# NPAC course on Astroparticles

# VI - PARTICLE ACCELERATION: DIFFUSIVE SHOCK ACCELERATION

# Particle acceleration

Ohm's law

**Static electric field ->** $$\vec{E} = \frac{\vec{j}}{\sigma} \approx 0$$

electric conductivity -> infinity in astrophysical plasmas!

Lorentz's force

**Static magnetic field ->** $$\vec{F} = q\,\vec{v} \times \vec{B}$$

no work done on the particle

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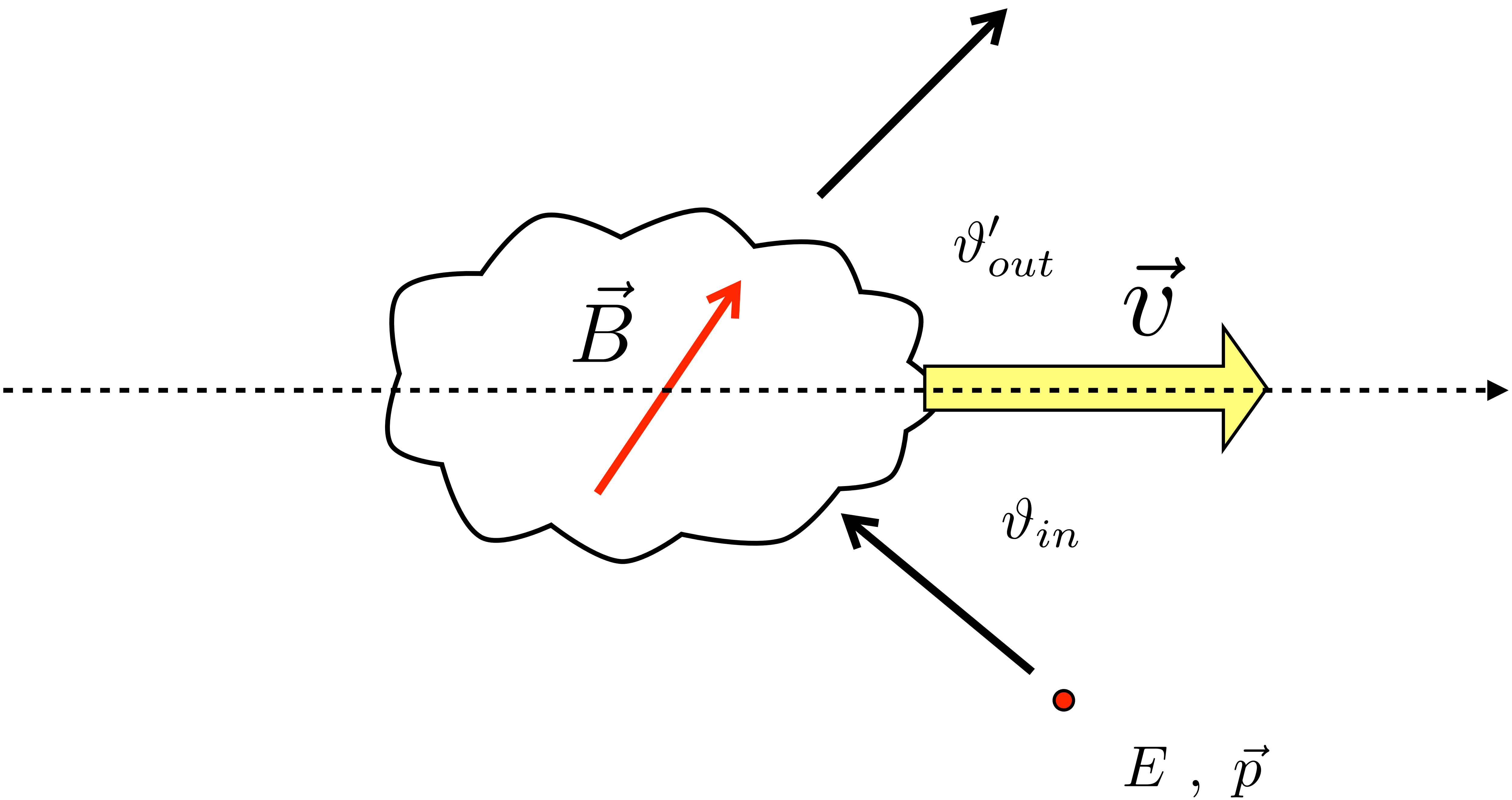
Induced E-field ->

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

Moving B-field ->

$$\vec{E} = -\frac{1}{c}\,\vec{v} \times \vec{B}$$

# Fermi's idea (1949, 1954)

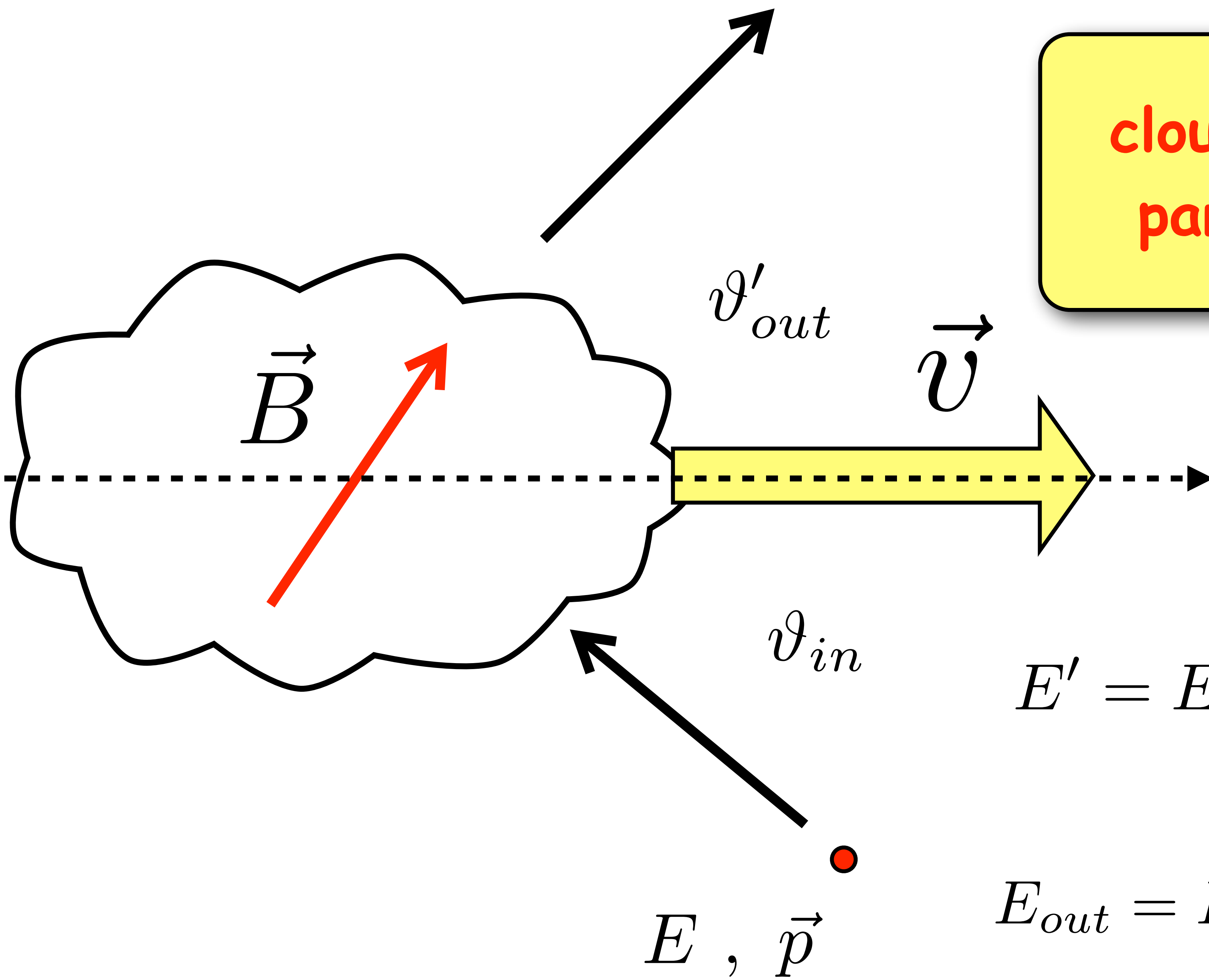



* primed quantities are measured in the rest frame of the cloud

# Energy gain (loss) per interaction

$$E' = \gamma_v \left(E - v\, p\, \cos(\vartheta_{in})\right)$$

**cloud -> non-relativistic**
**particle -> relativistic**




cloud speed

$$E' = E \left(1 - \beta \cos(\vartheta_{in}) + \frac{\beta^2}{2}\right)$$

$$E_{out} = E' \left(1 + \beta \cos(\vartheta'_{out}) + \frac{\beta^2}{2}\right)$$

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# Energy gain (loss) per interaction

$$\frac{\Delta E}{E} = \beta\left[\cos(\vartheta'_{out}) - \cos(\vartheta_{in})\right] + \beta^2\left[1 - \cos(\vartheta_{in})\cos(\vartheta'_{out})\right]$$

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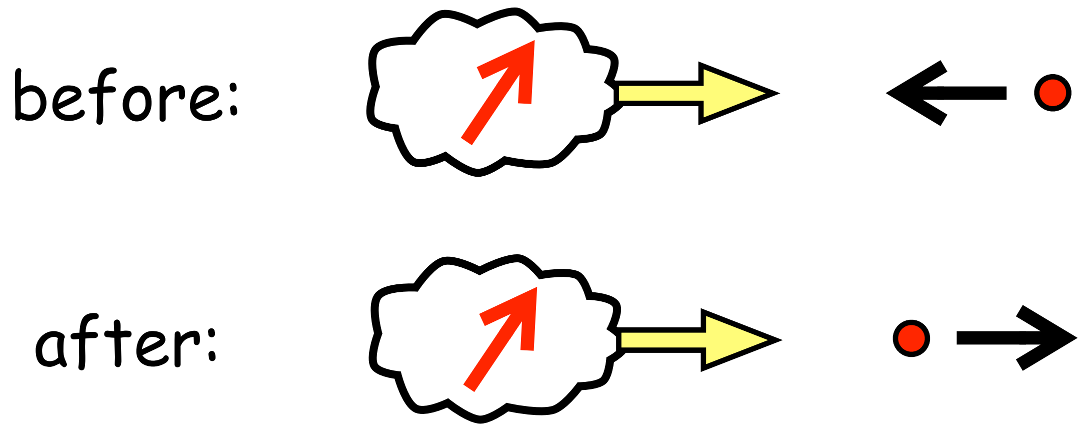
**head-on collision**

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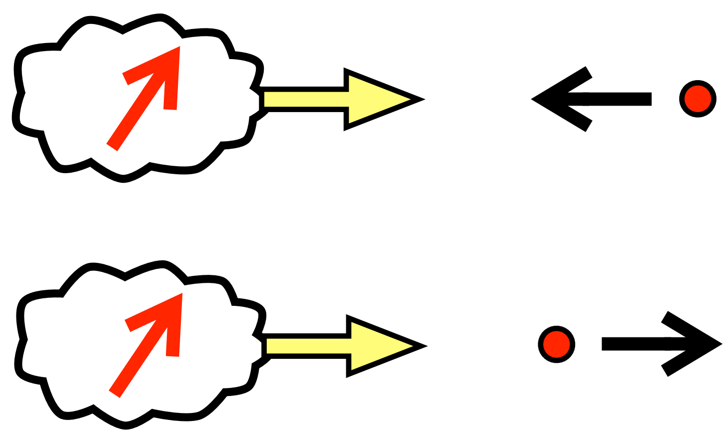
## head-on collision

before:

after:

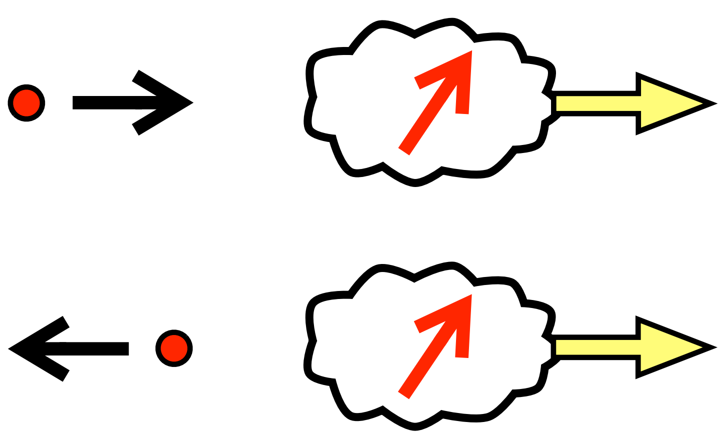


$$\frac{\Delta E}{E} = 2\,\beta\,(1+\beta)$$

## tail-on collision

before:

after:



$$\frac{\Delta E}{E} = -2\,\beta\,(1-\beta)$$

# Average energy gain

$$\frac{\Delta E}{E} = \beta \left[\cos(\vartheta'_{out}) - \cos(\vartheta_{in})\right] + \beta^2 \left[1 - \cos(\vartheta_{in}) \cos(\vartheta'_{out})\right]$$

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- isotropy of particles in the cloud frame ----> $\langle \cos(\vartheta'_{out}) \rangle = 0$
- # particles between $\vartheta_{in}$ and $\vartheta_{in} + \mathrm{d}\vartheta_{in}$ ----> $\propto \sin(\vartheta_{in}) \mathrm{d}\vartheta_{in}$
- rate at which particles enter the cloud prop. to ---> $\propto 1 - \beta \cos(\vartheta_{in})$

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$$\langle \cos(\vartheta_{in}) \rangle = -\frac{\beta}{3}$$

# Second order Fermi mechanism

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4}{3}\,\beta^2$$

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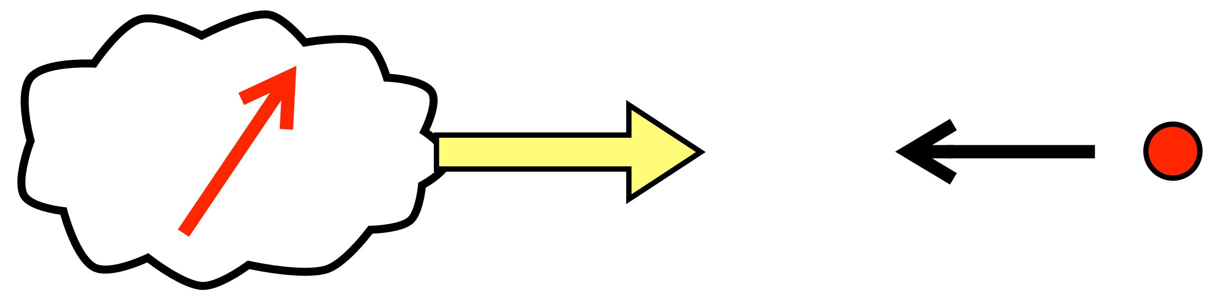
inefficient -> too slow...

# A very simple idea

head-on collision



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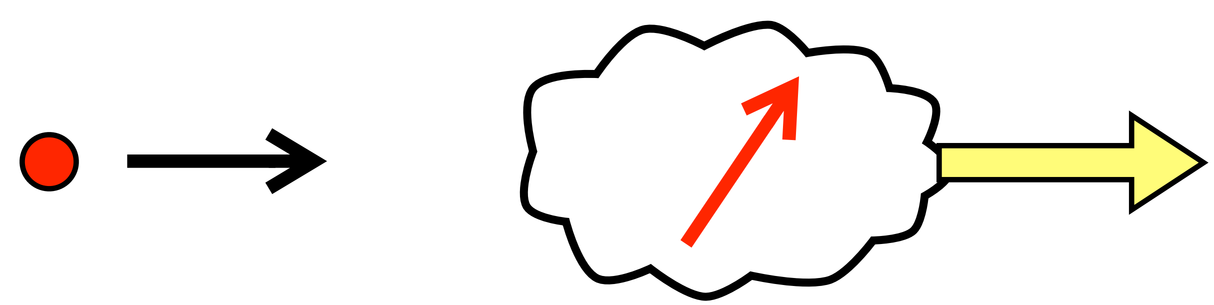

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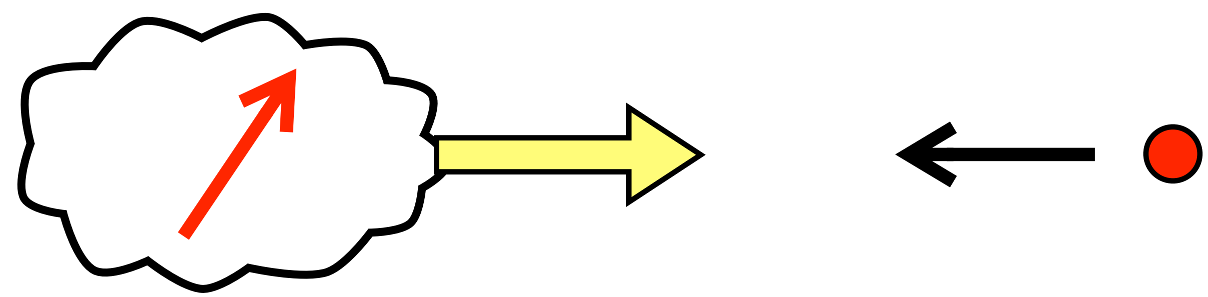

