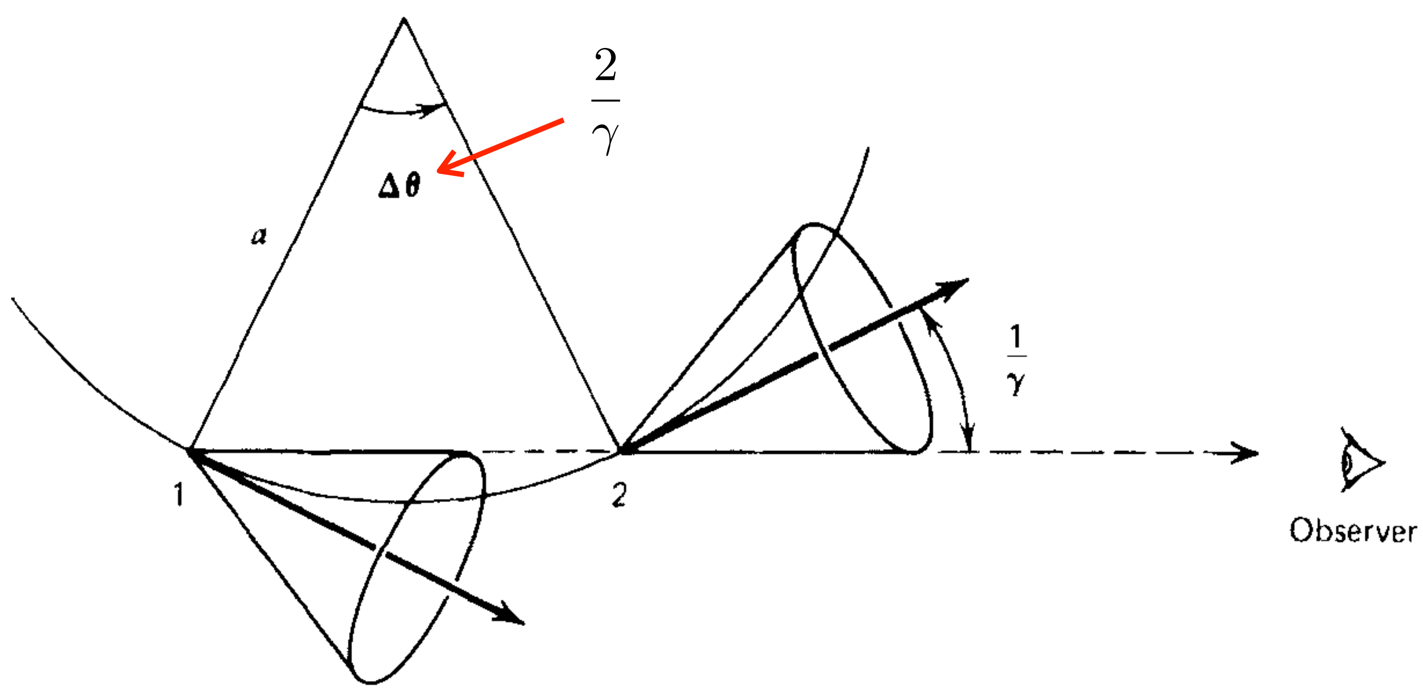


Figure 6.2 Emission cones at various points of an accelerated p trajectory.

photons that reach us are emitted in the time interval:

$$\Delta t_e = \frac{R_L \Delta\vartheta}{v_\perp} = \frac{1}{\pi\gamma\nu_B}$$

but the arrival time interval is shorter!

when a photon is emitted in 2, the photon emitted in 1 has traveled a distance: $c\ \Delta t_e$

$$\Delta t_a = \frac{c\Delta t_e - v_\perp \Delta t_e}{c} \approx \Delta t_e\,(1-\beta) = \Delta t_e \frac{1-\beta^2}{1+\beta} \approx \frac{\Delta t_e}{2\gamma^2}$$

Emission from one and many electrons

duration of the received pulse

$$\Delta t_a \approx \frac{\Delta t_e}{2\gamma^2} = \frac{1}{2\pi\gamma^3\nu_B} = \frac{1}{\gamma^2}\frac{mc}{qB}$$

characteristic synchrotron frequency

$$\nu_s = \frac{1}{2\pi\Delta t_a} = \gamma^3\nu_B = \gamma^2\frac{qB}{2\pi mc}$$

Larmor frequency ν_L

power emitted by one electron

$$P = \frac{4}{3}\sigma_T c\, U_B\gamma^2$$

particle energy distribution found most often in high energy astrophysics: POWER LAW

$$N(\gamma) = K\gamma^{-\delta}$$

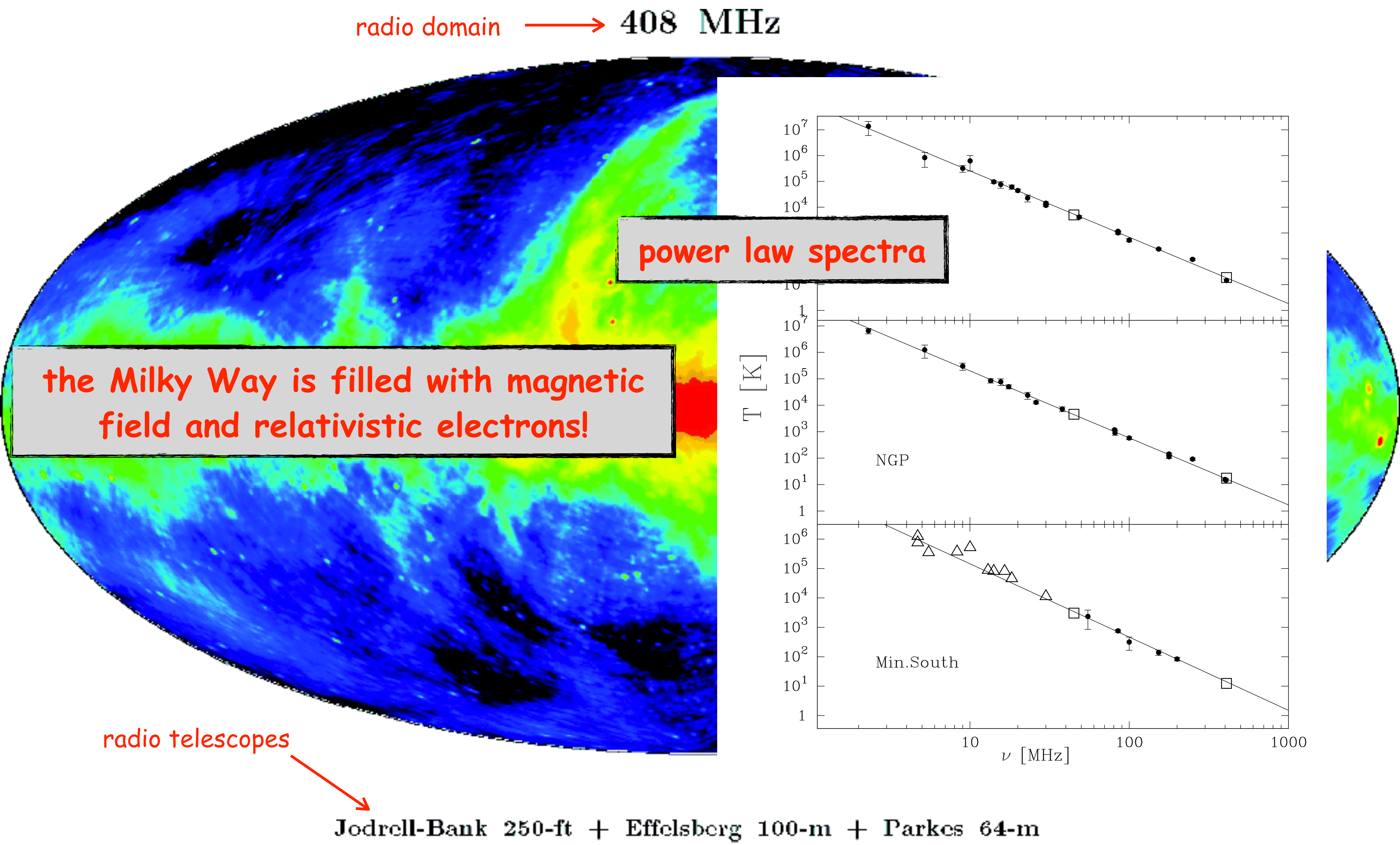
delta function approximation

POWER LAW

$$L_s(\nu) = \int \mathrm{d}\gamma N(\gamma)P(\gamma, B)\delta\left(\nu - \nu_s(\gamma, B)\right) \propto KB^{\frac{\delta+1}{2}}\nu^{-\frac{\delta-1}{2}}$$

