Cosmo TD, Ex 3

$a \sim t^p$

1) $H^2 = \frac{8\pi G}{3}\rho \sim \text{const}$ with $\rho \sim 10^{16}\ \text{GeV}^4$
$= 2.3 \times 10^{36}\ \text{kg}/\text{m}^3$

$\Rightarrow \frac{a_f}{a_i} \sim e^{H\Delta t}$ ← duration of infl^n

$= e^{N_*}$ ← def^n of infl^n

$\Rightarrow \Delta t = \frac{N_*}{H} = N_* \sqrt{\frac{3}{8\pi G\rho}} \sim 2.8\ \text{ps}$!

2) $dt = a(\eta)\, d\eta \Rightarrow \eta = \int \frac{dt}{a(t)} \propto \underbrace{\frac{t^{1-p}}{1-p}}_{<0}$

$\Rightarrow a(\eta) \sim (-\eta)^{1+\beta}$ with $\beta = \frac{2p-1}{1-p}$.

3) $\mathcal{H} = \frac{a'}{a} = -\frac{(1+\beta)}{\eta}$

4) Infl$^n \Leftrightarrow \ddot{a} > 0 \Rightarrow p(p-1) > 0 \Rightarrow \boxed{p > 1}$ since $p > 0$.

5)
a) Basic eq^{n}s
$$\begin{cases} H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\left(\frac{1}{2}\dot\phi^2 + V\right) & \text{— ①} \\ \dot H = -4\pi\dot\phi^2 & \text{— ②} \\ \ddot\phi + 3H\dot\phi + V' = 0. & \text{— ③} \end{cases}$$

(b) $H = \frac{\dot a}{a} = \frac{p}{t}$

So in ②, $\dot H = -\frac{p}{t^2} = -4\pi\dot\phi^2$

$$\Rightarrow \dot{\phi} = \sqrt{\frac{p}{4\pi}}\,\frac{1}{t} \qquad \Rightarrow \boxed{\phi = \frac{M_{pl}}{2}\sqrt{\frac{p}{\pi}}\,\ln t}$$

so $\phi/M_{pl} = 2\alpha \ln t$ — (*)

with $\alpha = \frac{1}{4}\sqrt{\frac{p}{\pi}}$

Hence from (1)

$$V = \frac{3H^2}{8\pi} - \tfrac{1}{2}\dot{\phi}^2$$

$$= \frac{3}{8\pi}\frac{p^2}{t^2} - \frac{1}{2}\frac{p}{4\pi}\frac{1}{t^2}$$

$$= \frac{(3p-1)\,p}{8\pi\,t^2}\,M_{pl}^4$$

Now we want V of the form

$$V = V_0 e^{-\phi/M_{pl}\alpha}$$

$$= V_0 e^{-2\ln t} \qquad \text{using (*)}$$

$$= \frac{V_0}{t^2}$$

So $\boxed{V = V_0 e^{-\phi/M_{pl}\alpha} \quad \text{with} \quad V_0 = \frac{p(3p-1)}{8\pi} M_{pl}^4}$

c) $$W = \frac{P}{\rho} = \frac{\tfrac{1}{2}\dot{\phi}^2 - V}{\tfrac{1}{2}\dot{\phi}^2 + V} = \frac{1 - 2V/\dot{\phi}^2}{1 + 2V/\dot{\phi}^2}$$

and $\frac{2V}{\dot{\phi}^2} = \frac{\cancel{2}(3p-1)\cancel{p}}{8\pi \cancel{t^2}} \frac{\cancel{4\pi}}{\cancel{p}} \cancel{t^2} = 3p-1$

$$\Rightarrow \boxed{w = -1 + \frac{2}{3p}} \xrightarrow[p \to \infty]{} -1.$$

$$\epsilon_V = \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2.$$

From 3, $V' = -\ddot{\phi} - 3H\dot{\phi} = \frac{1}{2}\sqrt{\frac{p}{\pi}}(1-3p)\frac{1}{t^2}$

$$\Rightarrow \frac{V'}{V} = \frac{1}{2}\sqrt{\frac{p}{\pi}}\cancel{(1-3p)}\frac{1}{\cancel{t^2}} \frac{8\pi \cancel{t^2}}{p\cancel{(3p-1)}}$$

$$= -\frac{4\pi}{p}\sqrt{\frac{p}{\pi}}$$

$$= -4\sqrt{\frac{\pi}{p}}.$$

$$\Rightarrow \boxed{\epsilon_V = \frac{1}{p}}$$

(or more simply! $V \propto e^{-\phi/M_{pl}\alpha}$

$\Rightarrow \frac{V'}{V} = -\frac{1}{M_{pl}\alpha}$.)

For $\boxed{\eta_V = M_{pl}^2\left(\frac{V''}{V}\right) = \frac{2}{p}}$.