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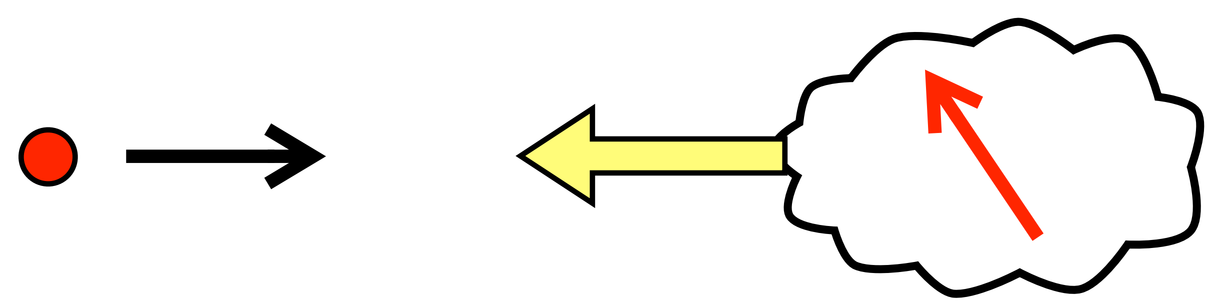
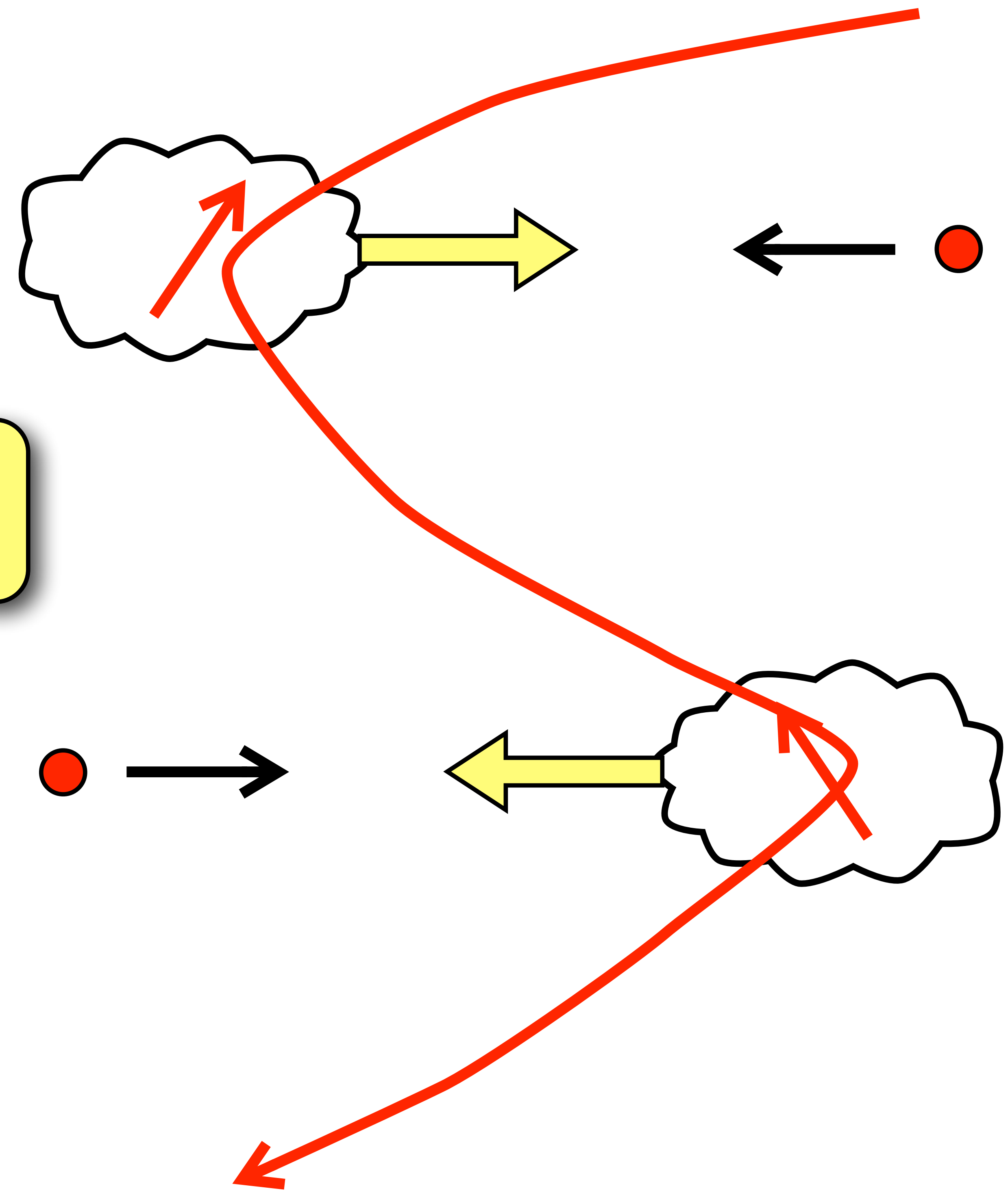


# A very simple idea

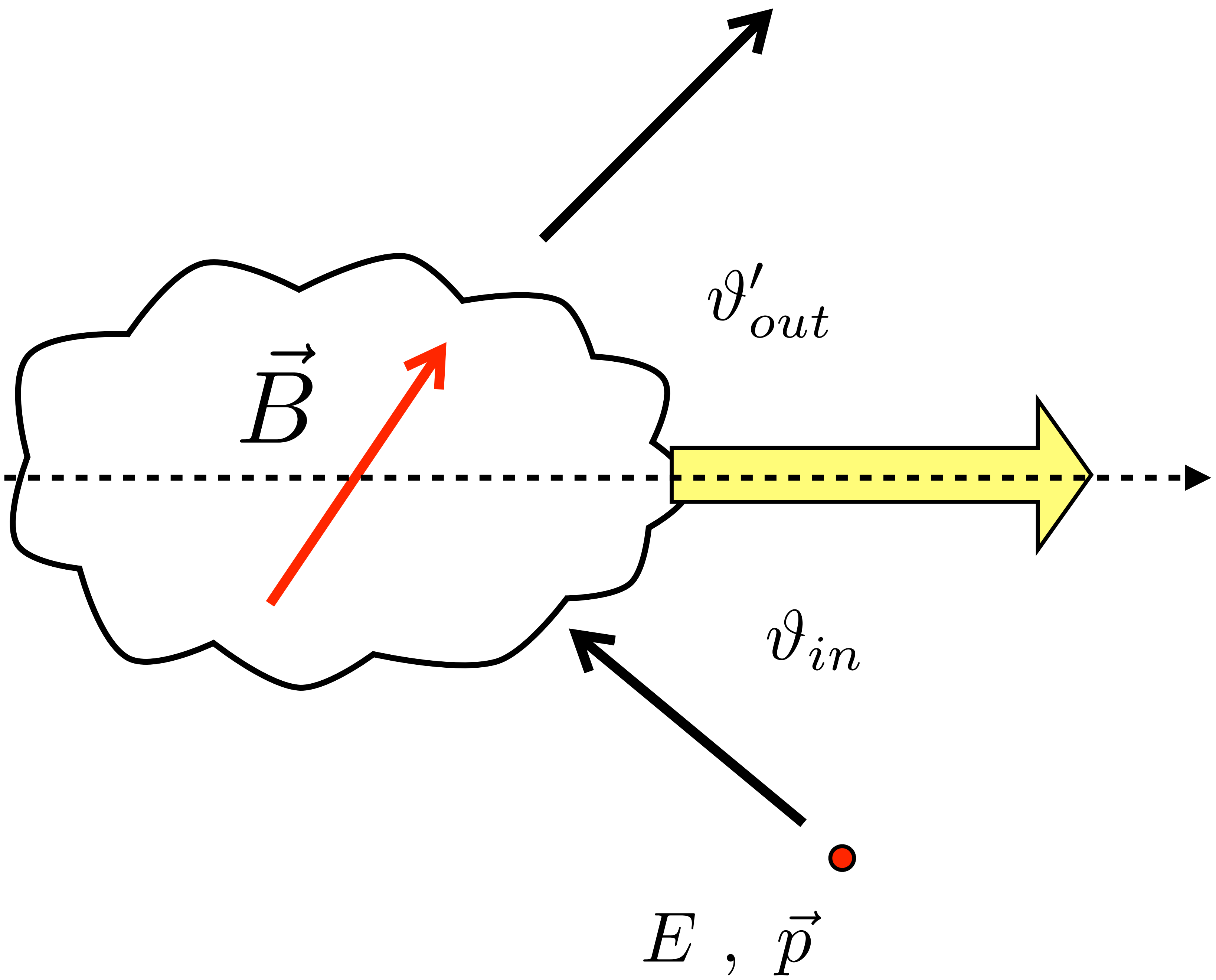
head-on collision

Converging walls

head-on collision



# First order Fermi mechanism



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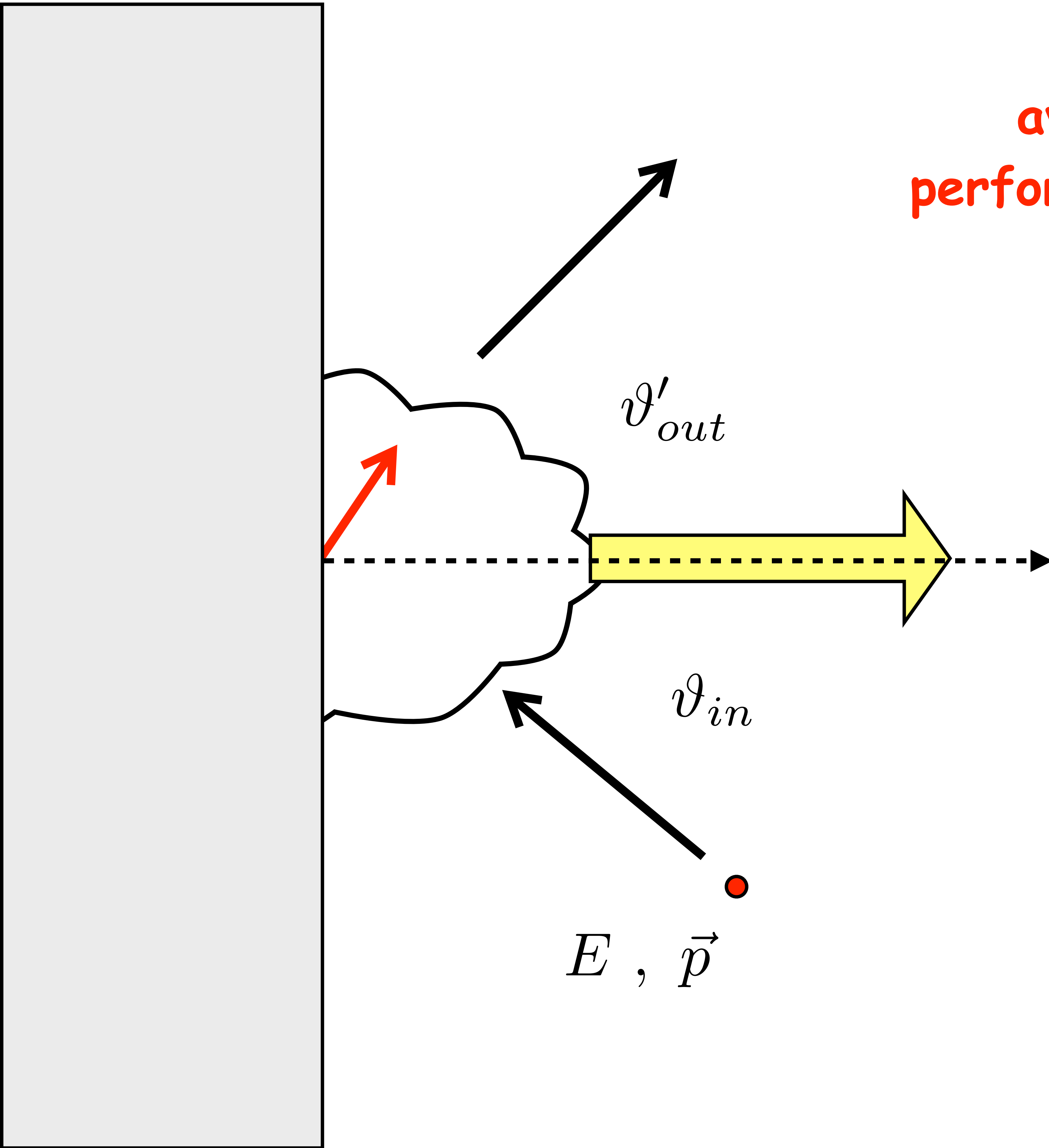


averages have to be performed over the interval:

$$0 < \vartheta < \frac{\pi}{2}$$

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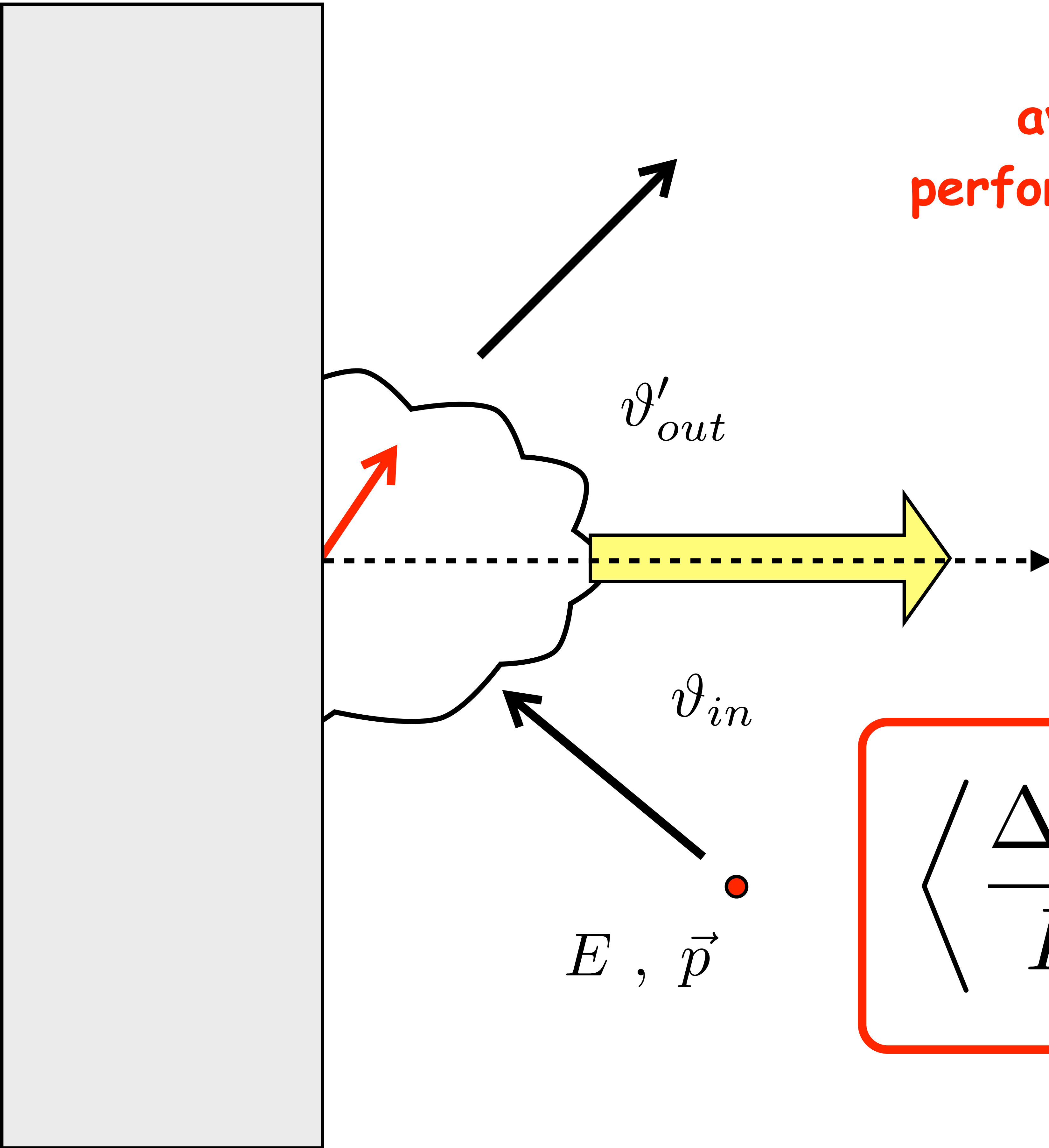



averages have to be performed over the interval:

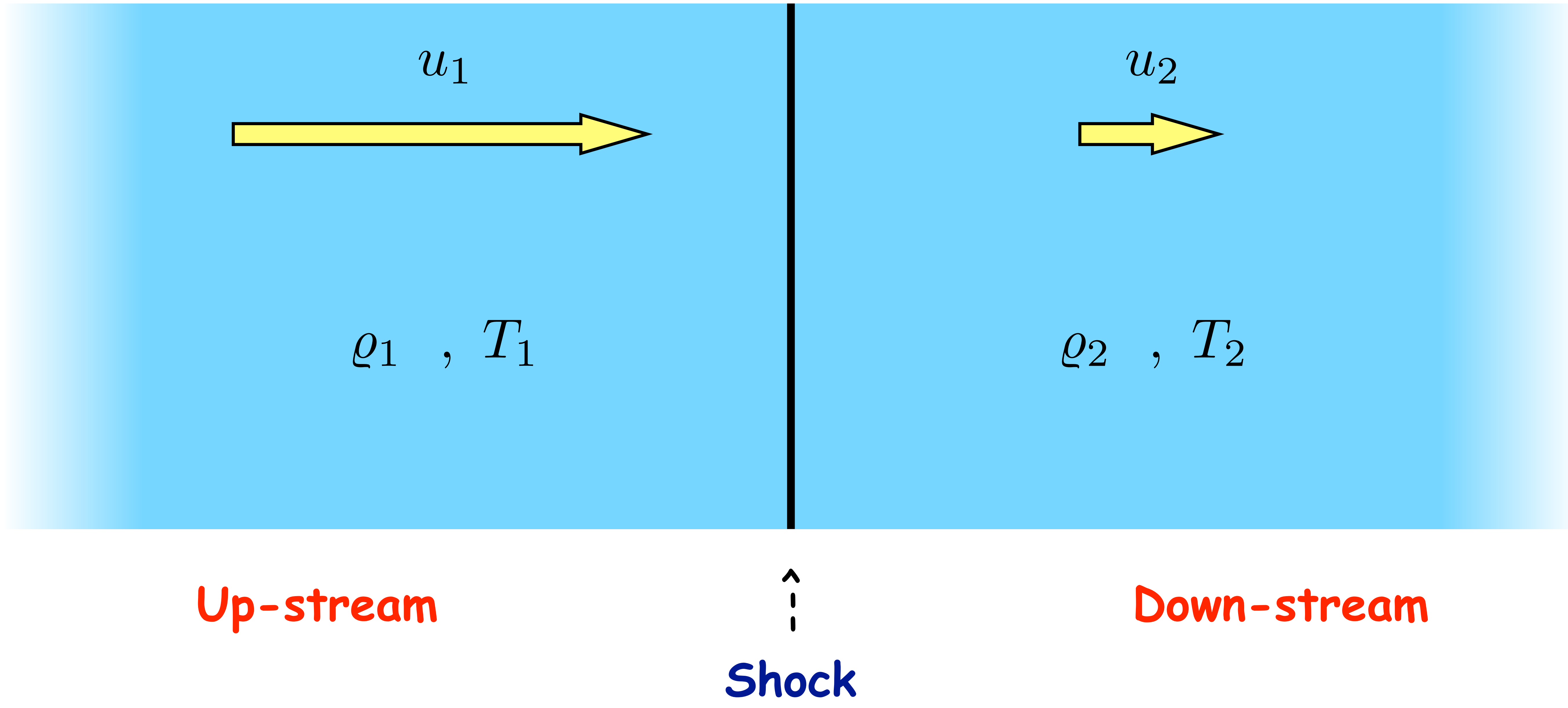
$$0 < \vartheta < \frac{\pi}{2}$$

First order!