$$\delta\left(\nu - \nu_s(\gamma, B)\right) = \delta\left(f(\gamma)\right) = \frac{\delta(\gamma-\gamma_0)}{|f'(\gamma_0)|} = \frac{\delta(\gamma-(\nu/\nu_L)^{1/2})}{2(\nu\nu_L)^{1/2}}$$

Synchrotron emission from the Milky Way



Synchrotron emission: final considerations

$$L(\nu) \propto KB^{\frac{\delta+1}{2}} \nu^{-\frac{\delta-1}{2}}$$

if we observe a radio flux F = L/4πd²

at a given frequency

we can estimate a combination of K and B, but not the two quantities separately!!!

several ways to measure B exist, and they indicate B ~ 3 μG in the Milky Way

$$\nu_s = \gamma^2 \frac{qB}{2\pi mc} \begin{cases} E_e = 10 \text{ GeV} \longrightarrow \nu_s \sim 3 \text{ GHz} \\ E_e = 100 \text{ TeV} \longrightarrow \nu_s \sim 1 \text{ keV} \end{cases}$$

radio

X-rays

Equipartition magnetic field

total energy in a synchrotron emitting source $W_{tot} = W_B + W_{CR}$

$$W_{CR} = W_e + W_p = \left(1 + \frac{W_p}{W_e}\right) W_e = \eta W_e$$

$$\begin{cases} W_B = V \times \dfrac{B^2}{8\pi} \\ W_e = \displaystyle\int_{E_{min}}^{E_{max}} \mathrm{d}E \; E \; K E^{-\delta} = \frac{K}{2-\delta} \left(E_{max}^{2-\delta} - E_{min}^{2-\delta}\right) \end{cases}$$

$$L(\nu) = C(\delta) K B^{\frac{\delta+1}{2}} \nu^{-\frac{\delta-1}{2}}$$

$$\nu_s = A \; E^2 B$$

Equipartition magnetic field

total energy in a synchrotron emitting source $W_{tot} = W_B + W_{CR}$

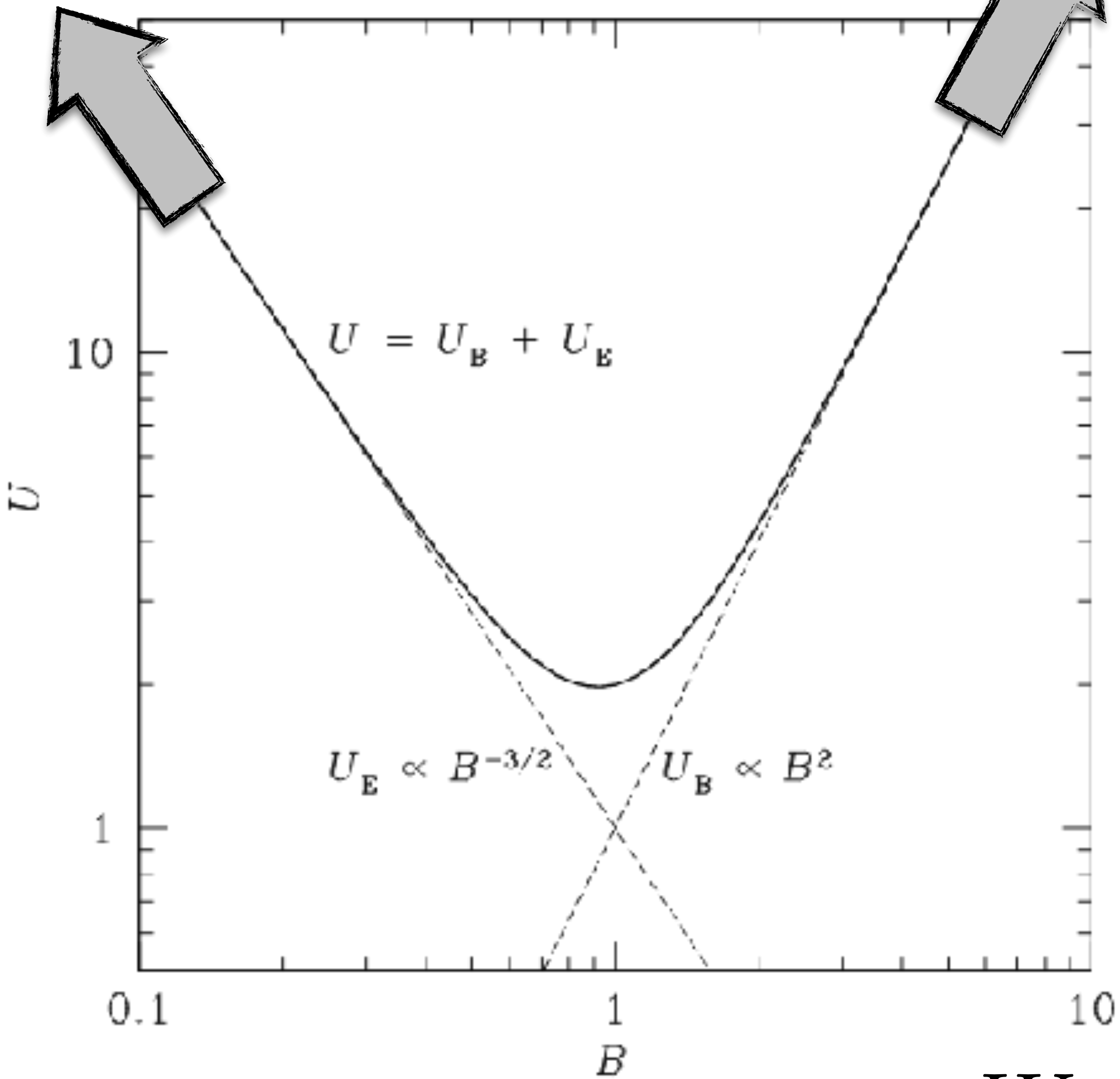
$$W_{CR} = W_e + W_p = \left(1 + \frac{W_p}{W_e}\right) W_e = \eta W_e$$

$$\begin{cases} W_B = V \times \dfrac{B^2}{8\pi} \quad \propto B^2 \\ W_e = \dfrac{L(\nu)\nu^{\frac{\delta-1}{2}}}{B^{\frac{\delta+1}{2}}(2-\delta)} \left[\left(\dfrac{\nu_{max}}{A\ B}\right)^{\frac{2-\delta}{2}} - \left(\dfrac{\nu_{min}}{A\ B}\right)^{\frac{2-\delta}{2}} \right] \propto B^{-3/2} \end{cases}$$

