Master NPAC

Correction of Cosmology Homework #2

December 13, 2022

1. Preliminary question:

$$\frac{da\chi}{dt} = \dot{a}\chi + a\dot{\chi} \quad \Rightarrow \frac{da\chi}{dt}(t_0) = a_0\dot{\chi}(t_0)$$

$d\tau/dt$ as a function of a and χ:

$$c^2 d\tau^2 = c^2 dt^2(1 - a^2\dot{\chi}^2/c^2) \quad \Rightarrow \quad \frac{d\tau}{dt} = (1 - \frac{a^2\dot{\chi}^2}{c^2})^{1/2}$$

2. Initial values of $dt/d\tau$ and $d\chi/d\tau$:

$$\frac{dt}{d\tau} = (1 - \frac{a^2\dot{\chi}^2}{c^2})^{-1/2}$$

$$\frac{d\chi}{d\tau}(t_0) = \frac{d\chi}{dt}\frac{dt}{d\tau}(t_0) = \frac{v_0}{a_0}\left(1 - \frac{a_0\dot{\chi}^2(t_0)}{c^2}\right)^{-1/2} = \frac{v_0}{a_0}\frac{1}{\sqrt{1-\frac{v_0^2}{c^2}}}$$

3 From the geodesics equation to the differential equations followed by $t(\tau)$ and $\chi(\tau)$:

The geodesics for the time coordinate is:

$$\frac{d^2t}{d\tau^2} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

If you explicit the sum and plug in the values of the Christoffel symbols (which we have computed in class), we get:

$$\frac{d^2t}{d\tau^2} + \Gamma^0_{11}\left(\frac{d\chi}{d\tau}\right)^2 = 0$$

which gives:

$$\frac{d^2t}{d\tau^2} + a\dot{a}\left(\frac{d\chi}{d\tau}\right)^2 = 0$$

For the spatial coordinate, the geodesics equation is:

$$\frac{d^2\chi}{d\tau^2} + 2\Gamma^1_{01}\left(\frac{dt}{d\tau}\right)\left(\frac{d\chi}{d\tau}\right) = 0$$

which gives

$$\frac{d^2\chi}{d\tau^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{dt}{d\tau}\frac{d\chi}{d\tau} = 0$$

or

$$\frac{d}{d\tau}\left(a^2\frac{d\chi}{d\tau}\right) = 0$$

4 Differential equation $d\chi/d\tau = f(a,\chi,t)$

$$\dot{\chi} = \frac{d\chi}{d\tau}\frac{d\tau}{dt}$$

Integrating the differential equation obtained at the previous step, we have:

$$a^2\frac{d\chi}{d\tau} = a^2\frac{d\chi}{dt}\frac{dt}{d\tau} = A \quad \text{with} \quad A = a_0 v_0\left(1 - \frac{v_0^2}{c^2}\right)^{-1/2}$$

or

$$a^2\dot{\chi}\left(1 - \frac{a^2\dot{\chi}^2}{c^2}\right)^{-1/2} = A$$

After a little bit of algebra, we obtain:

$$\dot{\chi} = \pm\frac{A}{a(a^2 + \frac{A^2}{c^2})^{1/2}}$$

Two cases:

$$v \to 0 \qquad A \sim a_0 v_0 \qquad \dot{\chi} \sim \frac{a_0 v_0}{a\left(a^2 + \frac{a_0^2 v_0^2}{c^2}\right)^{1/2}} \sim \frac{a_0 v_0}{a^2} \tag{1}$$

$$v \to c \qquad A \to \infty \qquad \dot{\chi} \sim \frac{A}{a\frac{A}{c}(1 + \frac{a^2 c^2}{A})^{1/2}} \sim \frac{c}{a} \tag{2}$$

5 Matter-dominated Universe: $a(t) = a_0(t/t_0)^{2/3}, t_0 = (2/3)H_0^{-1}$:

Now, it's just about integrating the equations we have obtained above:
First case ($v_0 \to 0$):

$$\frac{d\chi}{dt} = a_0 v_0 a_0^{-2} \left(\frac{t}{t_0}\right)^{-4/3} \tag{3}$$

$$d\chi = \frac{v_0}{a_0} t_0^{4/3} t^{-4/3} dt \tag{4}$$

$$\chi_1 = \frac{2v_0}{a_0} H_0^{-1} \left(1 - \left(\frac{t_0}{t_1}\right)^{1/3}\right) \tag{5}$$

Second case ($v_0 \to c$):

$$\frac{d\chi}{dt} = \frac{c}{a(t)} = \frac{c}{a_0} t_0^{2/3} t^{-2/3} \tag{6}$$

$$\chi_1 = \frac{2c}{a_0} H_0^{-1} \left[\left(\frac{t_1}{t_0}\right)^{1/3} - 1\right] \tag{7}$$

6 Vacuum-energy-dominated Universe: $a(t) = a_0 \exp H_0(t - t_0)$

Same as above:
First case ($v_0 \to 0$):

$$d\chi = \frac{v_0}{a_0} e^{-2H_0(t-t_0)} dt \tag{8}$$

$$\chi_1 = \frac{v_0}{a_0} e^{2H_0 t_0} \int_{t_0}^{t_1} e^{-2H_0 t} dt \tag{9}$$

$$= \frac{v_0}{a_0} \frac{1}{2H_0} \left(1 - e^{-2H_0(t_1 - t_0)}\right) \tag{10}$$

Second case ($v_0 \to c$):

$$d\chi = \frac{c}{a_0} e^{-H_0(t-t_0)} dt \tag{11}$$

$$\chi_1 = \frac{c}{a_0} e^{H_0 t_0} \int_{t_0}^{t_1} e^{-H_0 t} dt \tag{12}$$

$$= \frac{c}{a_0} \frac{1}{H_0} \left(1 - e^{-H_0(t_1 - t_0)}\right) \tag{13}$$

7 Just have a look at the equations: In a matter-dominated Universe, with $v \to c$, you can indeed beat the expansion. In a vacuum-dominated Universe, no way.