$$\left\langle \frac{\Delta E}{E} \right\rangle \approx \frac{4}{3} \beta$$

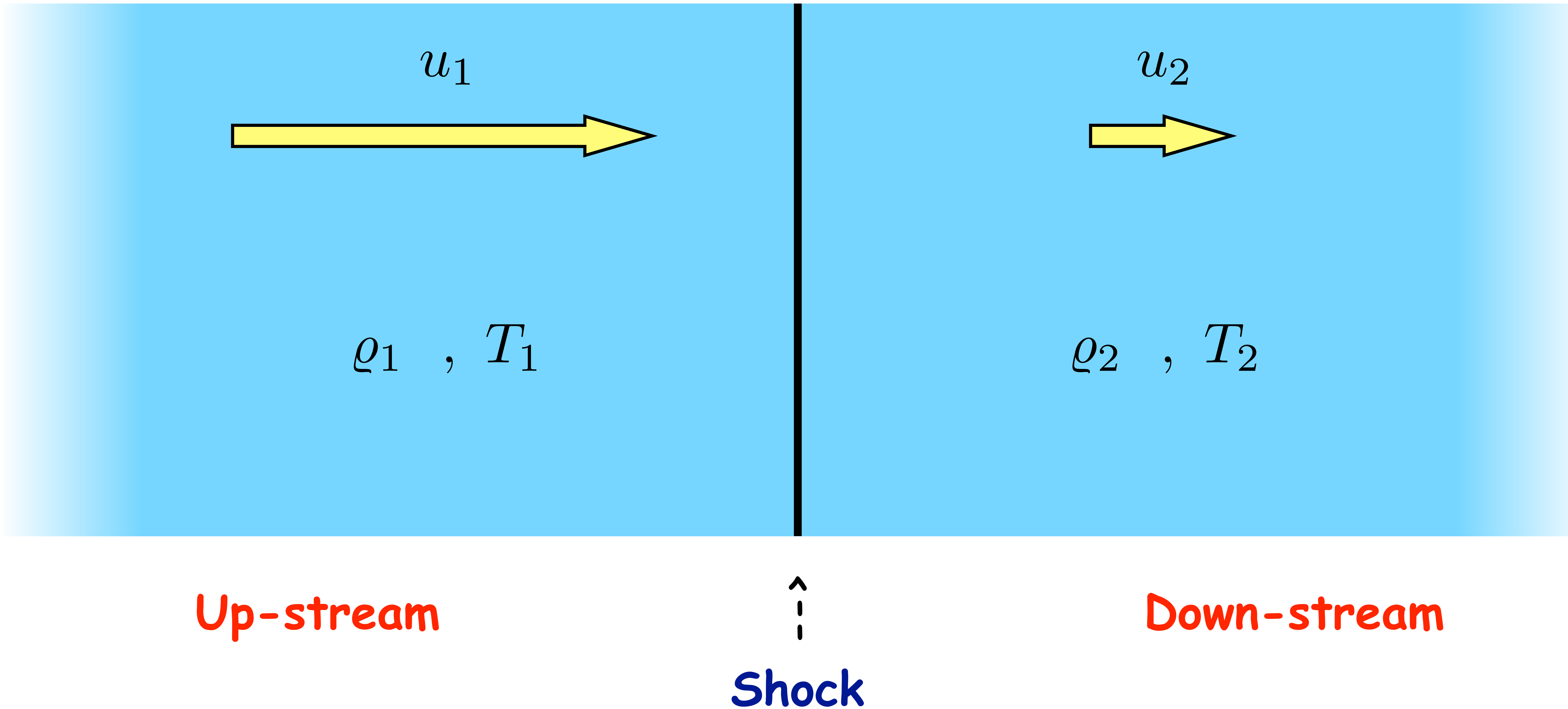
# Shock waves in one slide

Shock rest frame



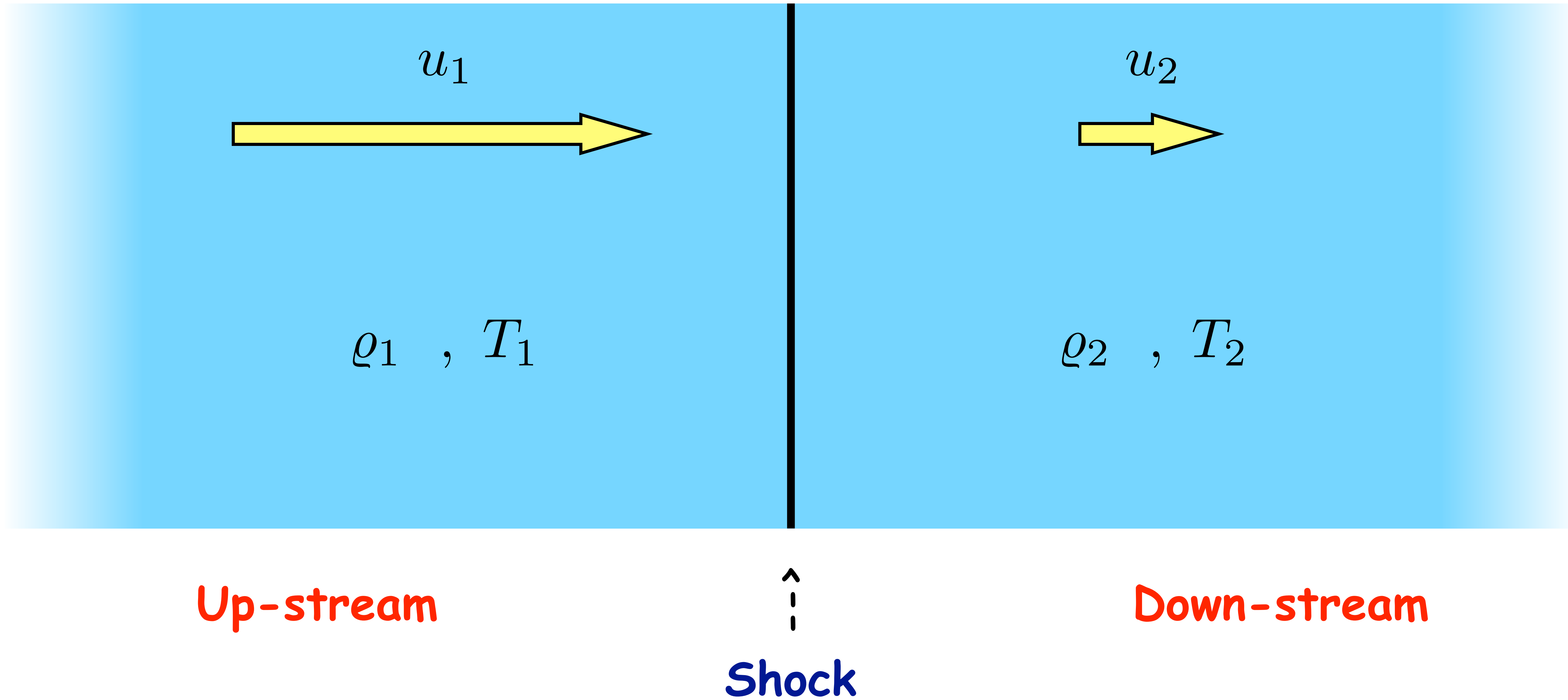
# Shock waves in one slide

## Shock rest frame




In 1960 (!) F. Hoyle (first?) suggests that shocks accelerate CRs

$$\frac{\varrho_2}{\varrho_1} = \frac{u_1}{u_2} = \frac{\gamma + 1}{\gamma - 1} = 4 \qquad p_2 = \frac{2}{\gamma + 1} \varrho_1 u_1^2$$

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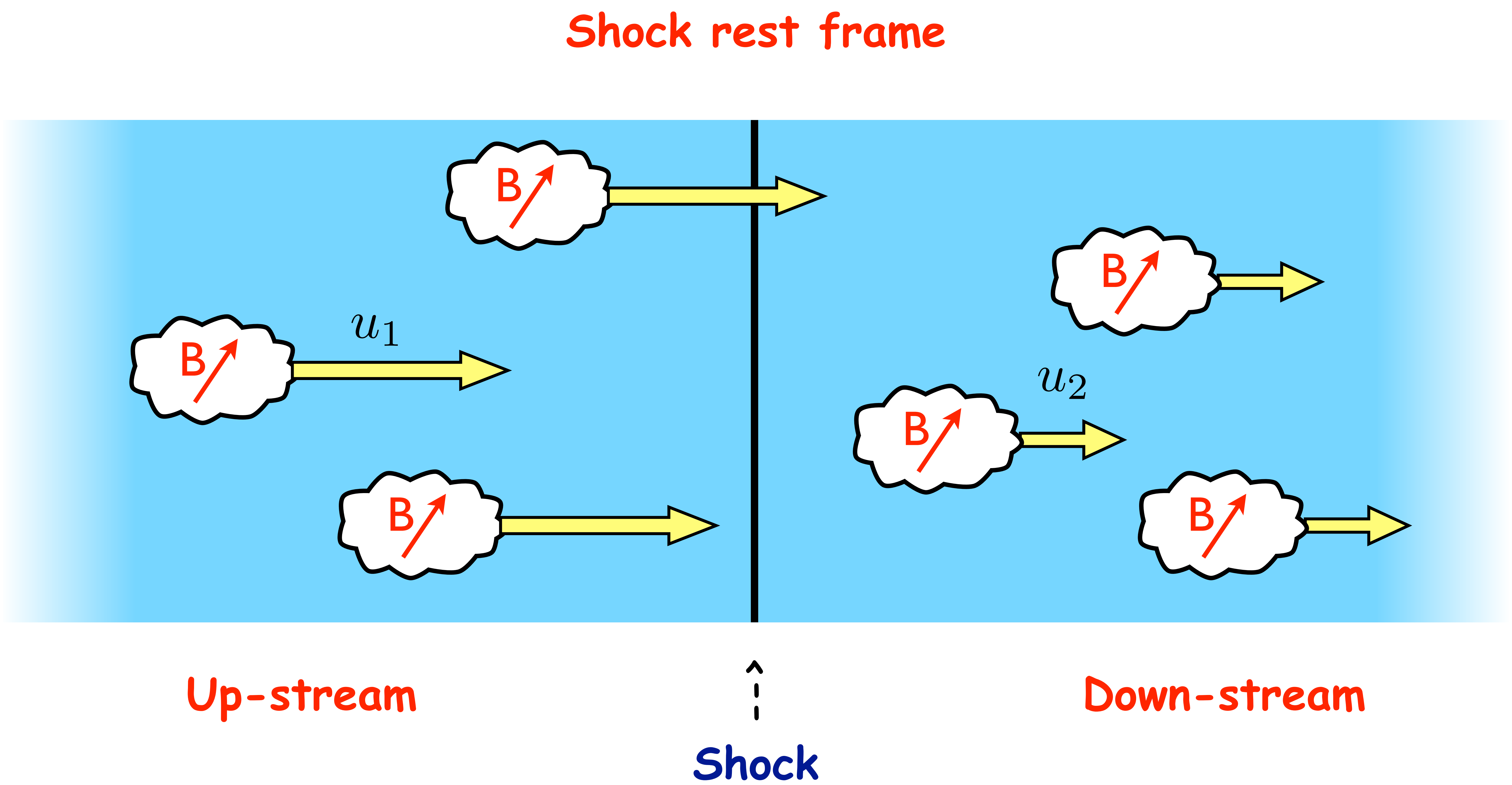
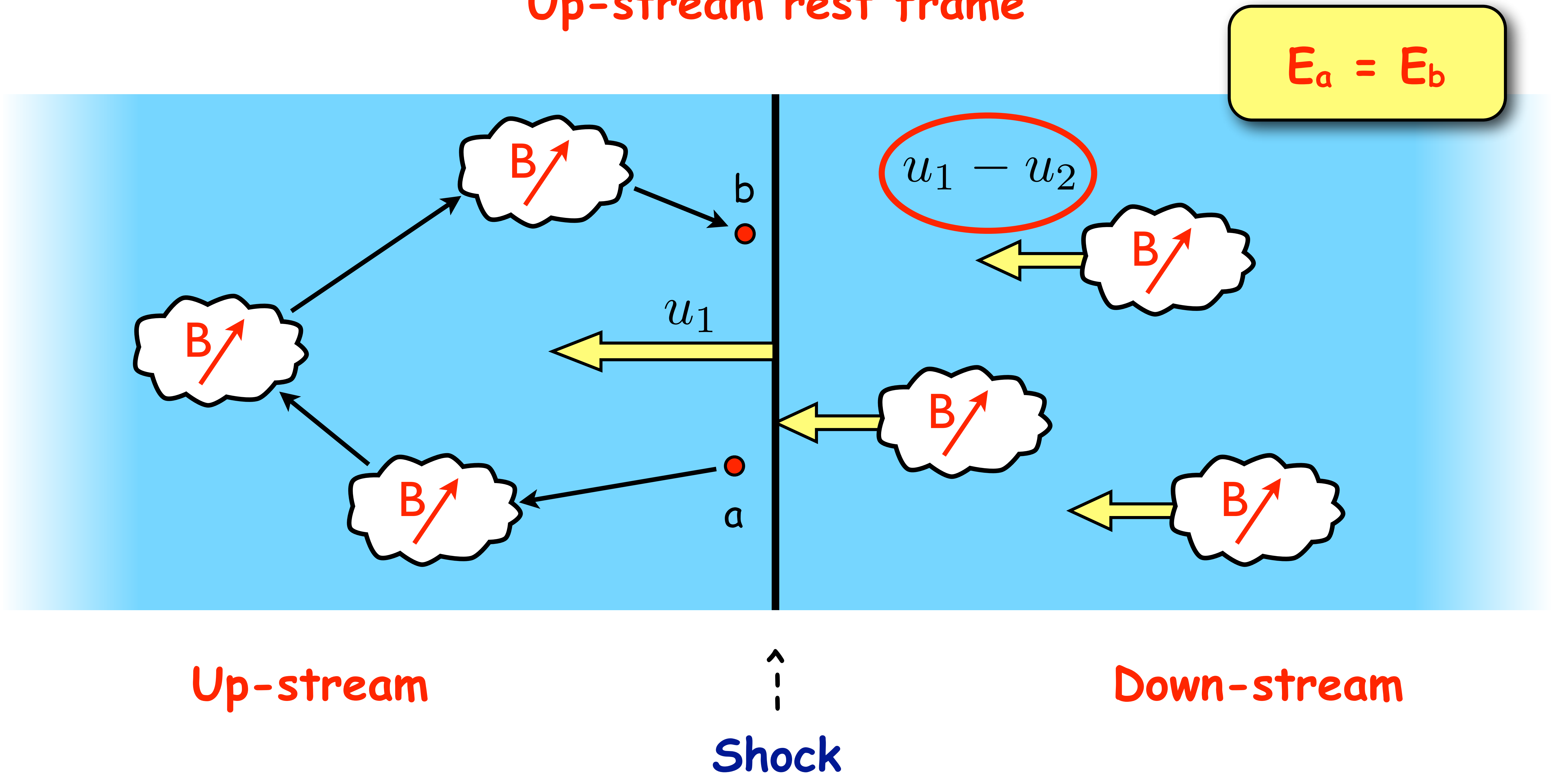
# Diffusive Shock Acceleration




Krymskii 1977, Axford et al. 1977, Blandford & Ostriker 1978, Bell 1978

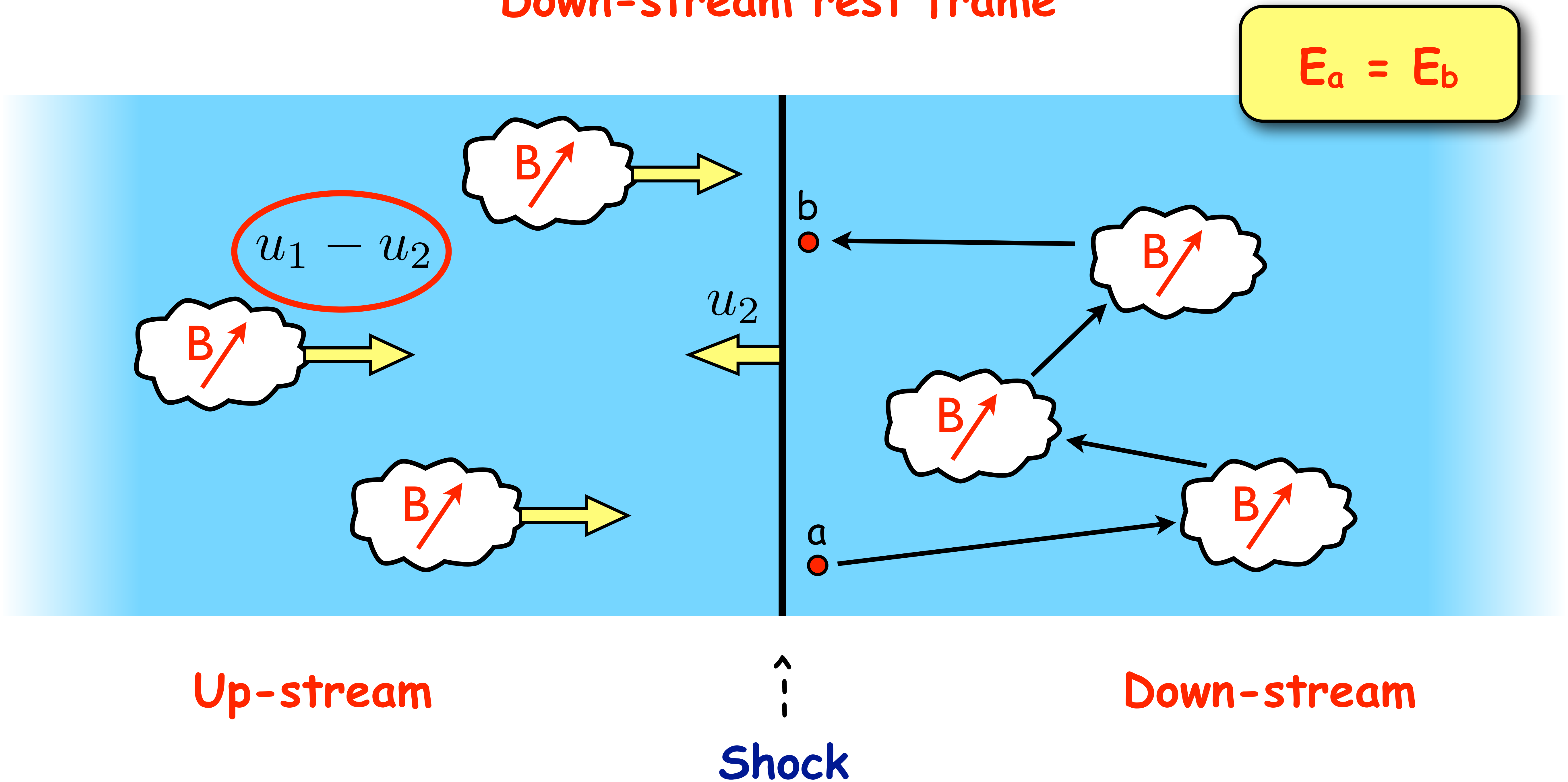
# Diffusive Shock Acceleration

Up-stream rest frame



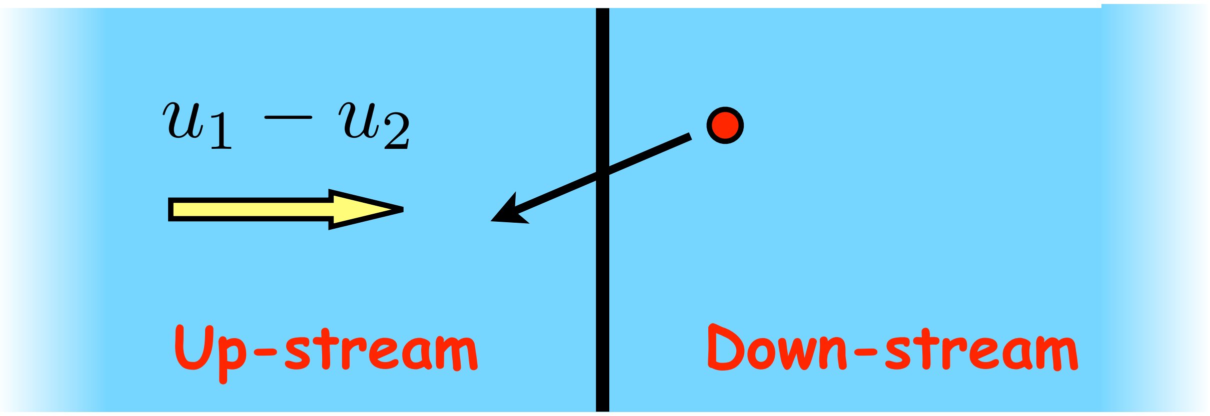
# Diffusive Shock Acceleration

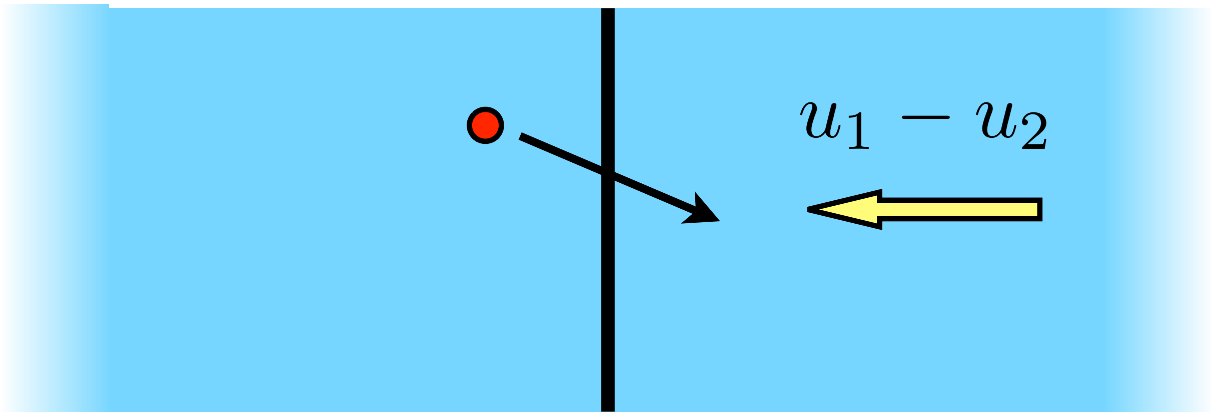
Down-stream rest frame



# Diffusive Shock Acceleration

## Symmetry

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# Diffusive Shock Acceleration