Equipartition magnetic field

too much energy!

too much energy!



$$W_{tot} = C_B B^2 + C_{CR} B^{-3/2}$$

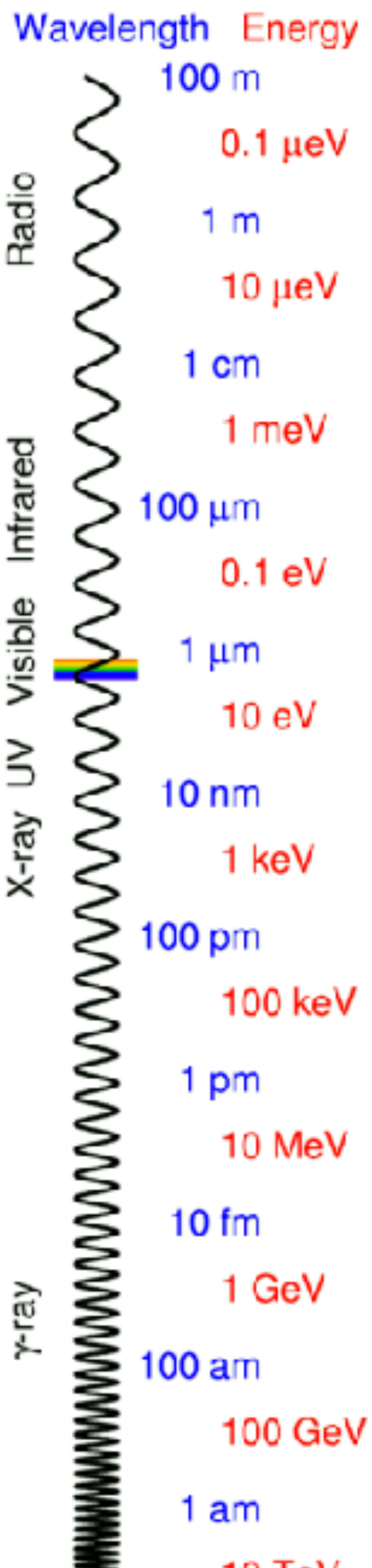
$$B_{min} = \left(\frac{4C_B}{3C_{CR}} \right)^{-2/7}$$

$$\frac{W_B}{W_{CR}} = \frac{3}{4}$$

EQUIPARTITION

in the absence of other estimates, the assumption of equipartition is used to estimate a reference value for the magnetic field

The electromagnetic spectrum



21 cm line -> atomic neutral Hydrogen (HI)
synchrotron emission of multi-GeV electrons in a 3 µG field

visible light -> stars

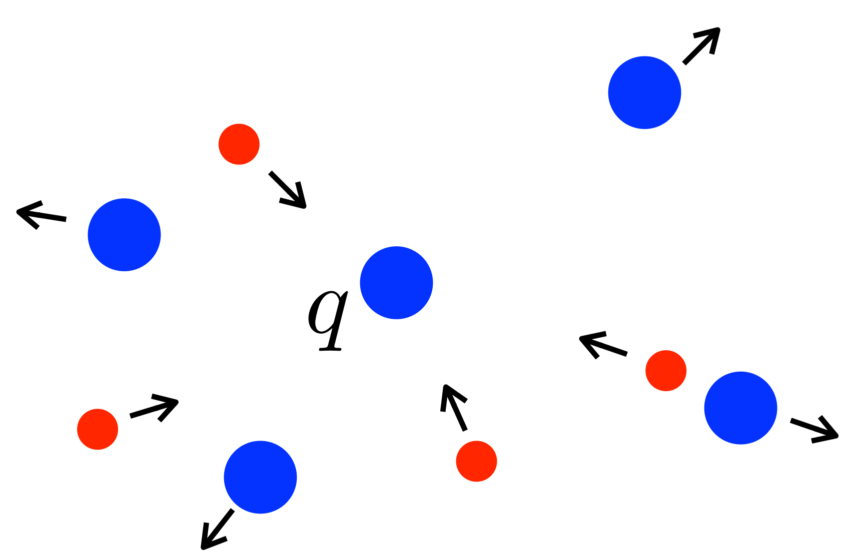
thermal emission from hot plasmas (millions of K)
synchrotron emission of 100 TeV electrons in a 3 µG field

interactions of cosmic ray protons with matter

Definition of plasma

Wikipedia: “Plasma is an electrically neutral medium of unbound positive and negative particles (i.e. the overall charge of a plasma is roughly zero)”.

fully ionised plasma global neutrality -> $N_p = N_e$ local neutrality -> $n_p = n_e$



electrostatic potential $\nabla^2 \phi = -4\pi \varrho_e$

$$\varrho_e = q \left[\delta(\vec{r}) - n_e e^{\frac{q\phi}{kT}} + n_p e^{-\frac{q\phi}{kT}} \right]$$

$$\xrightarrow[q\phi \ll kT]{} q\ \delta(\vec{r}) + \boxed{\frac{2 n_e q^2}{kT}} \phi$$

Debye length $L_D^{-2}/4\pi$

protons, electrons } thermal equilibrium $T_e = T_p = T$ -> Boltzmann distribution

Definition of plasma

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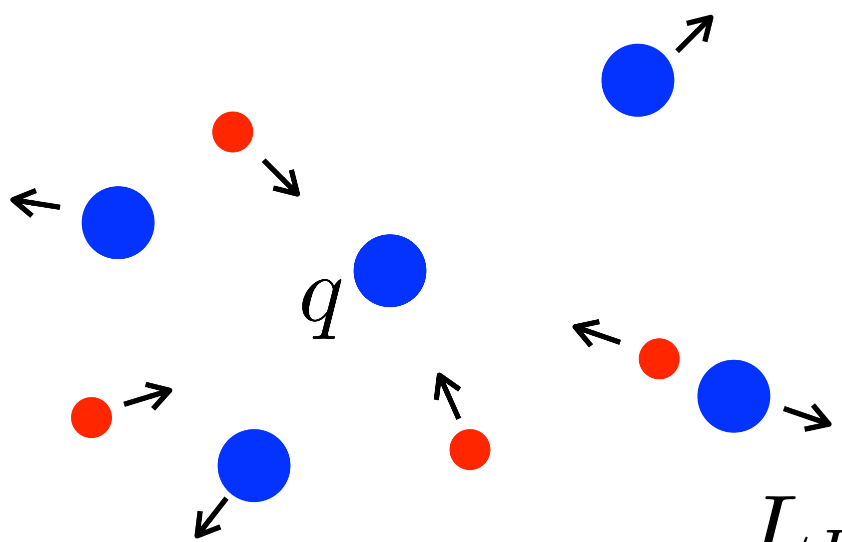


$$\nabla^2 \phi = -4\pi q \delta(\vec{r}) + \frac{\phi}{L_D^2} \rightarrow \phi = \frac{q}{r} e^{-\frac{r}{L_D}}$$

Debye shielding

$$L_D \sim 5 \times 10^2 \left(\frac{T}{10^4 \ \mathrm{K}} \right)^{1/2} \left(\frac{n}{\mathrm{cm}^{-3}} \right)^{-1/2} \mathrm{cm}$$

protons
electrons } thermal equilibrium $T_e = T_p = T$ -> Boltzmann distribution

Magnetohydrodynamics (MHD)

dynamics of electrically conducting fluids in the presence of magnetic fields

plasma motion —> electric fields —> currents —> magnetic fields —> ...