Since $(\eta, \epsilon) < 1$ for $p > 2$, infl^n never ends.

④

⑥ (a) $z = \frac{a\phi'}{\mathcal{H}} = \frac{a\dot{\phi}}{H}$ $\qquad ' = \frac{d}{d\eta} \neq \frac{d}{dt}$

& $\dot{\phi} = \frac{1}{2}\sqrt{\frac{p}{\pi}}\,\frac{1}{t} \propto \frac{1}{(-\eta)^{1/1-p}}$

$\Rightarrow \phi' = \frac{d\phi}{d\eta} = \dot{\phi}\frac{dt}{d\eta} \propto \frac{(-\eta)^{p/1-p}}{(-\eta)^{1/1-p}} \propto \frac{1}{\eta}$

Hence $z \propto (-\eta)^{\frac{p}{1-p}}\,\frac{1}{\eta}\,\eta$

$$\boxed{z \propto (-\eta)^{\frac{p}{1-p}}}$$

Hence $z' \propto \frac{p}{1-p}(-\eta)^{\frac{2p-1}{1-p}}$

$z'' \propto \frac{(2p-1)}{(1-p)^2}\,p\,(-\eta)^{\frac{3p-2}{1-p}}$

& finally $$\boxed{\frac{z''}{z} = \frac{(2p-1)p}{(1-p)^2}\,\frac{1}{\eta^2}}$$

(b) $v_k'' + \left(k^2 - \frac{z''}{z}\right)v_k = 0$

$\Rightarrow v_k'' + \left[k - \frac{1}{\eta^2}\left(\nu^2 - \frac{1}{4}\right)\right]v_k = 0$

with $$\boxed{\nu^2 = \frac{1}{4} + \frac{p(2p-1)}{(1-p)^2}}$$

Solution

$$v_k(\eta) = \sqrt{-\eta}\left[A(k)\,H_\nu^{(1)}(-k\eta) + B(k)\,H_\nu^{(2)}(-k\eta)\right] \qquad (4)$$

(c) In $k|\eta| \gg 1$ limit, on subhorizon scales $\Rightarrow$ can neglect curvature, and quantize as in Minkowski

$$v_k \longrightarrow \frac{e^{-ik\eta}}{\sqrt{2k}} \qquad (\hbar = 1)$$

(d) Now, as shown in Uzan + Peter,

$$H_\nu^{(1/2)}(z) \sim \sqrt{\frac{2}{\pi z}}\, e^{\pm i\left(z - \frac{\pi\nu}{2} - \frac{\pi}{4}\right)} \quad \text{as } |z| \to \infty$$

in limit $k|\eta| \to \infty$

$$\Rightarrow A(k)\sqrt{\frac{2}{-\pi k\eta}}\, e^{i(-k\eta)}\, e^{-\frac{i\pi\nu}{2}}\, e^{i\pi/4}$$

$$+ B(k)\sqrt{\frac{2}{-\pi k\eta}}\, e^{ik\eta}\, e^{\frac{i\pi\nu}{2}}\, e^{-i\pi/4}$$

$$= \frac{e^{-ik\eta}}{\sqrt{2k}}\frac{1}{\sqrt{-\eta}}$$

Hence $B(k) = 0$ &

$$A(k)\sqrt{\frac{2\cdot(2k)}{\pi k}}\, e^{-\frac{i\pi}{2}(\nu + 1/2)} = 1$$

$$e^{-i\pi/2} = -i$$

⑥

$$\Rightarrow A(k) = \sqrt{\frac{\pi}{4}}\; i^{\nu + 1/2}$$

$$B(k) = 0$$

& thus substituting into ④ $\Rightarrow$

$$\boxed{v_k(\eta) = \sqrt{\frac{-\pi\eta}{4}}\; i^{\nu+1/2}\, H_\nu^{(1)}(-k\eta)}$$

(e) $$P_\zeta(k) = \frac{k^3}{2\pi^2} |\zeta(k)|^2$$ exactly same as in lectures

$$= \frac{k^3}{2\pi^2} \left| \frac{v_k}{z} \right|^2$$

$$\sim k^3 \left| H_\nu^{(1)}(-k\eta) \right|^2.$$

For superhubble modes $k\eta \ll 1$, see Uzan + Pete

$$H_\nu^{(1)}(-k\eta) \sim \frac{\left(-\frac{k\eta}{2}\right)^{-\nu}}{\Gamma(1-\nu)\sin(\pi\nu)}$$

$$\Rightarrow P_\zeta(k) \sim k^{3-2\nu}$$

(g) lowest order slow-roll approx.

(f) $$n_s - 1 = \frac{d\ln P}{d\ln k} = 3 - 2\nu$$

$$\Rightarrow n_s = 2(2-\nu)$$