## Symmetry


Every time the particle crosses the shock (up -> down or down -> up), it undergoes an head-on collision with a plasma moving with velocity $u_1$-$u_2$

## Asymmetry

(Infinite and plane shock:) Upstream particles always return the shock, while downstream particles may be advected and never come back to the shock

# Universality
# of diffusive shock acceleration

Let's search for a test-particle solution

Assumption: scattering is so effective at shocks that the distribution of particles is isotropic

-> an universal solution of the problem can be found

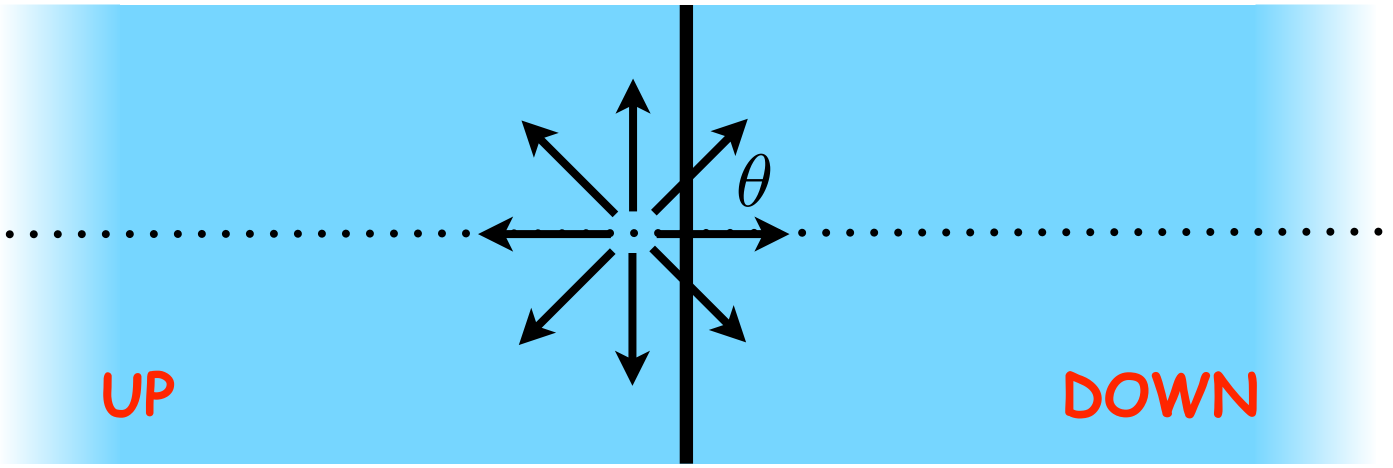
# Rate at which particles cross the shock

**Let's calculate $R_{in}$...**

n -> density of accelerated particles close to the shock

n is isotropic: $$dn = \frac{n}{4\pi}\, d\Omega$$

velocity across the shock: $c\ \cos(\theta)$



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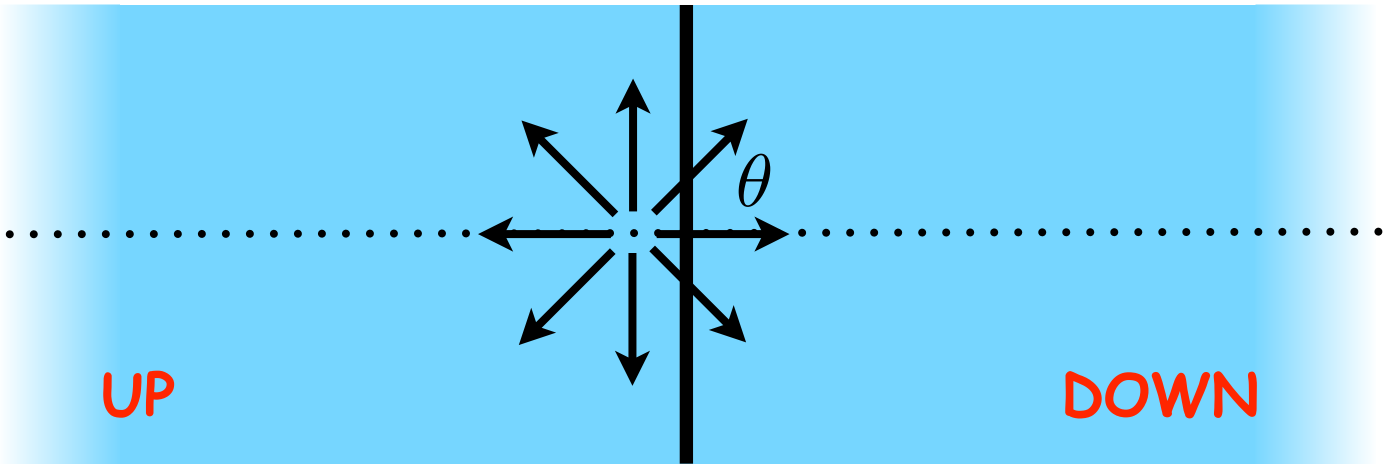


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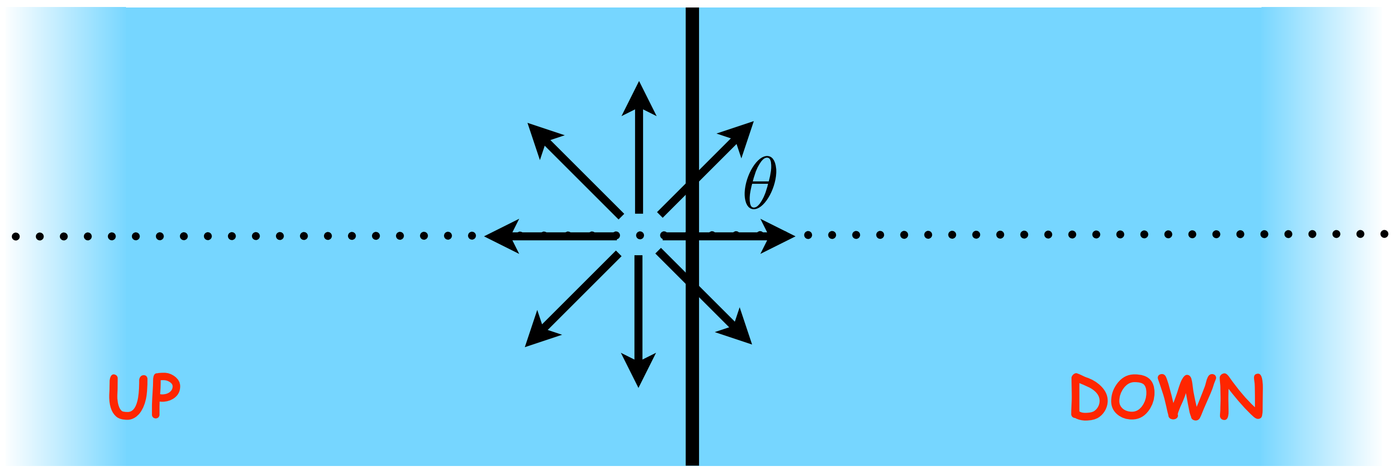


$$R_{in} = \int_{up\rightarrow down} \mathrm{d}n\ c\ \cos(\theta) = \frac{n\ c}{4\pi}\int_0^{\frac{\pi}{2}} \cos(\theta)\sin(\theta)\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\psi = \frac{1}{4}\ n\ c$$

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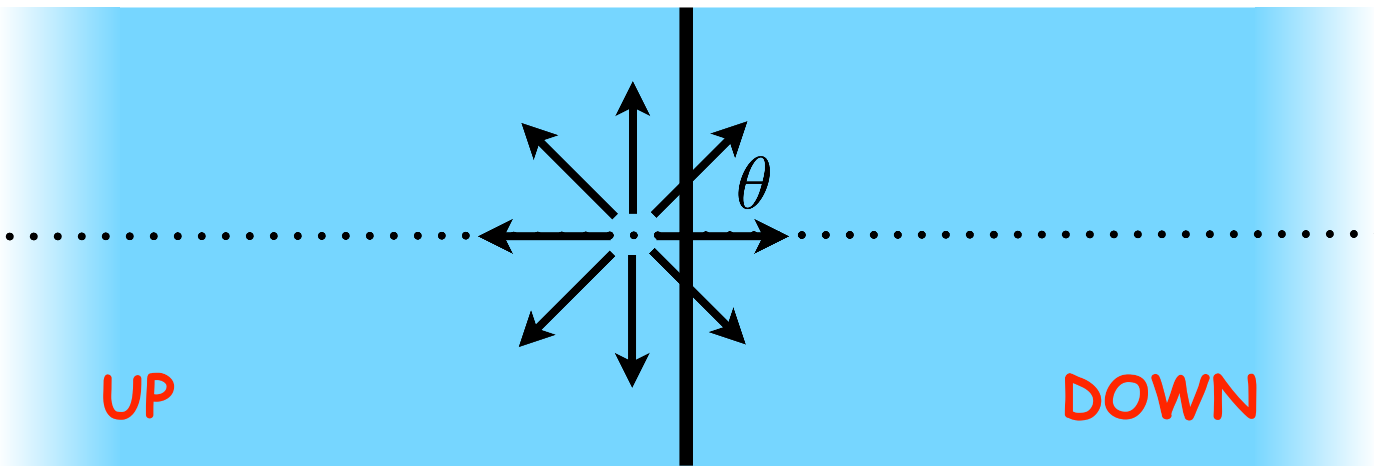
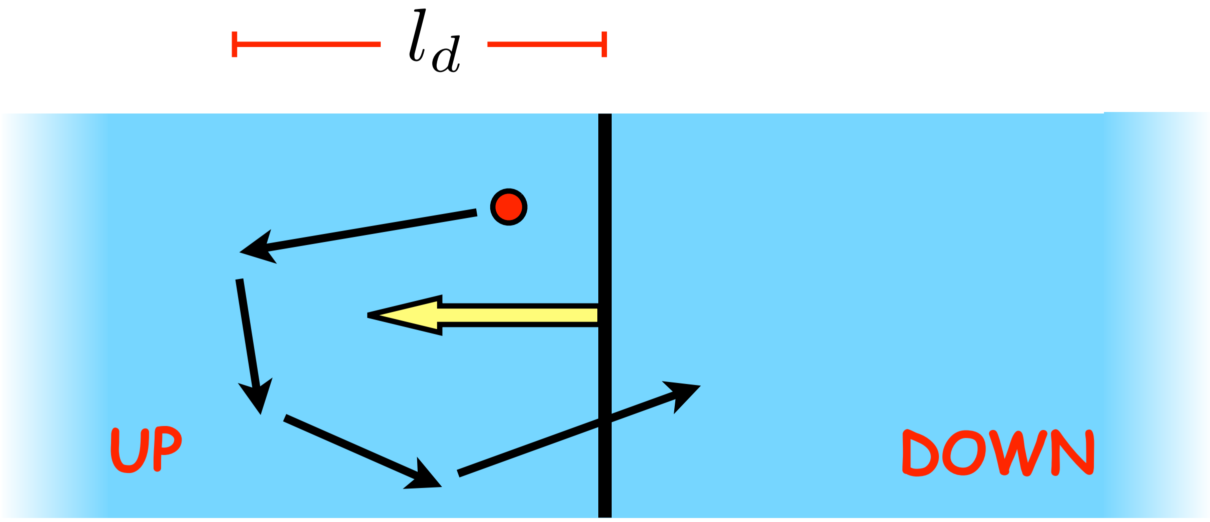



$$R_{in} = \int_{up \rightarrow down} dn \; c \; \cos(\theta) = \frac{n \; c}{4\pi} \int_0^{\frac{\pi}{2}} \cos(\theta) \sin(\theta) d\theta \int_0^{2\pi} d\psi = \frac{1}{4} \, n \; c$$

-> the same result is obtained for down -> up

# Residence time upstream

-> let's find the STEADY STATE solution upstream of the shock




behavior of particles is diffusive
D(E) -> diffusion coefficient

very poorly constrained (from both observations and theory)

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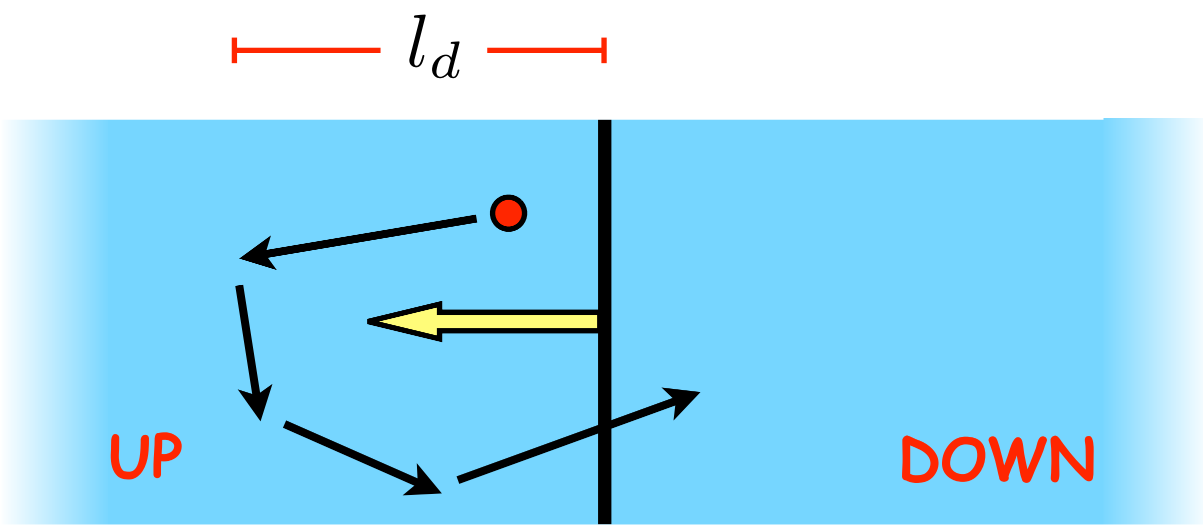


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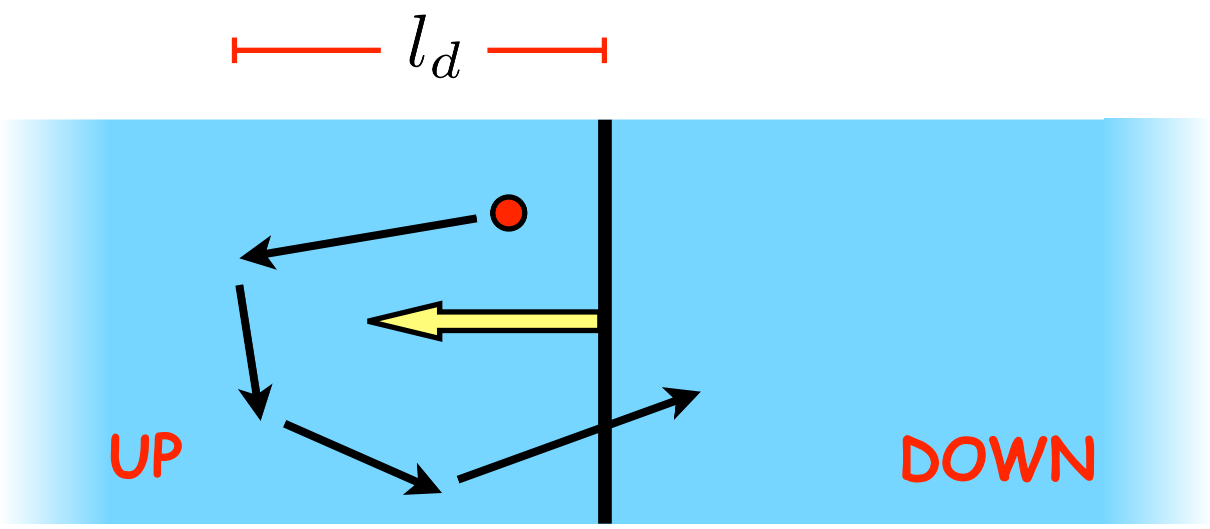


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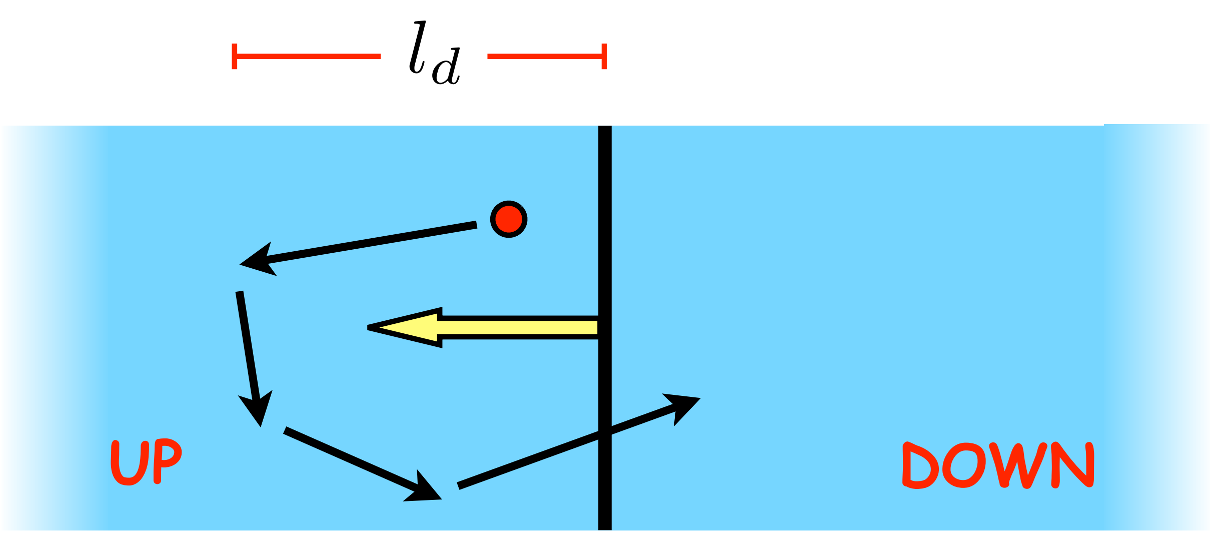