Maxwell equations

$$\nabla \vec{E} = 4\pi \varrho \;\; = 0$$ —> plasma neutrality

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \longrightarrow E \approx \frac{L}{c}\frac{B}{T}$$

characteristic length scale

characteristic time scale

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \longrightarrow \frac{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}{\nabla \times \vec{B}} \approx \frac{\frac{L}{c^2 T^2}B}{\frac{B}{L}} \approx \left(\frac{U}{c}\right)^2 \ll 1$$

characteristic velocity

displacement current

electric currents —> only source of B-field

Magnetohydrodynamics (MHD)

$n_p = n_e$-> this does not prevent the plasma from possessing electromagnetic properties

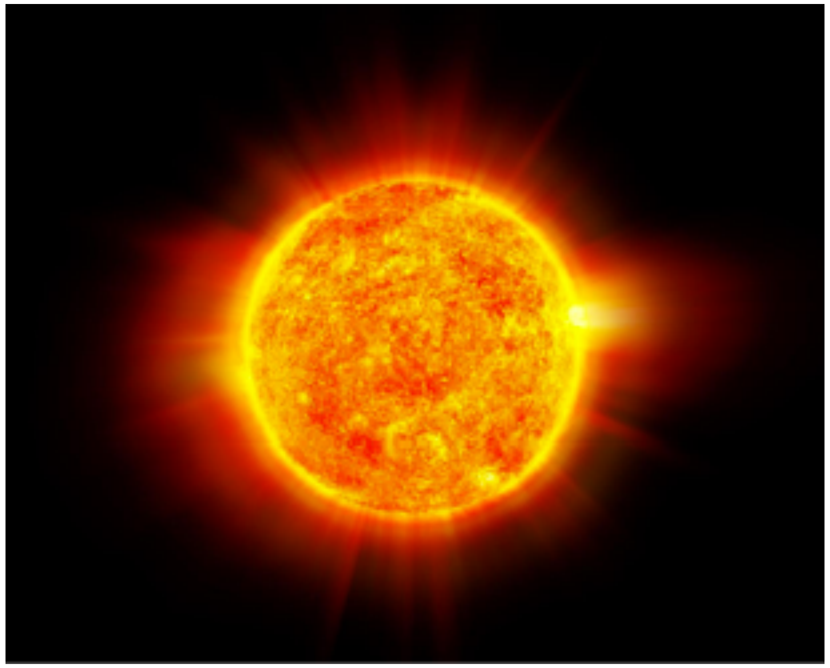
electric current

$$\vec{j} = q\left(n_i \vec{u}_i - n_e \vec{u}_e\right) = q n_i \vec{v}_{ei}$$

Ampere law

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} \longrightarrow j \approx \frac{Bc}{4\pi L} \longrightarrow v_{ei} \approx \frac{B\ c}{4\pi q n_i L}$$

the Sun



- B fields up to 10^3 G!
- generated in a convective region of L ~ 2 x 10^{10} cm
- average electron density $n_e \sim 10^{23}$ cm^{-3}

$$v_{ei} \approx 10^{-12} \mathrm{cm/s}$$

very small!!!!

for any practical purpose we can consider a 1-component plasma
electrons and ions are fully coupled

MHD equation for the magnetic field

Ohm's law: relates the electric current j to the other variables of the problem

electric conductivity

$$\vec{j}' = \sigma \vec{E}' = \vec{j}$$

primed quantities -> rest frame where the plasma is at rest

$$\vec{j} = q\left(n_i \vec{u}_i - n_e \vec{u}_e\right) = q n_i \vec{v}_{ei}$$

invariant under Galilean transformation

$$\vec{E}' = \vec{E} + \frac{\vec{u}}{c} \times \vec{B}$$

Lorentz transformation for $u/c \ll 1$

Ampere law

$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j} = \frac{4\pi\sigma}{c}\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right)$$

Faraday law

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

⇒

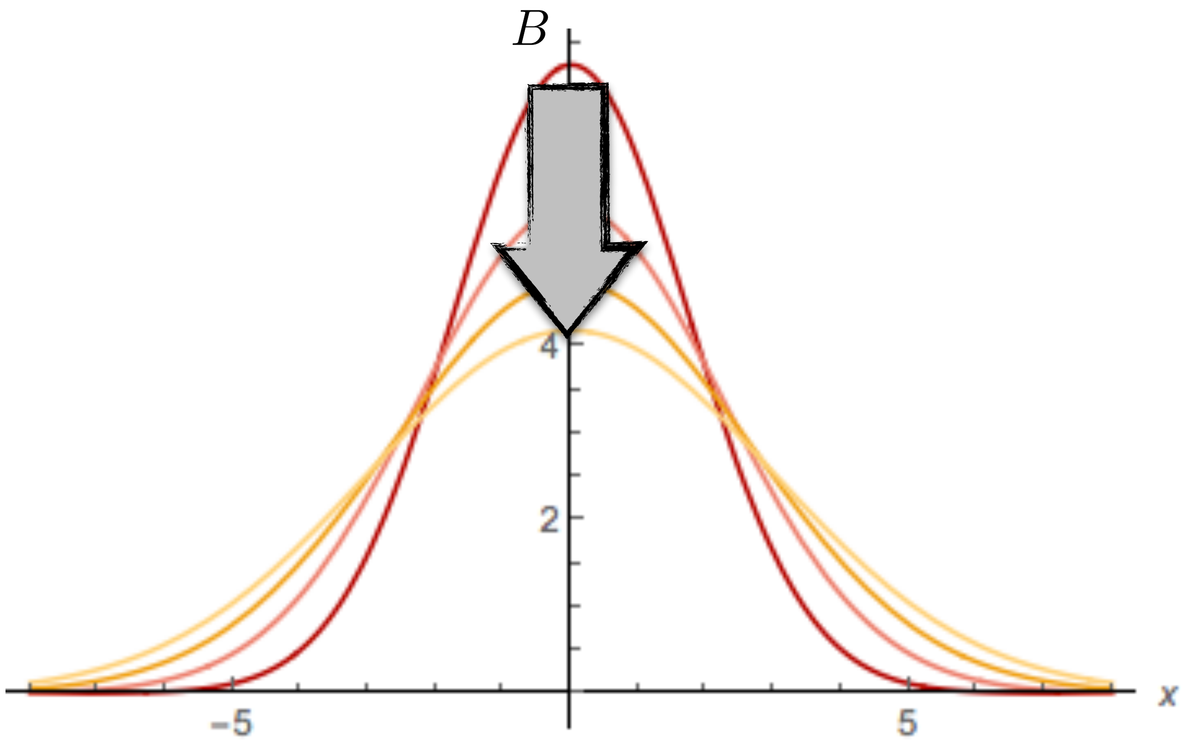
$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{B}\right) - \nabla \times \left(\frac{c^2}{4\pi\sigma}\nabla \times \vec{B}\right)$$

MHD equation for the magnetic field

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{B}\right) - \nabla \times \left(\frac{c^2}{4\pi\sigma} \nabla \times \vec{B}\right)$$

using the vectorial identity $\nabla \times \left(\nabla \times \vec{B}\right) = \nabla \left(\nabla \vec{B}\right) - \nabla^2 \vec{B}$ and $\nabla \vec{B} = 0$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{B}\right) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}$$



magnetic diffusion

$$T_d \approx \frac{L^2}{\eta} \qquad \eta = \frac{c^2}{4\pi\sigma}$$

magnetic conductivity

Mass, momentum, and energy

mass

$$\frac{\partial \varrho}{\partial t} + \nabla \left(\varrho \vec{u} \right) = 0$$

momentum

Lorentz force

$$\varrho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = -\nabla P + \frac{1}{c} \vec{j} \times \vec{B}$$

$$\nabla \times \vec{B} = \frac{4\pi}{c} \vec{j} \longrightarrow \qquad = -\nabla P + \frac{1}{4\pi} \left(\nabla \times \vec{B} \right) \times \vec{B}$$

energy

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(P \varrho^{-\gamma} \right) = 0$$

adiabatic condition

MHD equations

$$\frac{\partial \varrho}{\partial t} + \nabla \left(\varrho \vec{u} \right) = 0$$

$$\varrho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = \nabla P + \frac{1}{4\pi} \left(\nabla \times \vec{B} \right) \times \vec{B}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(P \varrho^{-\gamma} \right) = 0$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{B} \right) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}$$

8 equations for 8 variables: $\varrho \quad P \quad \vec{u} \quad \vec{B}$

we got rid of $\vec{E}$ and $\vec{j}$!

