Cosmo, ex 2

1) $H^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2}$ (with $\Lambda = 0$)

$\Rightarrow$ $1 = \Omega_\rho + \Omega_K$ —— ⓪

with $\Omega_K = -\frac{K}{H^2 a^2} = -\frac{K}{\dot{a}^2}$ —— ①   & $\Omega_\rho = \frac{8\pi G \rho}{3H^2}$

• $\frac{d\Omega_K}{d\ln a} = \frac{d\Omega_K}{dt} \frac{1}{\frac{d(\ln a)}{dt}}$ $\frac{d\ln a}{dt} = \frac{1}{a}\dot{a} = H$

$= \frac{1}{H}\frac{d\Omega_K}{dt}$

& $\frac{d\Omega_K}{dt} \overset{①}{=} \frac{2K}{\dot{a}^3}\ddot{a}$ —— ②

Now from 2nd Friedman eqⁿ $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$

$= -\frac{4\pi G}{3}\rho(1+3w)$ —— ③

subs ③ into ②:

$\Rightarrow \frac{d\Omega_K}{dt} = -\frac{8K}{\dot{a}^3}\frac{\pi G}{3}\rho(1+3w)a = \underbrace{\left(-\frac{K}{\dot{a}^2}\right)}_{\Omega_K}\left(\frac{a}{\dot{a}}\right)\underbrace{\frac{8\pi G}{3}\rho}_{=\Omega_\rho H^2}(1+3w)$

$= \Omega_K\left(\frac{1}{H}\right)(\Omega_\rho H^2)(1+3w)$

$\Rightarrow \frac{d\Omega_K}{d\ln a} = \frac{1}{H} \times \frac{d\Omega_K}{dt} = \Omega_K\Omega_\rho(1+3w)$

$\boxed{\frac{d\Omega_K}{d\ln a} = \Omega_K(1-\Omega_K)(1+3w)}$ (using ⓪)

2) Assume w is a constant.

$$\Rightarrow \quad \frac{d\Omega_k}{\Omega_k(1-\Omega_k)} = (1+3w)\, d\ln a \qquad \int_a^{a_0}$$

$$\Rightarrow \quad \int_{\Omega_k}^{\Omega_{k0}} \left(\frac{1}{\Omega_k} + \frac{1}{1-\Omega_k} \right) d\Omega_k = (1+3w)\left(\ln a_0 - \ln a\right)$$

$$\Big[\ln \Omega_k + \ln(1-\Omega_k)\Big]_{\Omega_k}^{\Omega_{k0}} = (1+3w)\ln\left(\frac{a_0}{a}\right)$$

$$\Rightarrow \quad \ln\big(\Omega_k(1-\Omega_k)\big)\Big|_{\Omega_k}^{\Omega_{k0}} = \ln\left(\left(\frac{a_0}{a}\right)^{1+3w}\right)$$

$$\Rightarrow \quad \frac{\Omega_{k0}(1-\Omega_{k0})}{\Omega_k(1-\Omega_k)} = \left(\frac{a_0}{a}\right)^{1+3w}$$

$$\Rightarrow \quad \Omega_k(1-\Omega_k) = \Omega_{k0}(1-\Omega_{k0})\left(\frac{a}{a_0}\right)^{1+3w}.$$

3) or inversely

$$\Omega_{k0}(1-\Omega_{k0}) = \Omega_k(1-\Omega_k)\left(\frac{a_0}{a}\right)^{1+3w}$$

4) For $\Omega_{k0} \to 0$ need $\left(\frac{a_0}{a}\right)^{1+3w} \to 0$ as $a \to 0$

$\Rightarrow 1 + 3w < 0$

$\Rightarrow 3w < -1$

$\Rightarrow w < -1/3$.

3). Today $|\Omega_{k0}| = 10^{-4}$.

Rad^n dominated $\Rightarrow w = 1/3$ as $p = 1/3\rho$.

$$\Rightarrow \Omega_k(1-\Omega_k) \approx 10^{-4}\left(\frac{a}{a_0}\right)^2$$

and $a \propto t^{1/2}$

$$\Omega_k(1-\Omega_k) \simeq 10^{-4}\left(\frac{t}{t_0}\right)^{1/2}$$

LHS $\sim \Omega_k$ as RHS is a tiny number

$$\Rightarrow \Omega_k \simeq 10^{-4}\left(\frac{t}{t_0}\right)^{1/2}$$

Subst t_0 = age of universe today $\rightarrow \Omega_k$

t = Planck time $\rightarrow \Omega_k$.