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$$l_d \approx \frac{D}{u_1}$$

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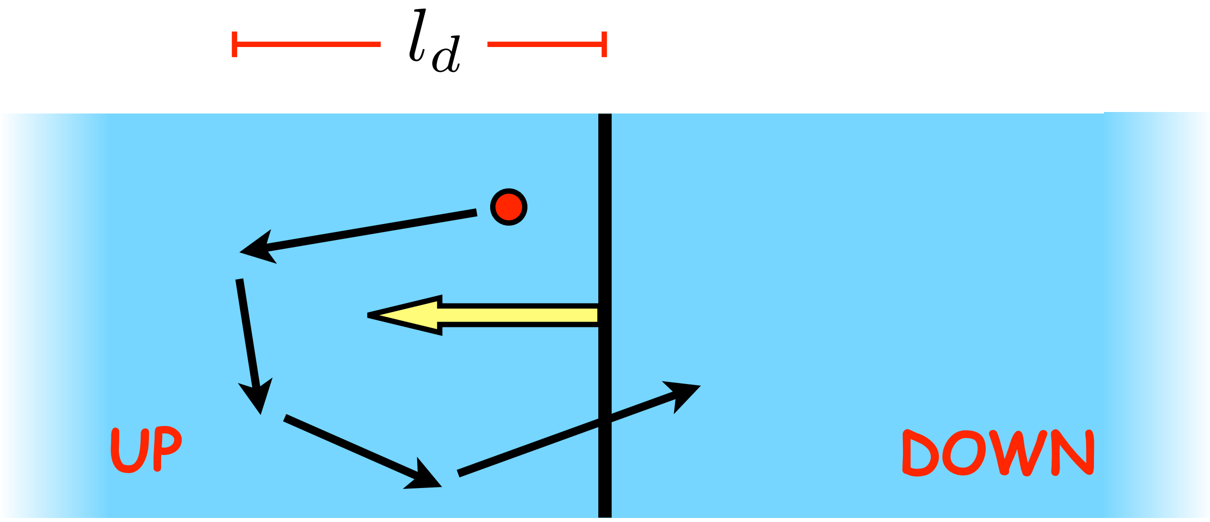


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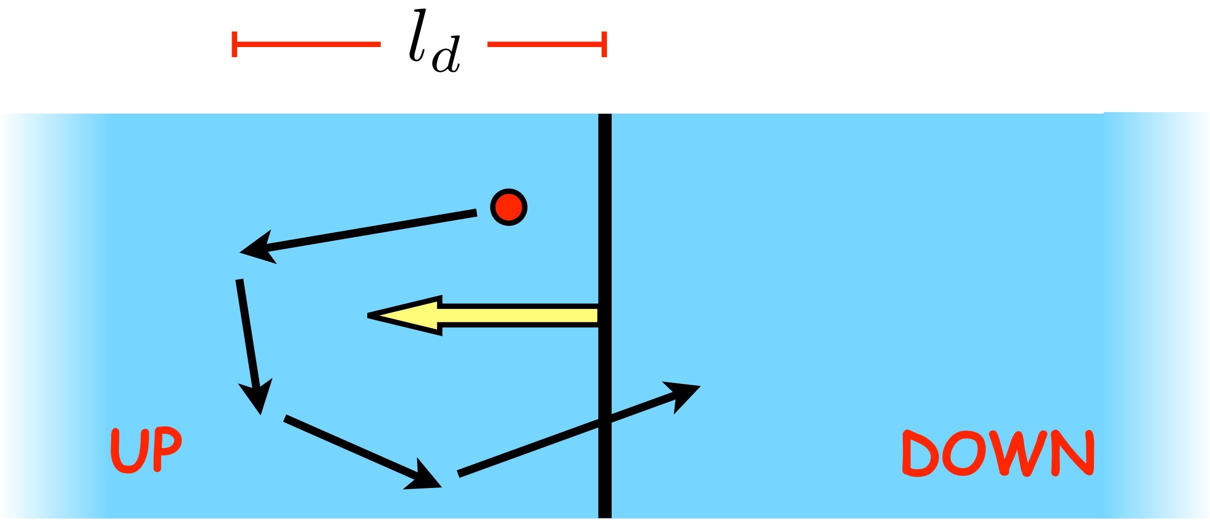


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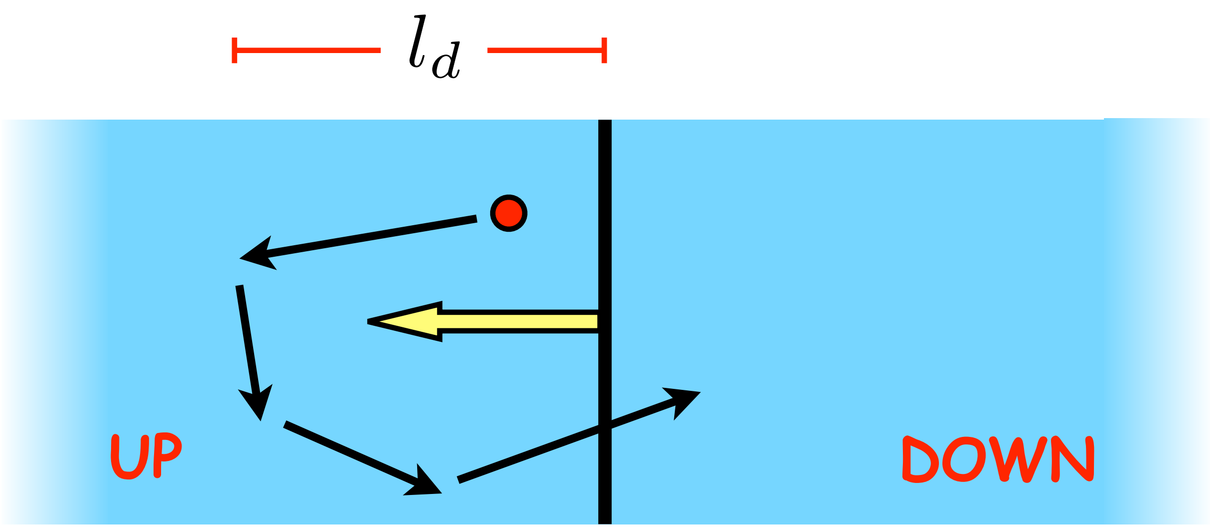


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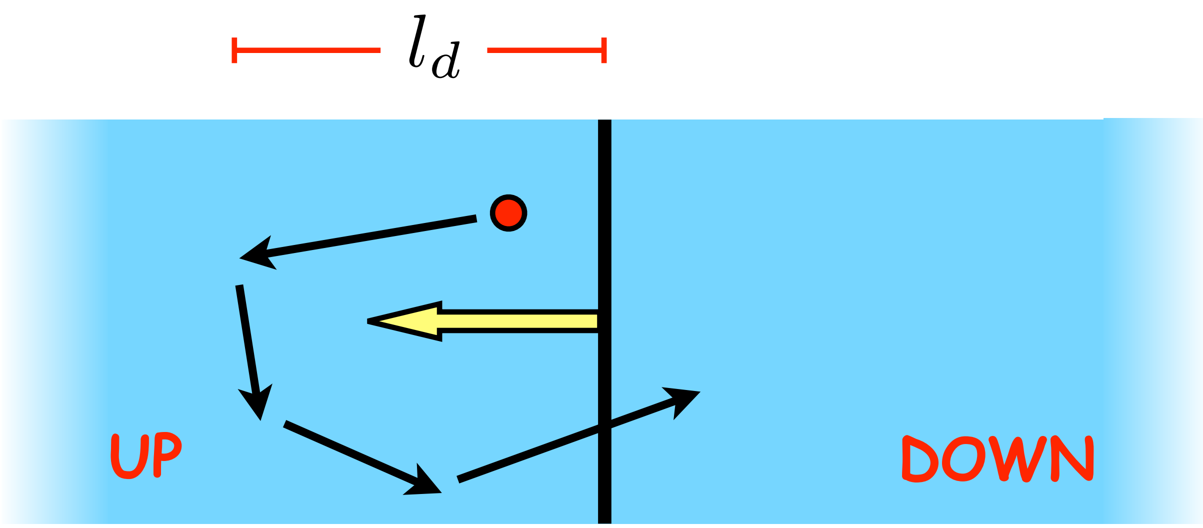


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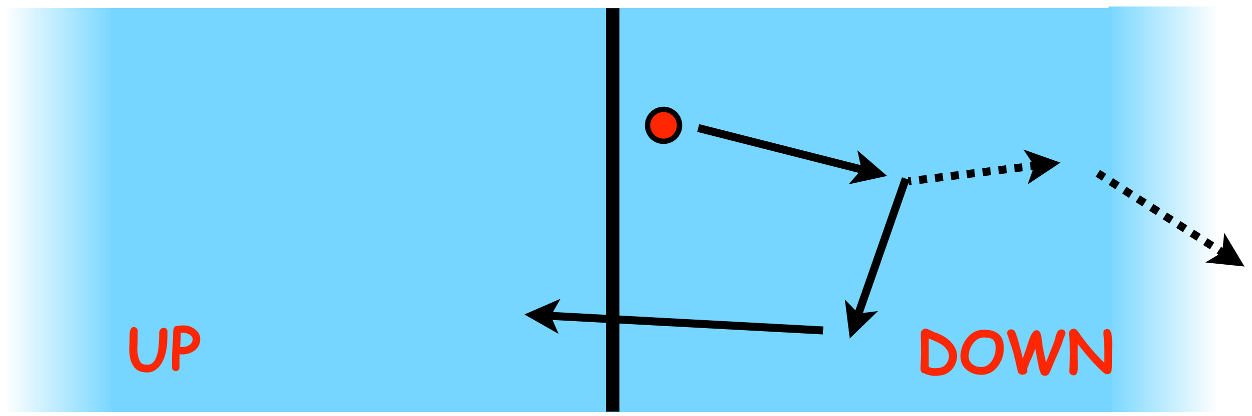
# Residence time downstream

-> a bit more subtle...

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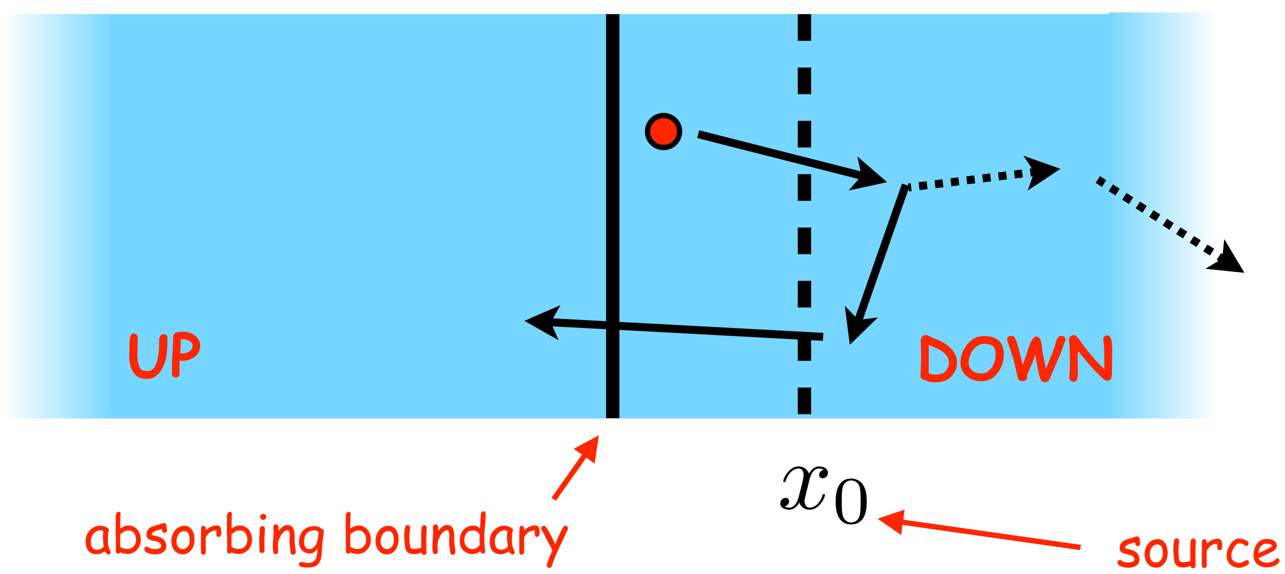


$n$ is constant downstream of the shock

$$u_2 \frac{\partial n}{\partial x} = D \frac{\partial^2 n}{\partial x^2} + Q \, \delta(x - x_0)$$

$$n(0) = 0$$

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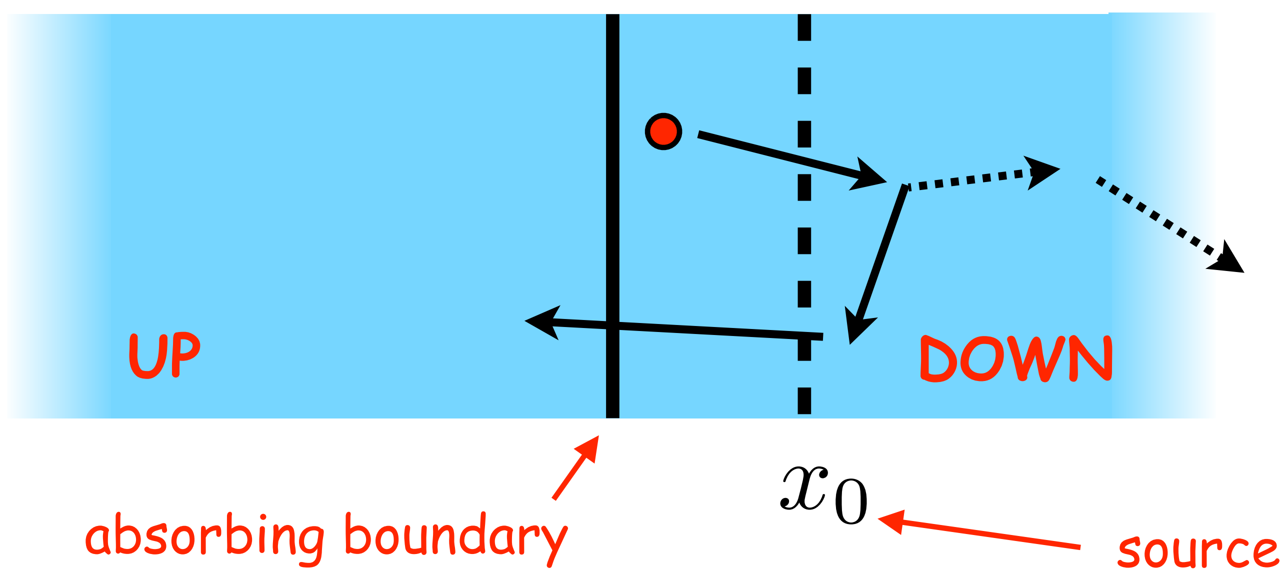


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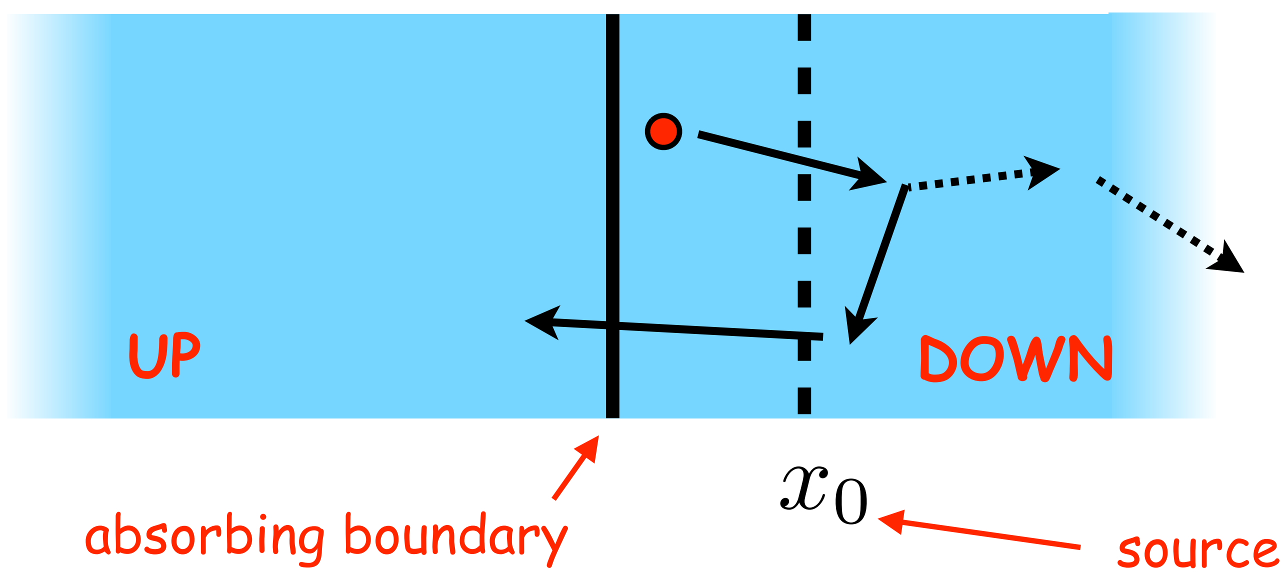


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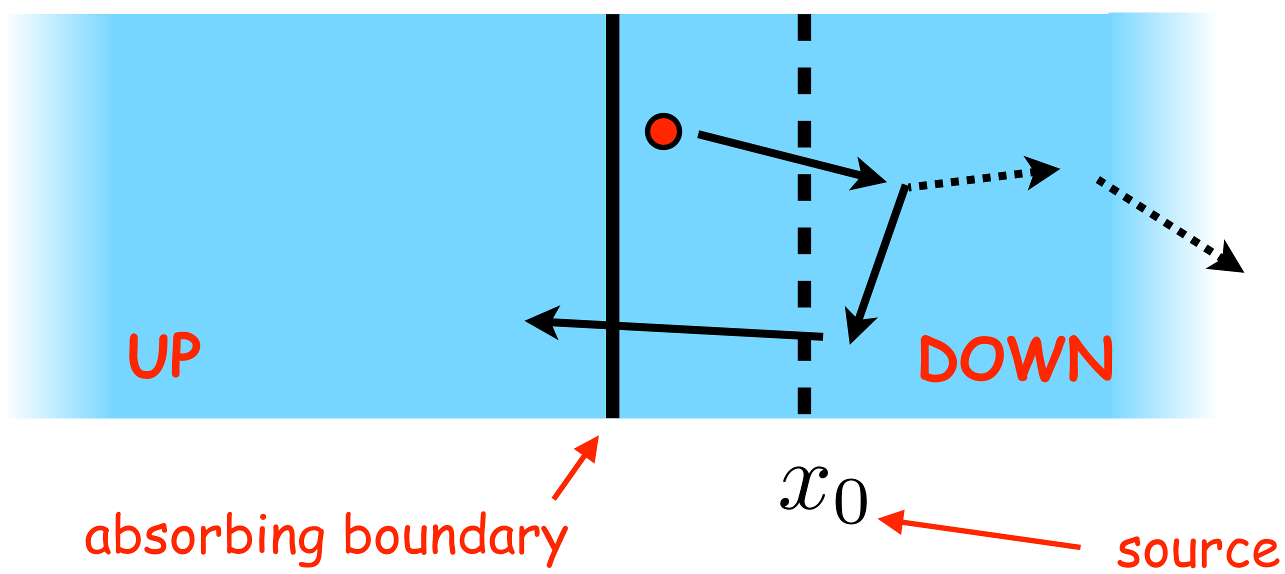