Ideal MHD equations

$$\frac{\partial \varrho}{\partial t} + \nabla \left(\varrho \vec{u} \right) = 0$$

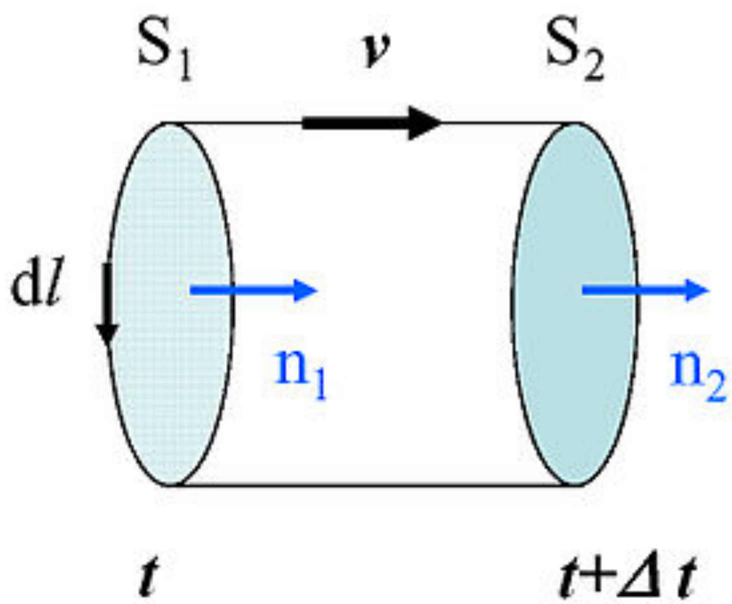
$$\varrho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = \nabla P + \frac{1}{4\pi} \left(\nabla \times \vec{B} \right) \times \vec{B}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(P \varrho^{-\gamma} \right) = 0$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times \left(\vec{u} \times \vec{B} \right) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{B}$$

under most astrophysical conditions $T_d \approx \frac{L^2}{\eta} \longrightarrow \infty$

Magnetic flux freezing



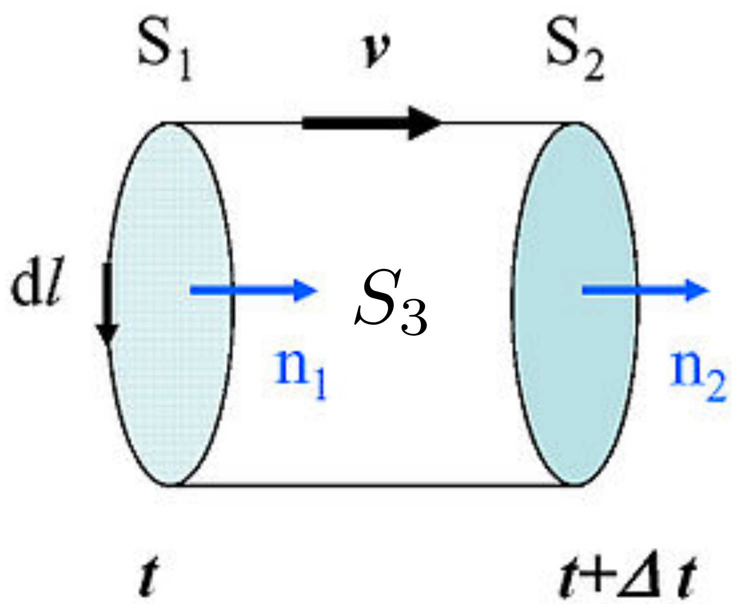
$$\Phi_1 = \int_{S_1} \vec{B}(\vec{x}, t) \cdot \mathrm{d}\vec{S}_1$$

S_2 differs from S_1 by terms of order Δt

$$\Phi_2 = \int_{S_2} \vec{B}(\vec{x} + \vec{v}\Delta t, t + \Delta t) \cdot \mathrm{d}\vec{S}_2$$

$$\approx \int_{S_2} \vec{B}(\vec{x} + \vec{v}\Delta t, t) \cdot \mathrm{d}\vec{S}_2 + \Delta t \int_{S_{\cancel{2}1}} \frac{\partial}{\partial t} \vec{B}(\vec{x} + \vec{v}\Delta t, t) \cdot \mathrm{d}\vec{S}_{\cancel{2}1}$$

Magnetic flux freezing



$$\Phi_1 = \int_{S_1} \vec{B}(\vec{x},t)\cdot \mathrm{d}\vec{S}_1$$

$$\Phi_2 \approx \int_{S_2} \vec{B}(\vec{x}+\vec{v}\Delta t,t)\cdot \mathrm{d}\vec{S}_2 + \Delta t \int_{S_1} \frac{\partial}{\partial t}\vec{B}(\vec{x}+\vec{v}\Delta t,t)\cdot \mathrm{d}\vec{S}_1$$

Gauss theorem

$$-\int_{S_1} \vec{B}\cdot \mathrm{d}\vec{S}_1 + \int_{S_2} \vec{B}\cdot \mathrm{d}\vec{S}_2 + \int_{S_3} \vec{B}\cdot \mathrm{d}\vec{S}_3 = \oint \vec{B}\cdot \mathrm{d}\vec{S}_{tot} = \int \nabla\cdot\vec{B}\,\mathrm{d}V = 0$$

$$\Phi_1 - \Phi_2 = -\Delta t \int_{S_1} \frac{\partial \vec{B}}{\partial t}\cdot \mathrm{d}\vec{S}_1 + \int_{S_3} \vec{B}\cdot \mathrm{d}\vec{S}_3$$

Magnetic flux freezing

$$\Phi_1 - \Phi_2 = -\Delta t \int_{S_1} \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{S}_1 + \int_{S_3} \vec{B} \cdot \mathrm{d}\vec{S}_3$$

$$\mathrm{d}\vec{S}_3 = \mathrm{d}\vec{x} \times \vec{v}\Delta t$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$= -\Delta t \int_{S_1} \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{S}_1 + \Delta t \int_{S_3} \vec{B} \cdot (\mathrm{d}\vec{x} \times \vec{v})$$

$$= -\Delta t \left[\int_{S_1} \frac{\partial \vec{B}}{\partial t} \cdot \mathrm{d}\vec{S}_1 - \oint_{S_1} \mathrm{d}\vec{x} \cdot \left(\vec{v} \times \vec{B} \right) \right]$$

curl theorem

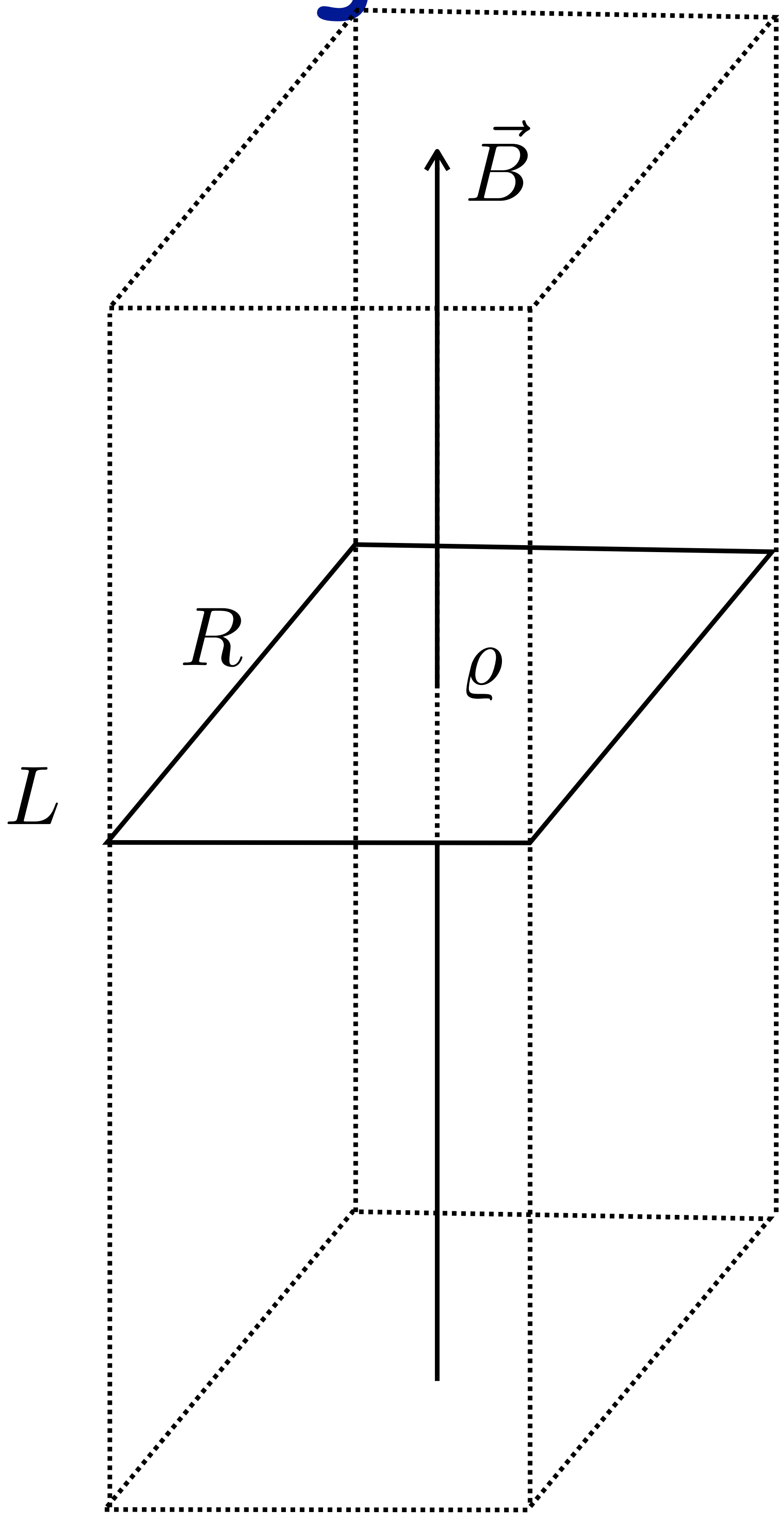
$$= -\Delta t \left[\int_{S_1} \left(\frac{\partial \vec{B}}{\partial t} - \nabla \times \left(\vec{v} \times \vec{B} \right) \right) \cdot \mathrm{d}\vec{S}_1 \right] = 0$$

ideal MHD

the magnetic flux is constant across a surface moving with the plasma!