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we need to know the returning flux $D \frac{\partial n}{\partial x}\Big|_{x=0} \longrightarrow P_{ret} = \frac{D \frac{\partial n}{\partial x}\Big|_{x=0}}{Q}$

$$P_{ret} = \exp\left(-\frac{x_0 u_2}{D}\right)$$

# Residence time downstream

number of downstream particles that will return to the shock:

$$\int_0^\infty \mathrm{d}x \; P_{ret}(x) \; n \; = \; \frac{D \; n}{u_2}$$

**same expression upstream!**

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**mean residence time upstream <-> mean residence time downstream**

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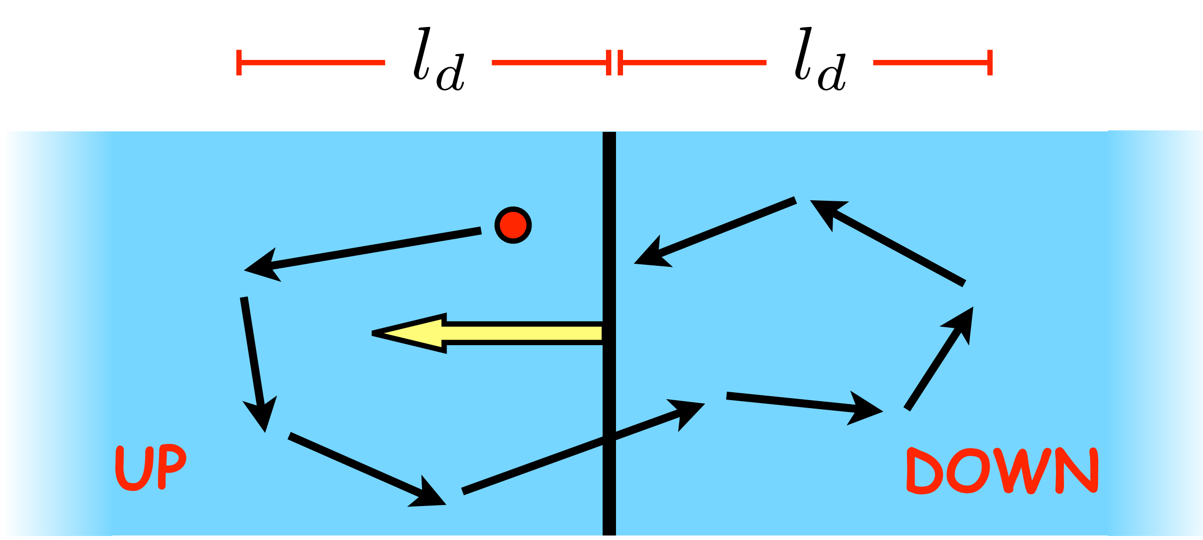


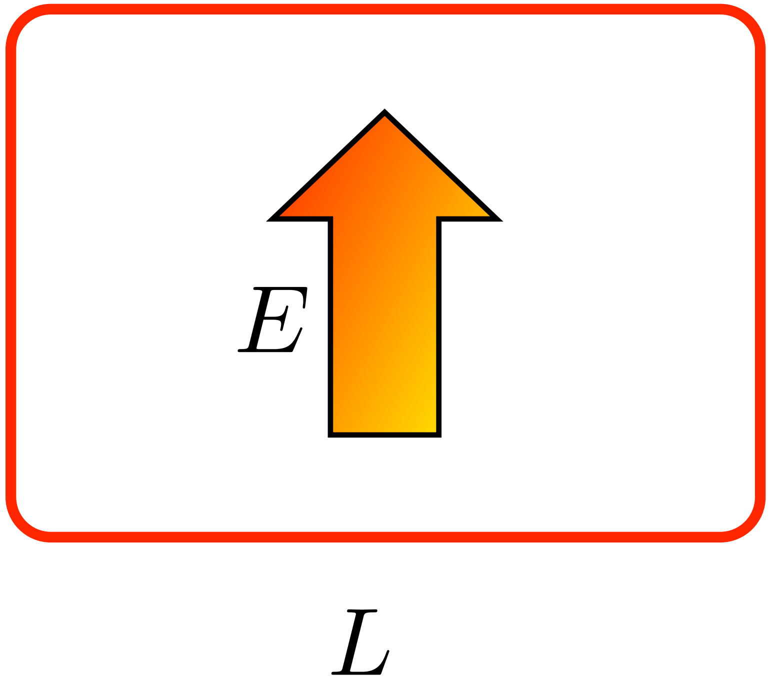
# Acceleration rate

$L$

everything happens within a "box" of size:

$$L = \frac{D}{u_1} + \frac{D}{u_2} \quad = \; c \; \frac{\tau_{up} + \tau_{down}}{4}$$

# Acceleration rate

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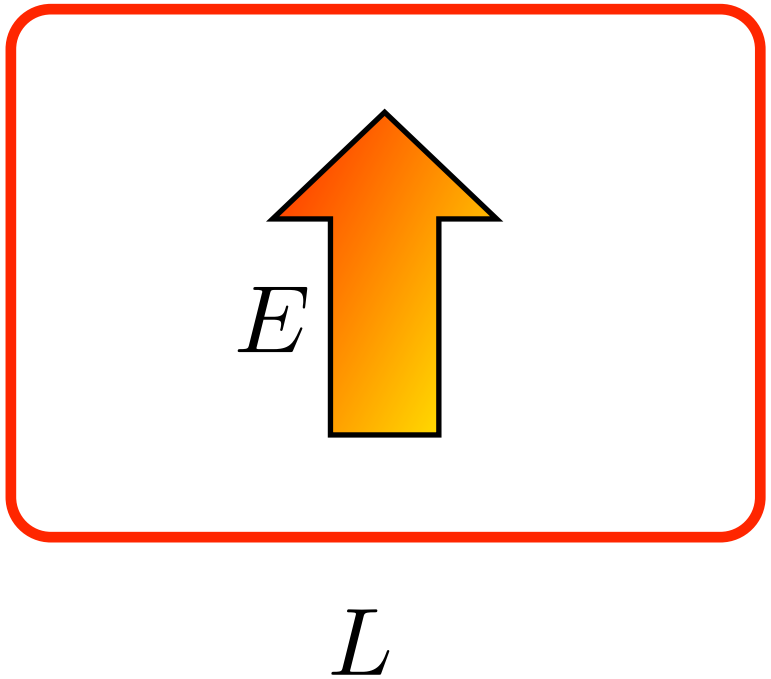


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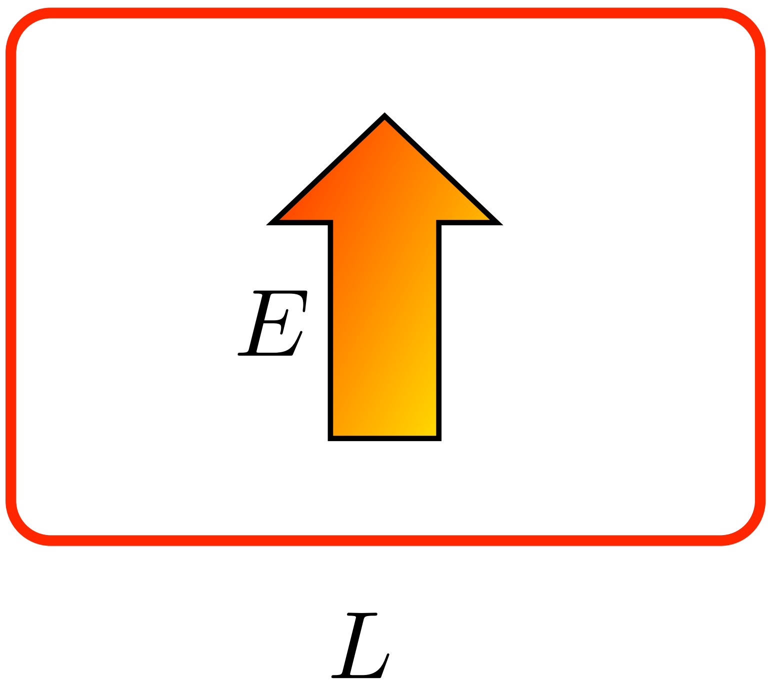
# Box model for shock acceleration

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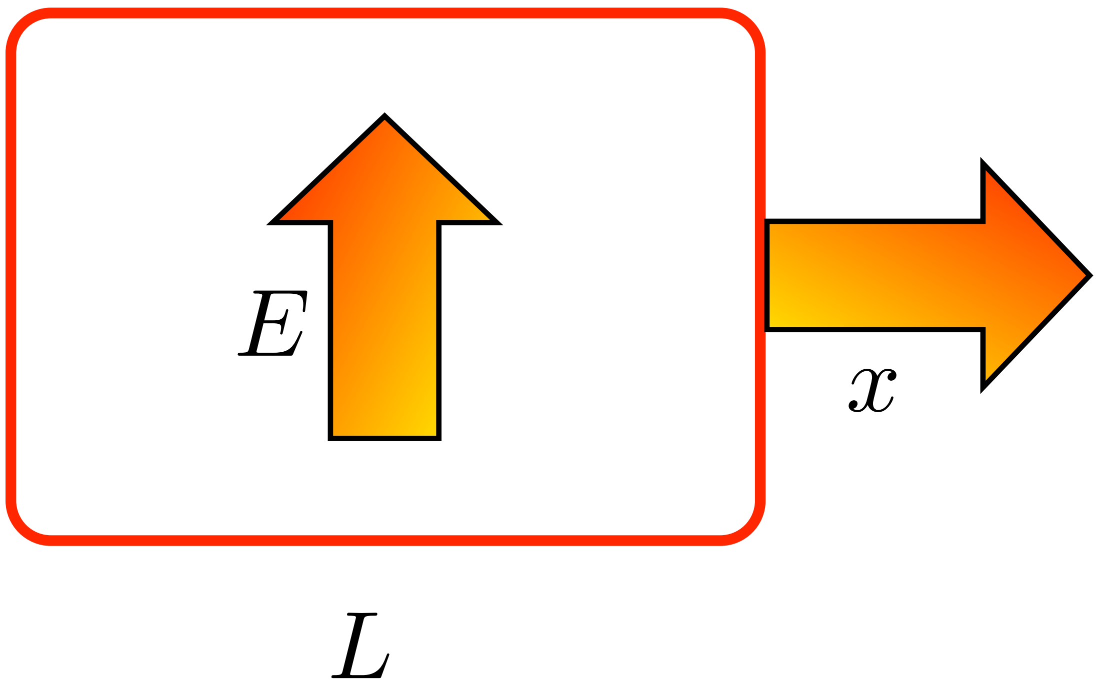


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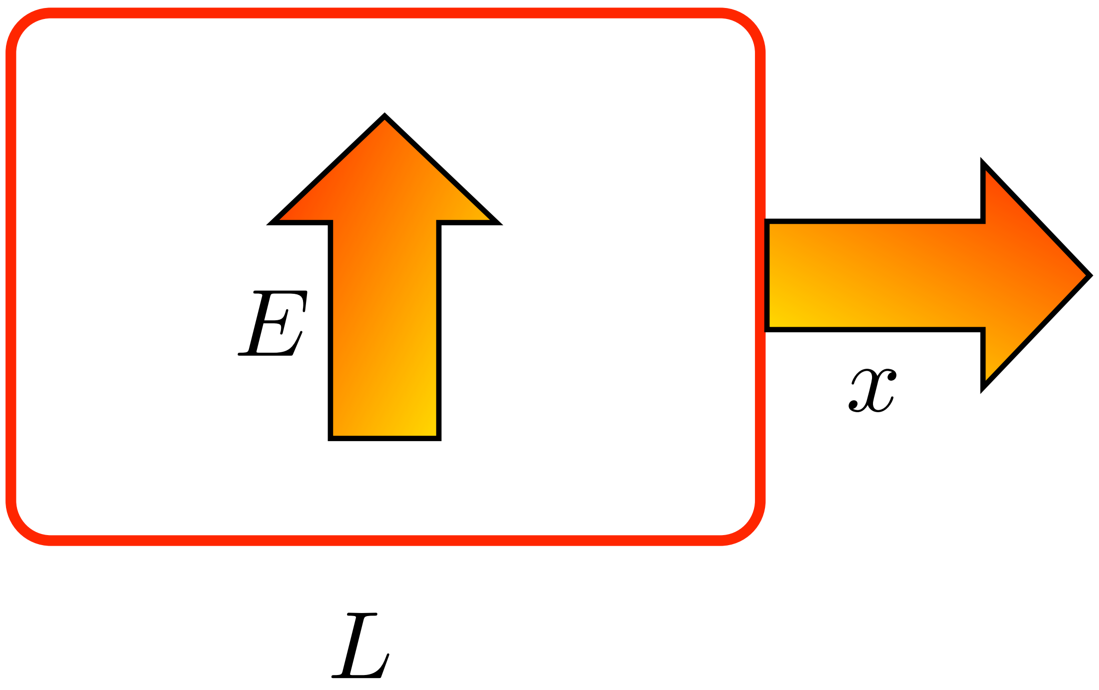


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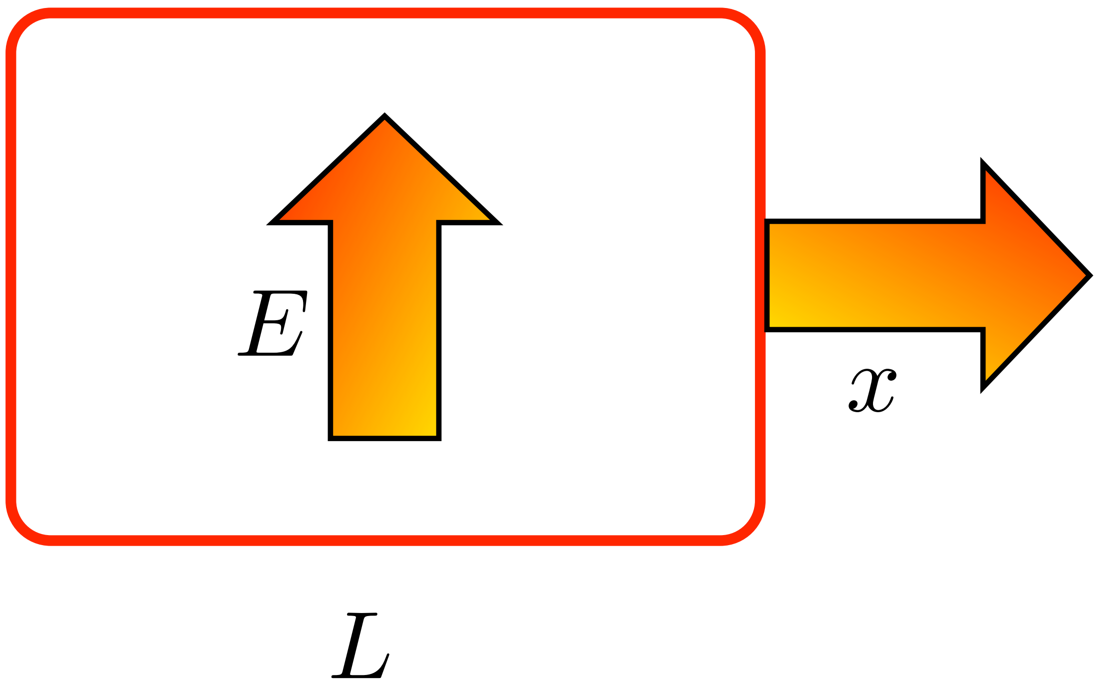


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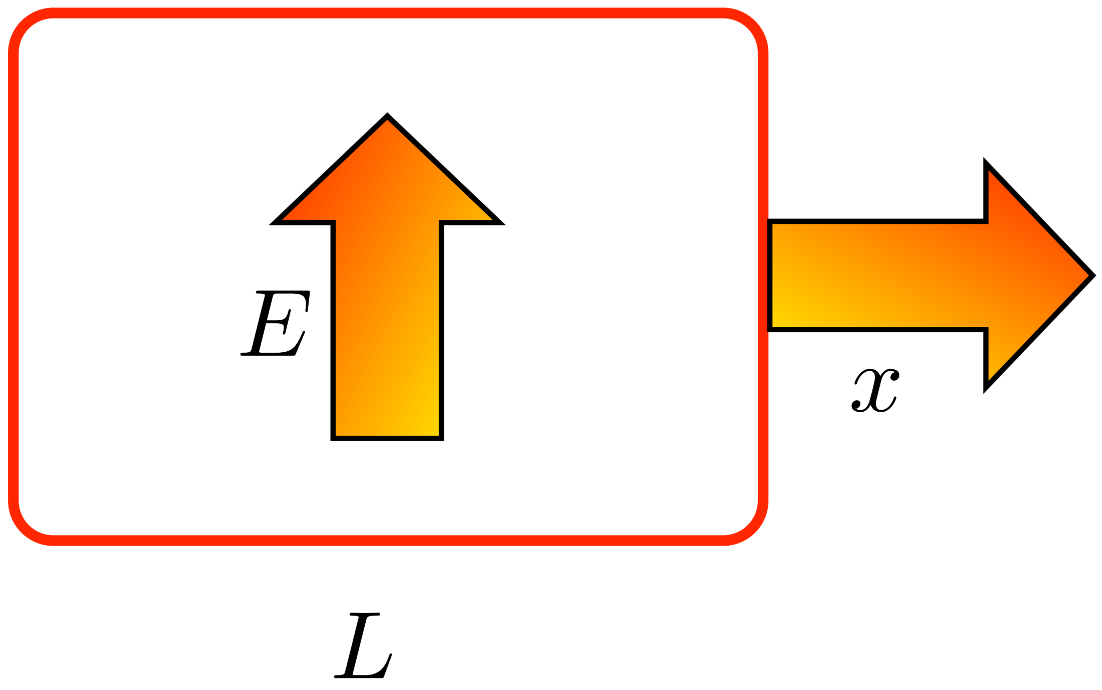


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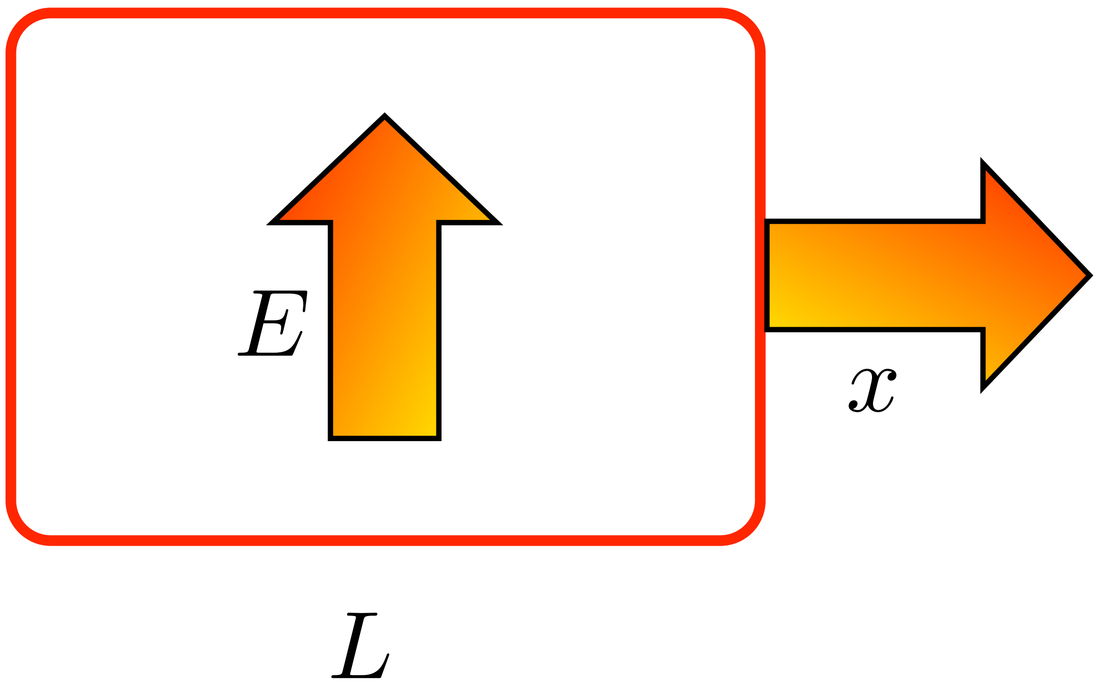