Magnetic flux freezing —> B-field line move with the plasma

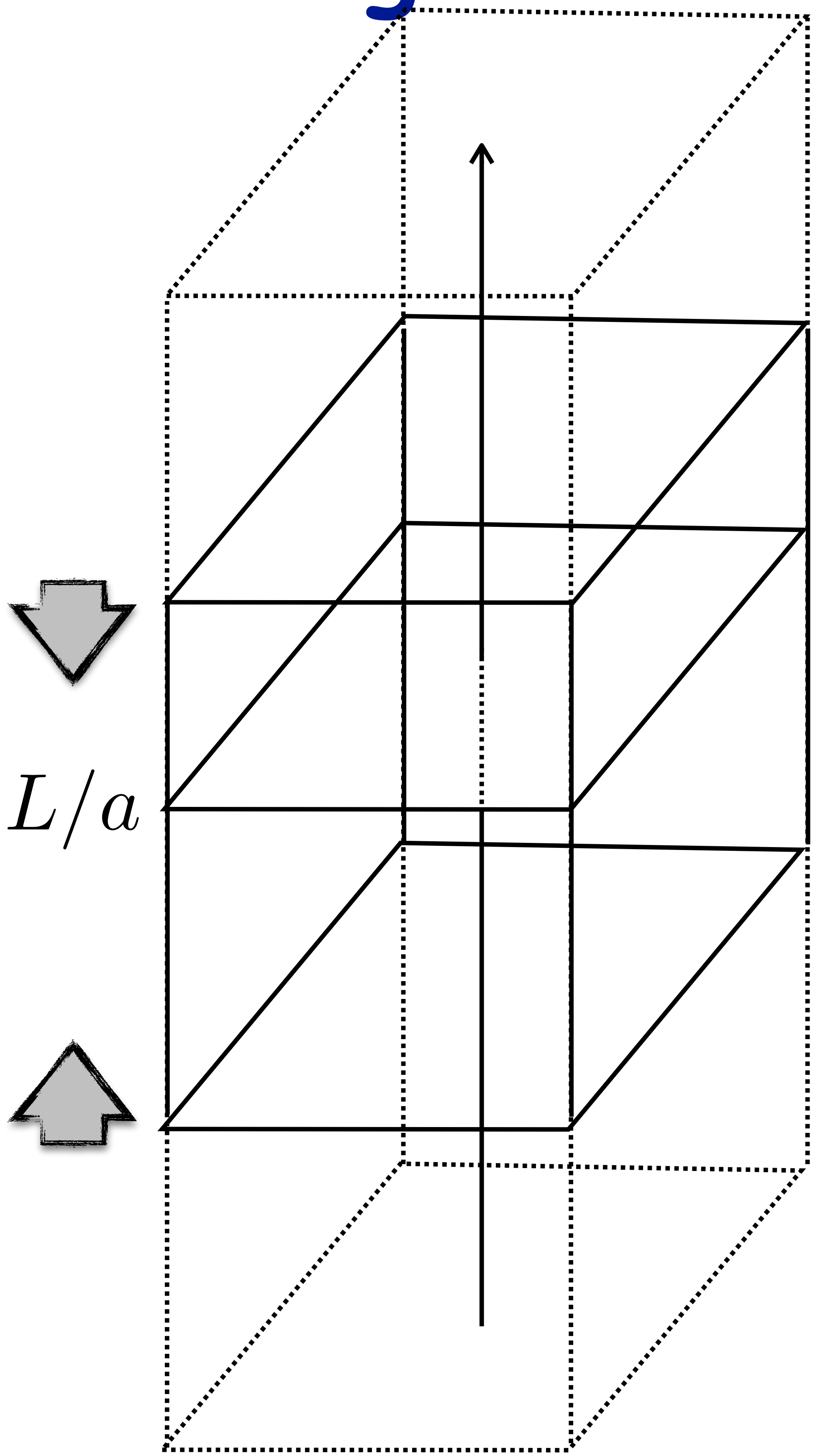
Magnetic flux freezing: examples



magnetic flux $\Phi = BR^2$

gas density ϱ

Magnetic flux freezing: examples

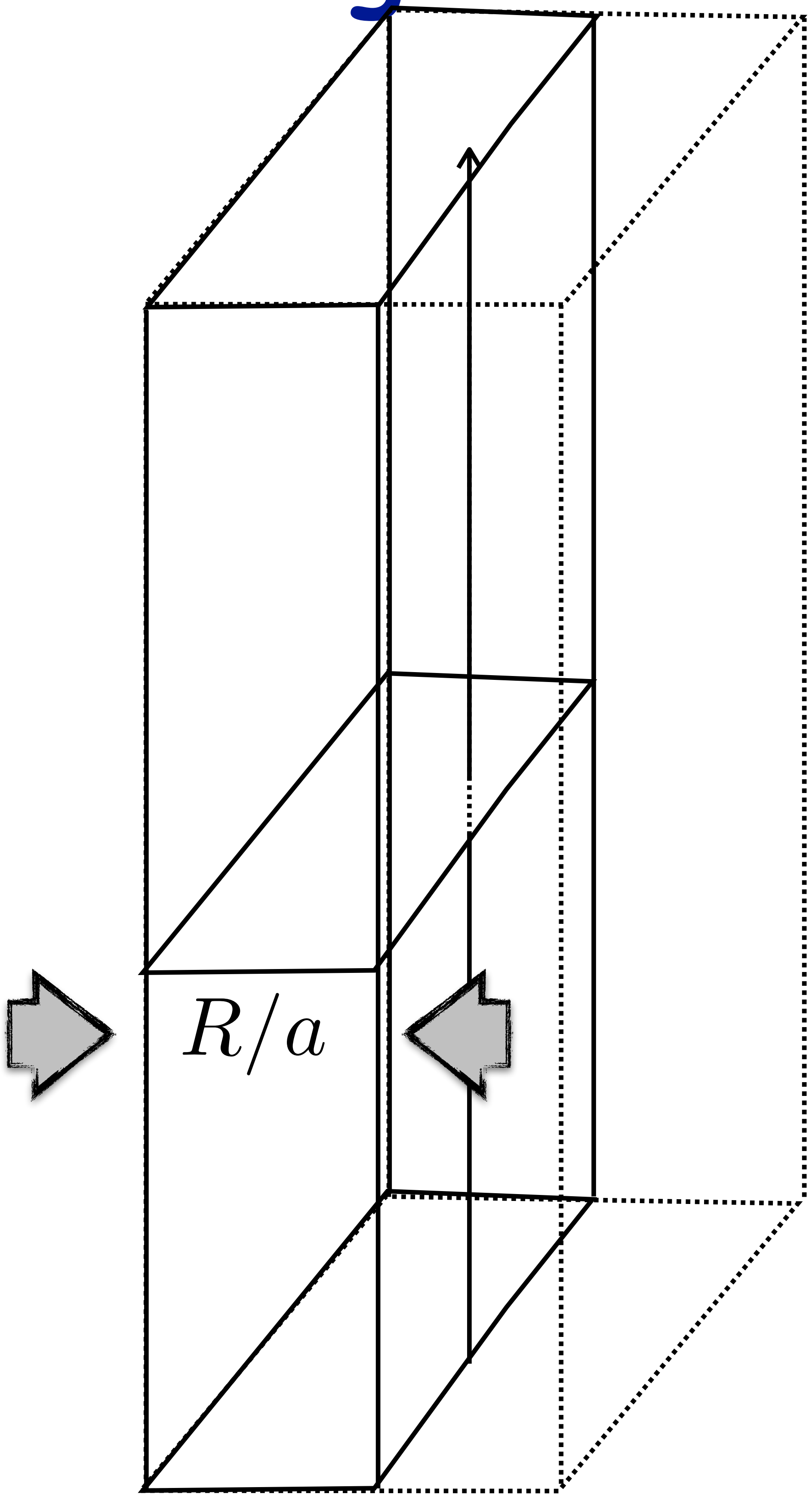


magnetic flux $\Phi = BR^2 \longrightarrow BR^2$

gas density $\varrho \longrightarrow a\varrho$

B-field $B \longrightarrow B$

Magnetic flux freezing: examples



magnetic flux	$\Phi = BR^2 \longrightarrow BR^2 a^{-1}$
gas density	$\varrho \longrightarrow a\varrho$
B-field	$B \longrightarrow aB$
in general:	$\frac{B}{\varrho^n} = const$

- n = 0 —> parallel contraction/expansion
- n = 1 —> perpendicular contraction/expansion
- n = 2/3 —> isotropic contraction or expansion

Magnetic pressure and tension

Lorentz force

$$F_L = \frac{1}{c}\,\vec{j} \times \vec{B} \;=\; \frac{1}{4\pi}\left(\nabla \times \vec{B}\right) \times \vec{B}$$

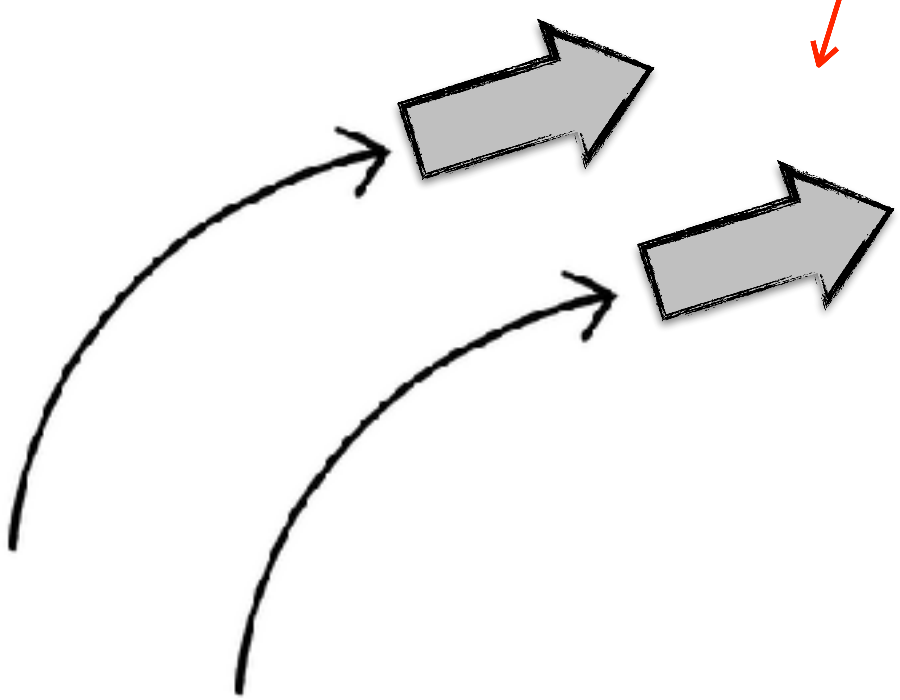
Ampere law

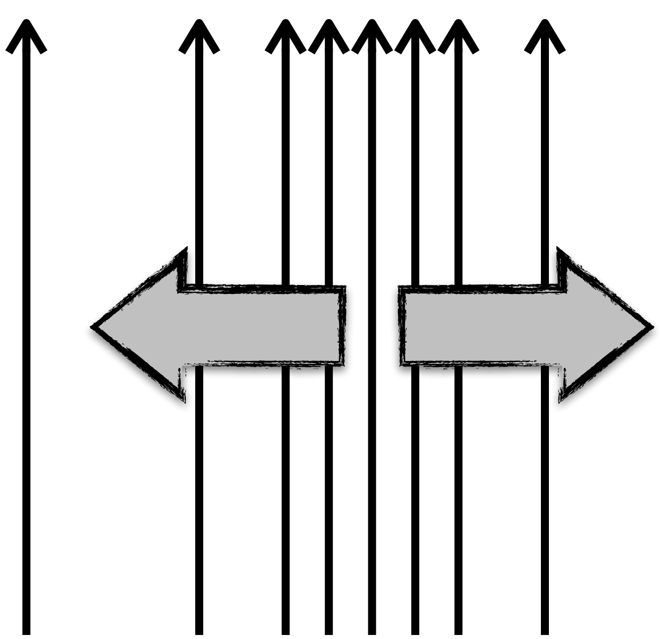
$$\nabla \times \vec{B} = \frac{4\pi}{c}\vec{j}$$

magnetic tension

gradient of magnetic pressure

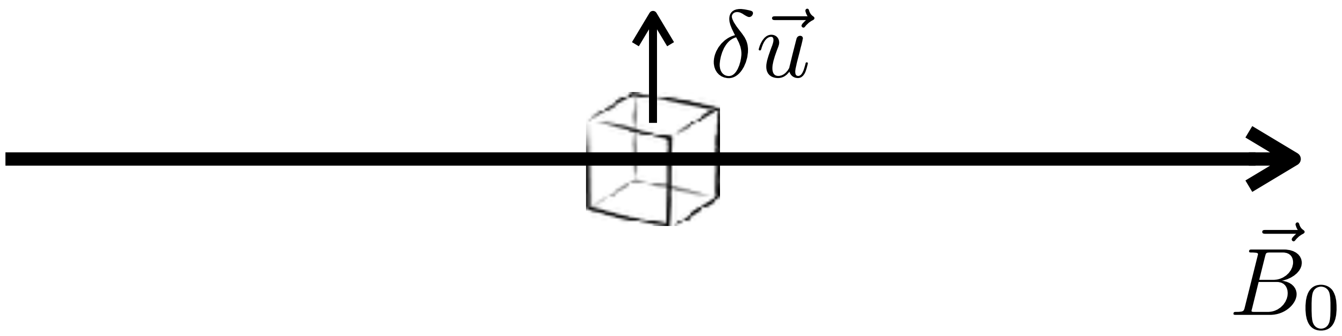
$$F_L = \frac{1}{4\pi}\left(\vec{B}\cdot\nabla\right)\vec{B} - \nabla\left(\frac{\vec{B}^2}{8\pi}\right)$$





Alfven waves

$\vec{u} = 0$



- displace a fluid element orthogonally to B_0

kinetic energy

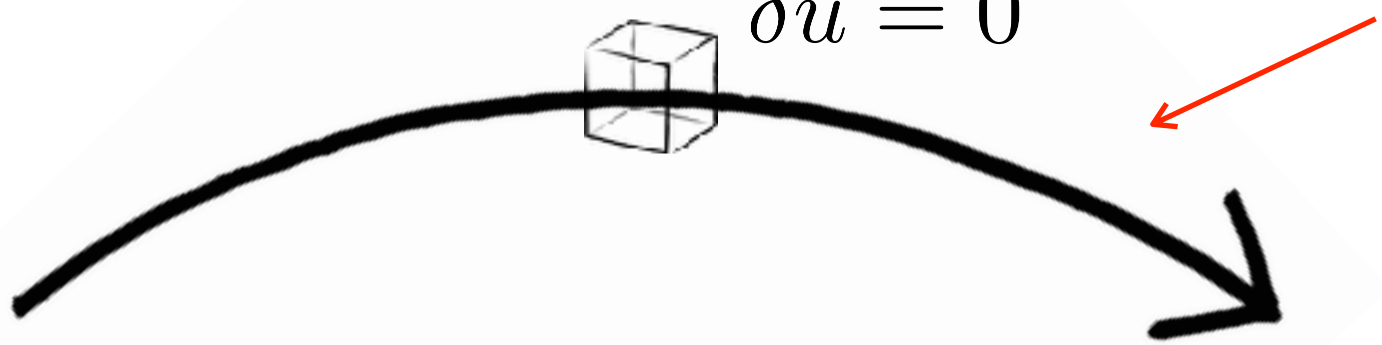
Alfven waves

$$\vec{u} = 0$$

$$\delta\vec{u} = 0$$

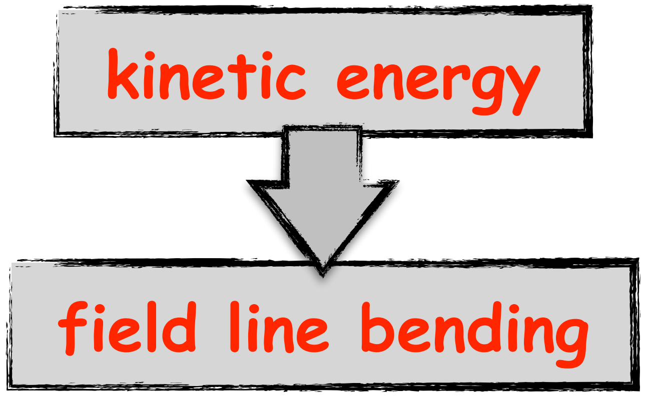
$$\vec{B}_0 + \delta\vec{B}$$

$$\delta\vec{B} \perp \vec{B}_0$$



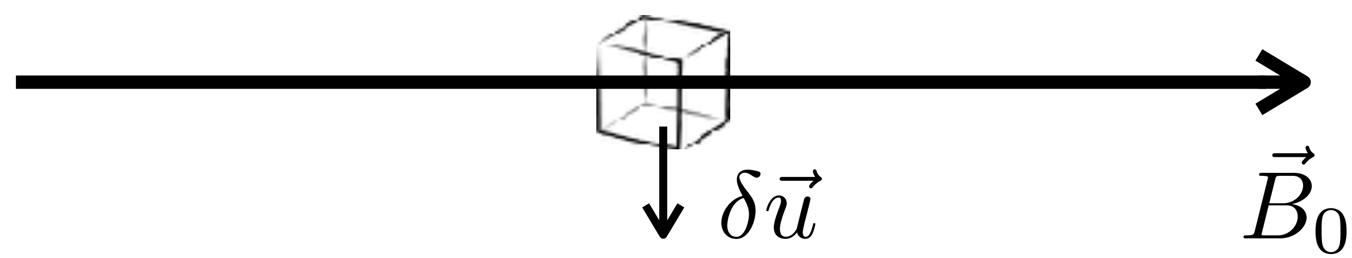
- displace a fluid element orthogonally to B_0
- the field is bent by plasma motion (freezing)

  -> work is needed to bend B



Alfven waves

$$\vec{u} = 0$$



- displace a fluid element orthogonally to B_0
- the field is bent by plasma motion (freezing)

  -> work is needed to bend B
- magnetic tension will push the plasma back

  into motion

kinetic energy

↓

field line bending

↓

kinetic energy

↓

WAVE MOTION

Alfven speed

Alfven waves propagate along magnetic field lines like waves on a string

$$c_W^2 = \frac{tension}{linear\ density} \longrightarrow v_A^2 = \frac{B^2/4\pi L}{\varrho_i/L}$$

in the ISM: $$v_A = \frac{B}{\sqrt{4\pi\varrho_i}} = 20 \left(\frac{B}{3\ \mu\mathrm{G}}\right) \left(\frac{n_i}{0.1\ \mathrm{cm}^{-3}}\right)^{-1/2} \mathrm{km/s}$$

$v_A \ll v_{SNR}$ SNR shocks are super-Alfvenic

$v_A \ll c$ cosmic rays "see" Alfven waves at rest

component of wave vector k along B_0

dispersion relation $$\omega = k_{||} v_A$$