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Assumption: scattering is so effective at shocks that the distribution of particles is isotropic

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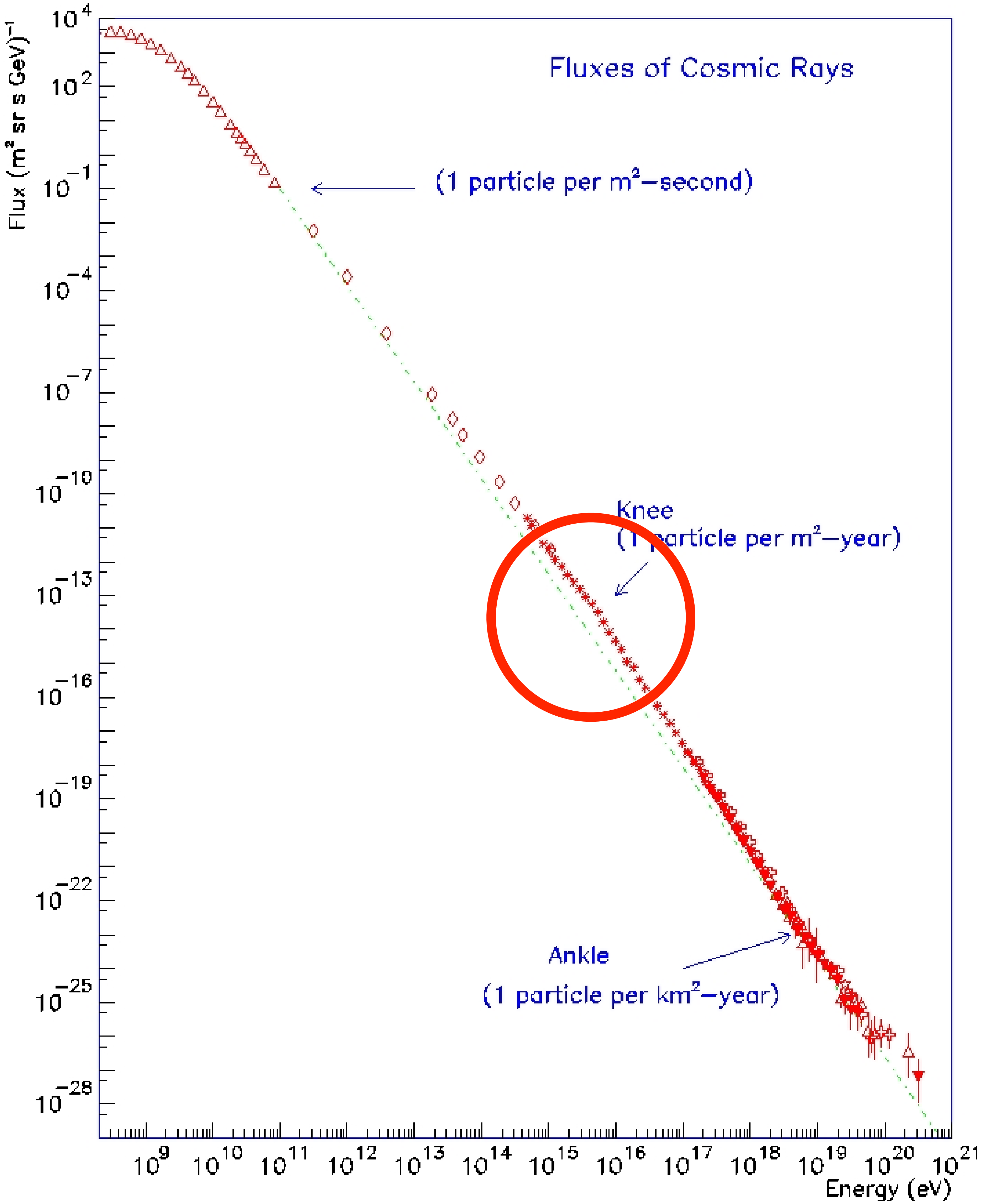


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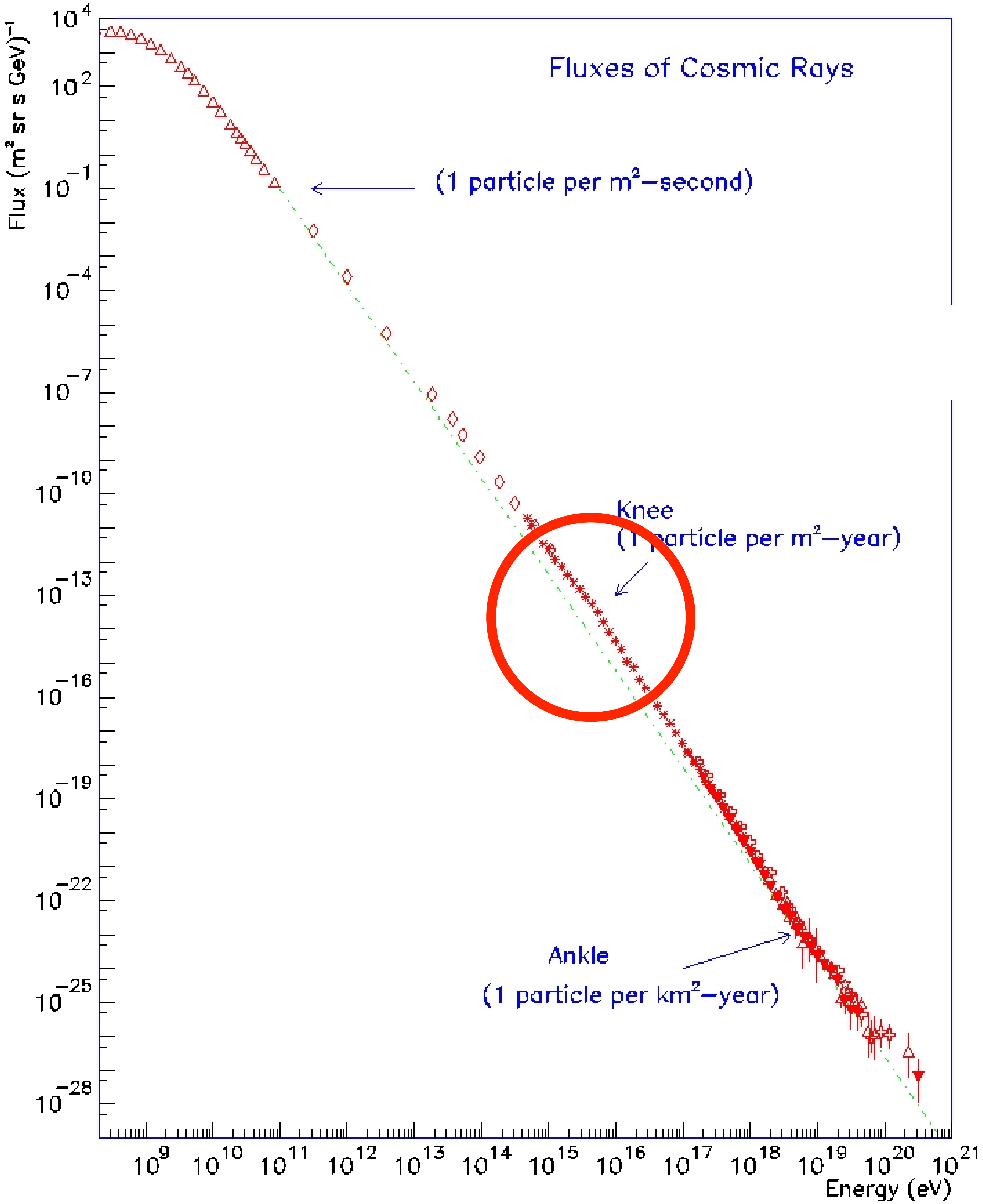


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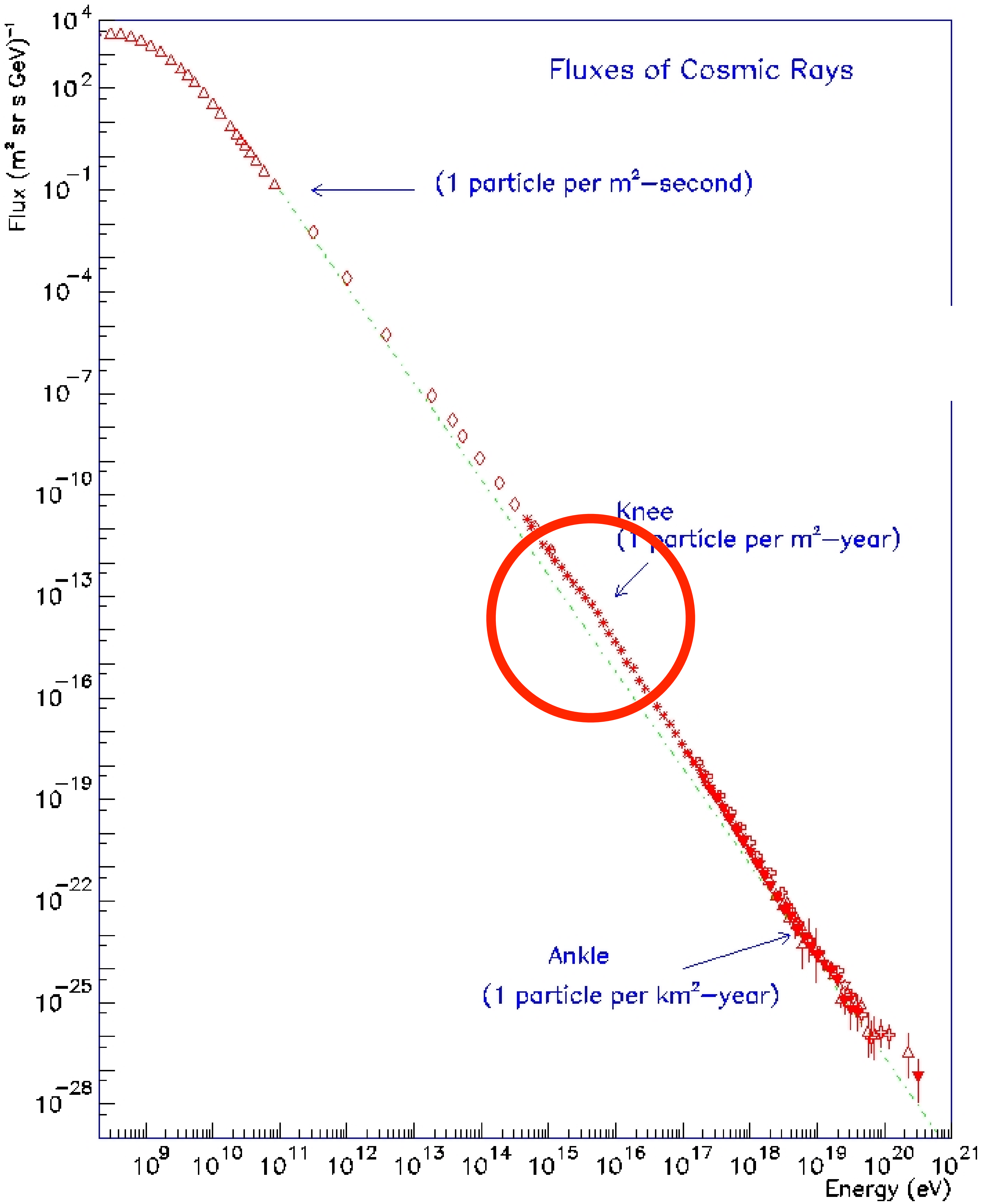


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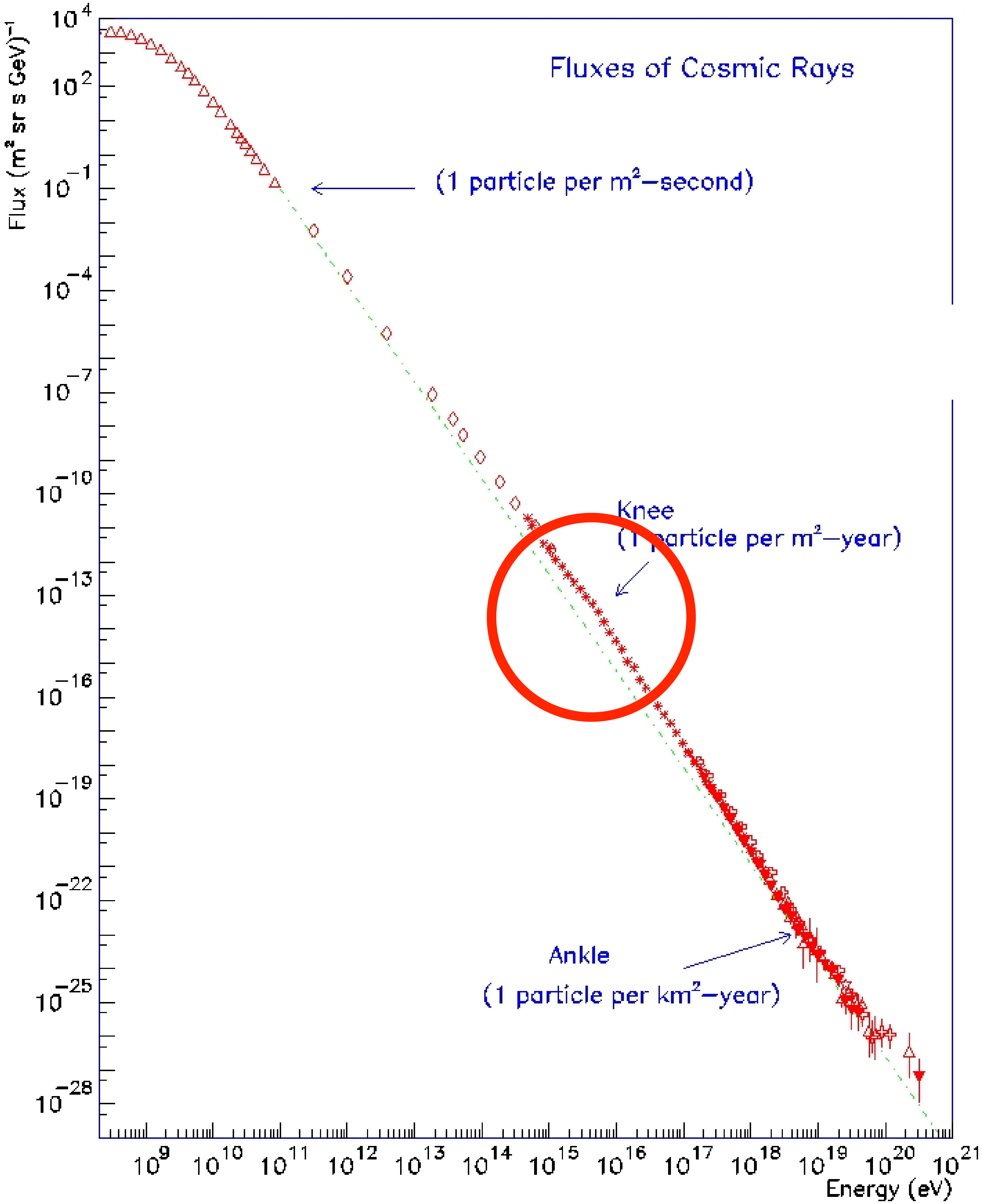


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>10 times below the knee

# How to solve the problem

horribly oversimplified, for a proper treatment see Bell 2004

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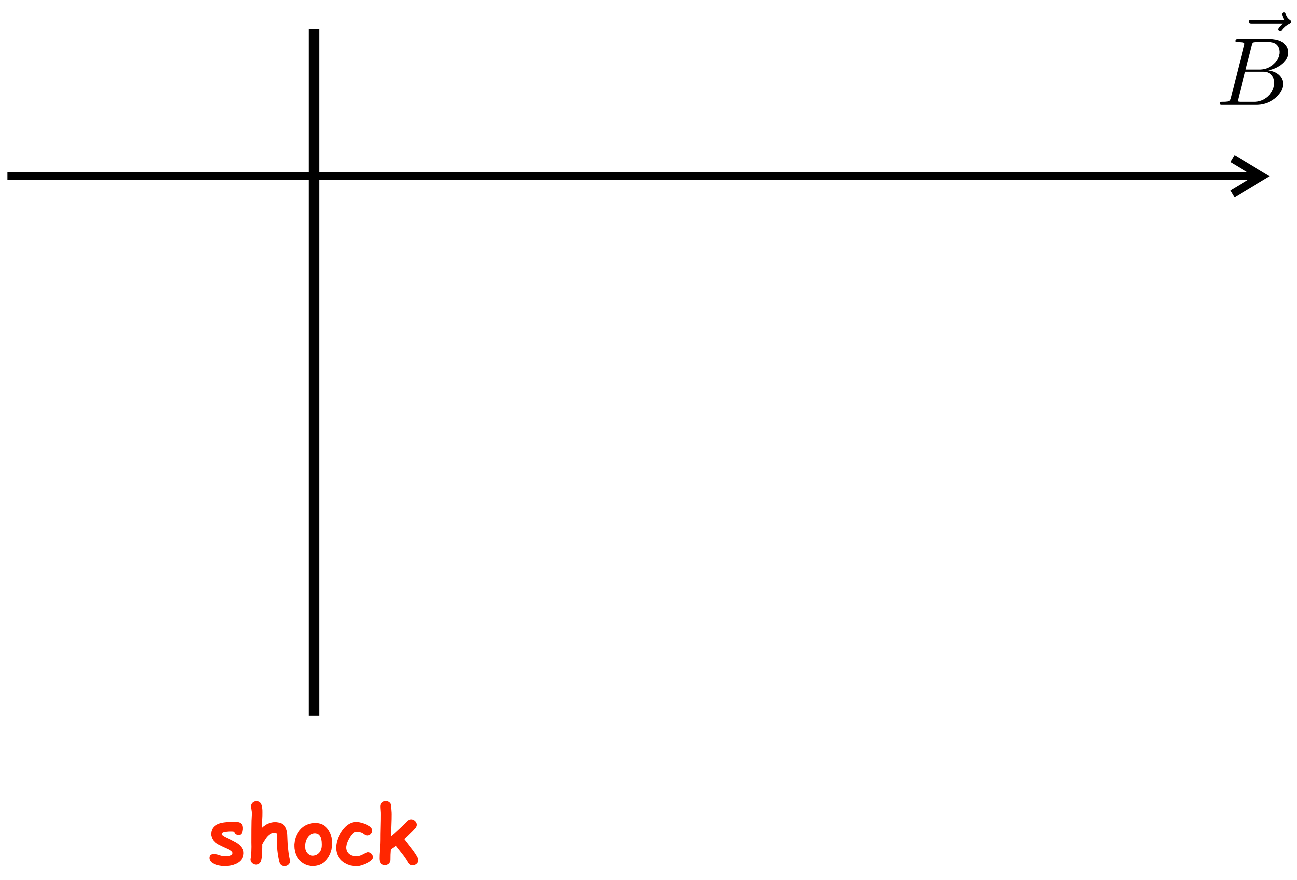

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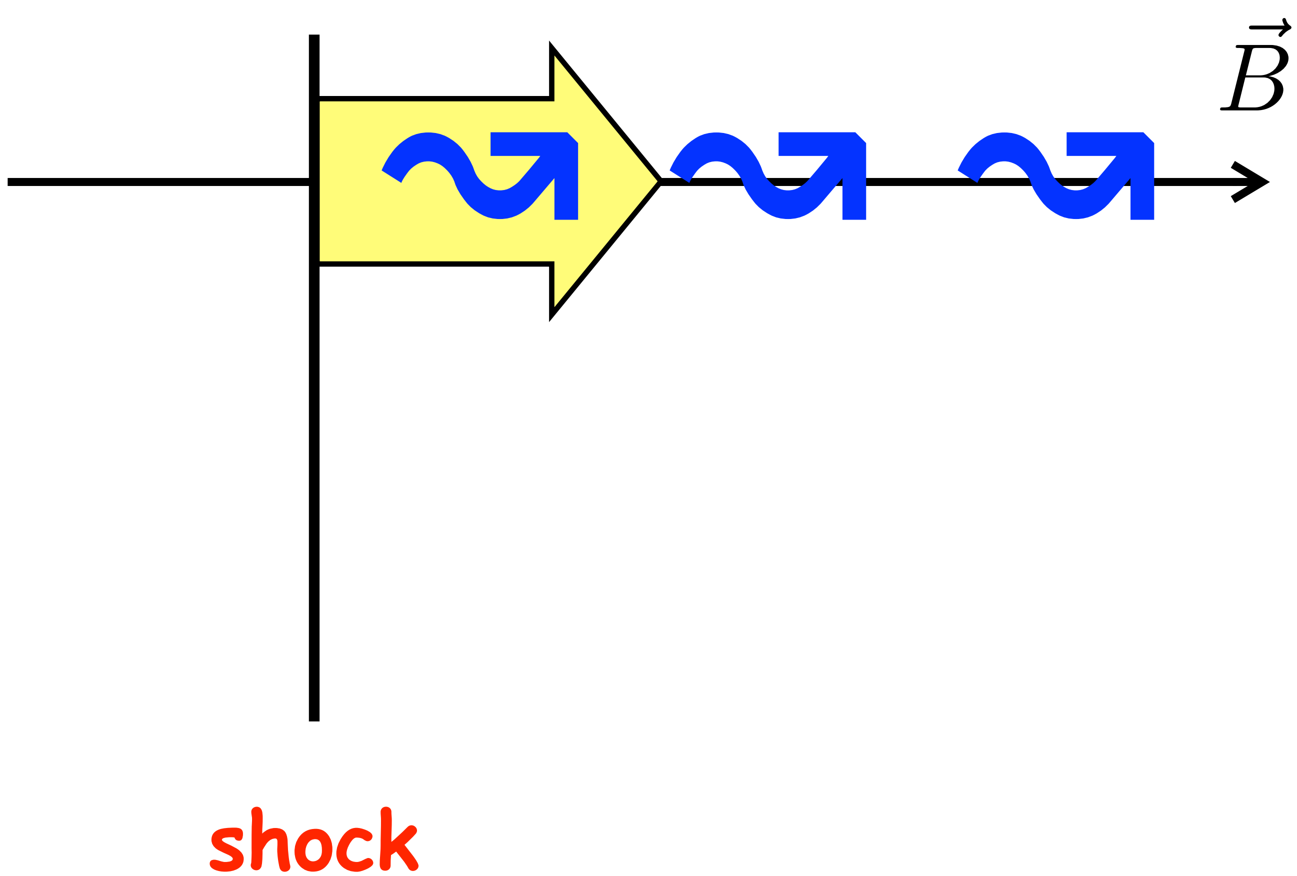


Alfven speed

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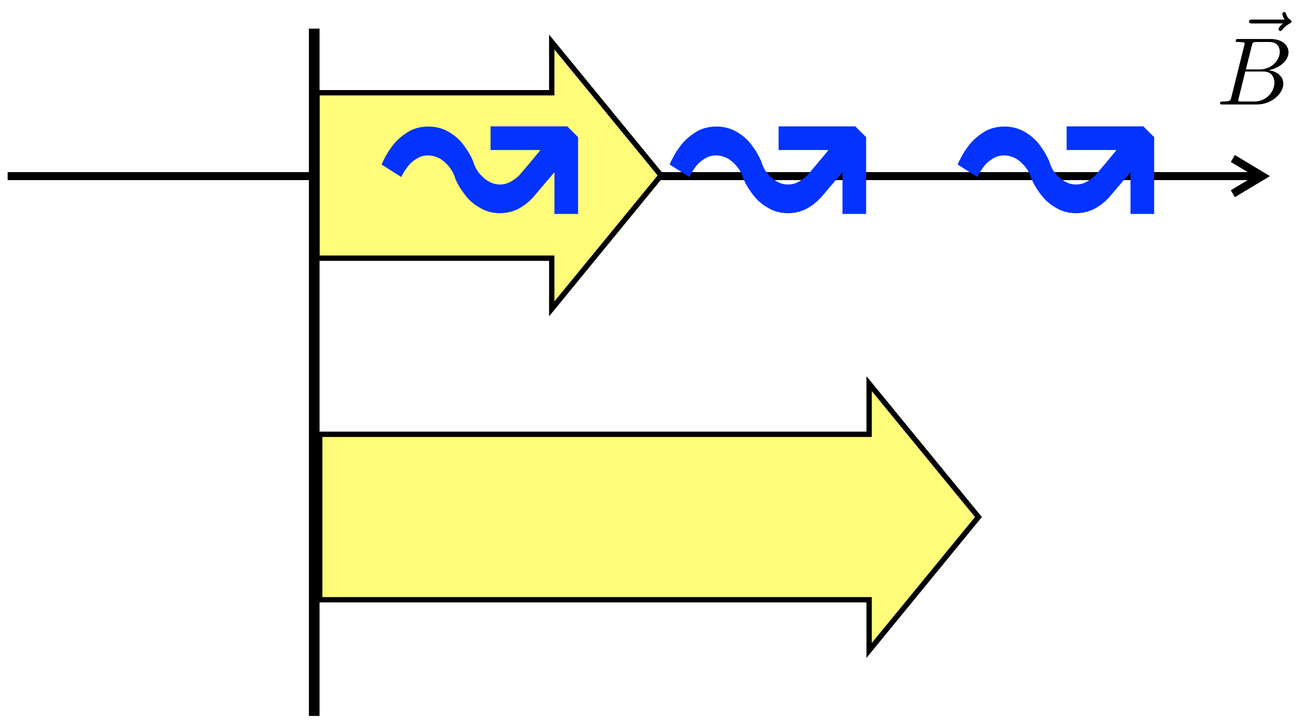


shock

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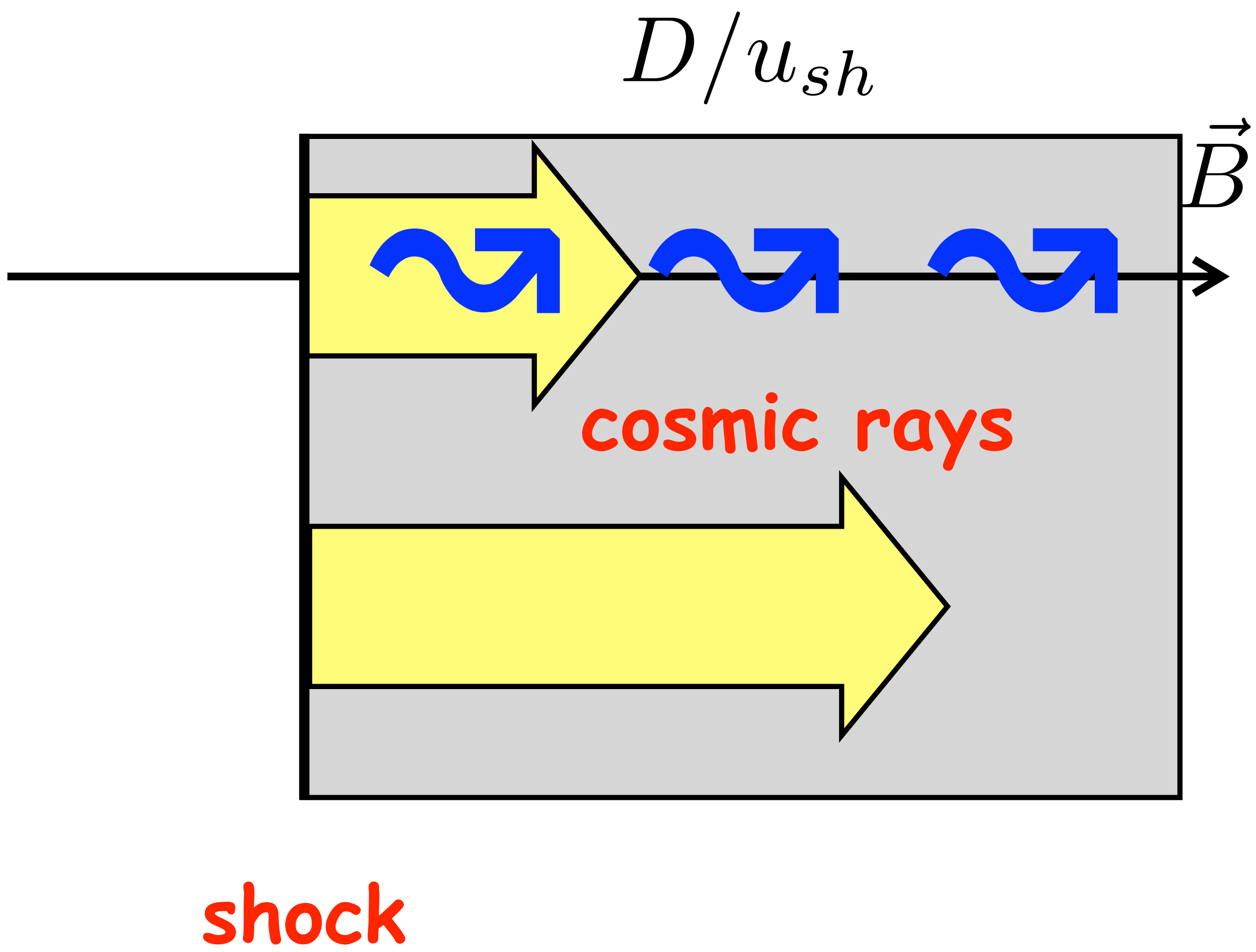


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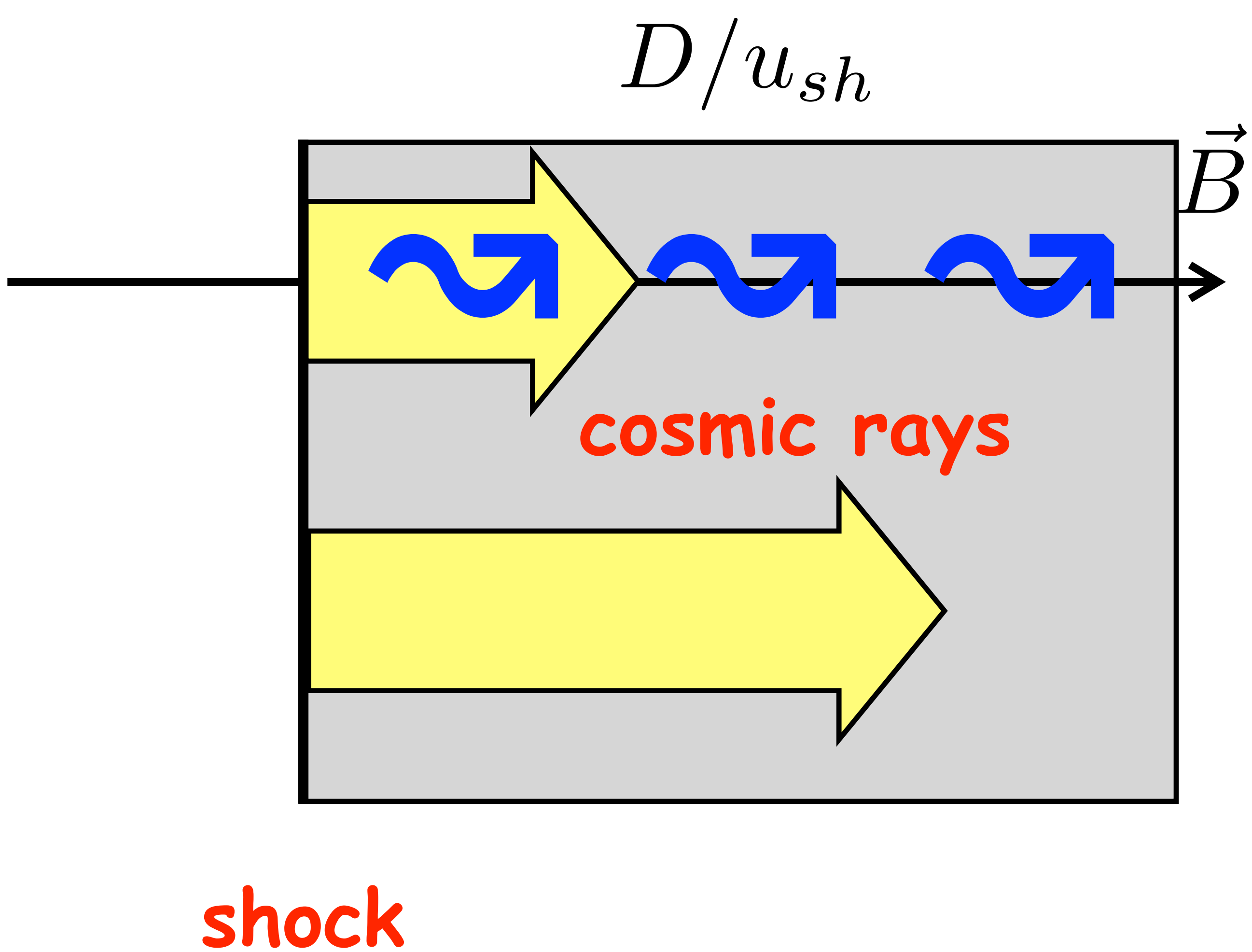


Alfven speed

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-> CRs move with the shock -> faster than waves -> CR and waves strongly coupled -> $V_A$ increases -> **B increases!**

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depends on velocity only!!!

$$u_s \approx 10^3 \mathrm{km/s} \longrightarrow E_{syn}^{max} \approx 1 \mathrm{kev}$$

X-rays!

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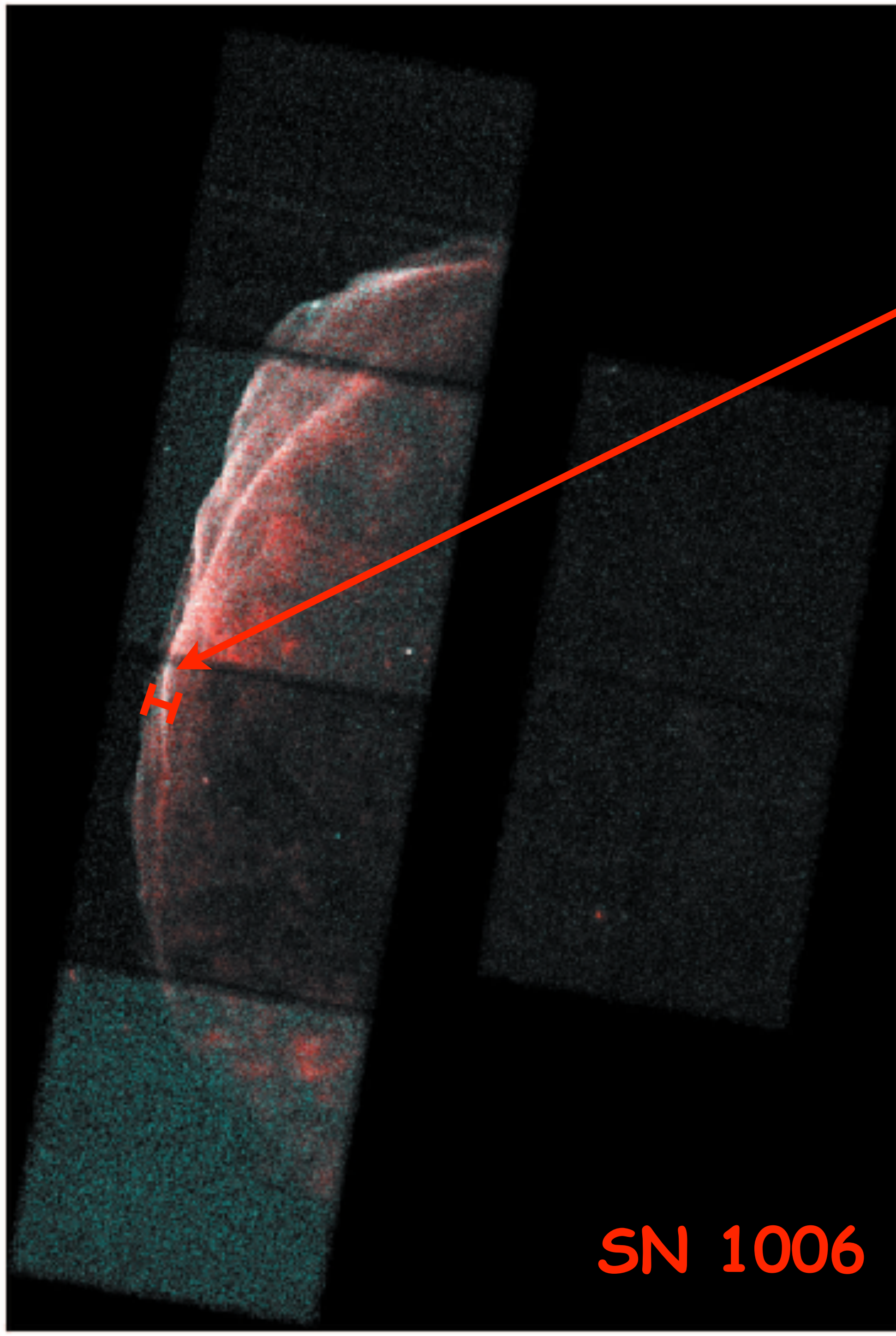


$$\Delta l_X \sim \tau_{syn} u_2 \sim B^{-3/2}$$

**B ~ hundreds of microGauss !**

# The supernova remnant paradigm: does it work?

- diffusive transport of cosmic rays in the galaxy —> **ISOTROPY**
- slope of the spectrum —> **$E^{-2}$ is too hard!**
  - what we see from gamma ray **observations** of SNRs seems to suggest that shock accelerate **steeper spectra**
  - theoreticians proposed tricks (modification of the diffusive shock acceleration theory) to explain this
- if **magnetic field amplification** operates at shocks (???) —> protons can be accelerated **up to the knee (~$10^{15}$ eV)**
- things we did not discuss: chemical composition, electrons, …