Master NPAC

Cosmology homework #2

December 1, 2022

Answers are to be sent by email to nicolas.regnault-AT-lpnhe.in2p3.fr by Thursday December 8th, 2022. Please use the following format (given as an example):

```
<your name>
 1. A
 2. B
 3. C
 4. D
 5. A
 6. B
 7. A
```

We have seen that, in a homogeneous, isotropic Universe in expansion, the metric, in comoving coordinates (t, χ, θ, ϕ) can be written as:

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + a^2(t)\left(d\chi^2 + r^2(\chi)\left(d\theta^2 + \sin\theta d\phi^2\right)\right) \tag{1}$$

General Relativity tells us that free particles follow geodesics $(x^\mu(\lambda))$, which are solution of the geodesics equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{2}$$

where the Γ coefficients are the christoffel connections. For the metric above, we remind you that:

$$\Gamma^0_{00} = \Gamma^i_{00} = \Gamma^0_{0i} = 0 \quad \Gamma^i_{0j} = \frac{\dot a}{a}\delta^i_j \quad \Gamma^0_{ij} = a\dot a \tilde g_{ij} \quad \Gamma^1_{11} = \Gamma^\chi_{\chi\chi} = 0 \tag{3}$$

In this homework, we study the motion of a spacecraft that leaves our Galaxy $(\chi = 0)$ today $(t = t_0)$, with an initial velocity $\frac{d\ell_{\rm phys}}{dt} = \frac{da(t)\chi(t)}{dt} = v_0$. We assume that all its fuel is used to generate the spacecraft's initial momentum, and that it follows a purely radial trajectory: $(t(\tau), \chi(\tau), \theta = \text{const}, \phi = \text{const})$.

1. Show that

$$\dot{\chi}(t_0) = \frac{d\chi}{dt}(t_0) = \frac{v_0}{a_0} \tag{4}$$

Use $-c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ to express $d\tau/dt$ as a function of a and χ:

(A) $\frac{d\tau}{dt} = (1 - \frac{a^2\dot{\chi}^2}{c^2})^{1/2}$

(B) $\frac{d\tau}{dt} = (1 - a^2\chi^2)^{1/2}$

(C) $\frac{d\tau}{dt} = (1 - a\chi/c)^{1/2}$

(D) None of the above

2. Express the initial values (at t_0) of $dt/d\tau$ and $d\chi/d\tau$

(A) $\frac{dt}{d\tau}_{|t_0} = 1 \quad \frac{d\chi}{d\tau} = v_0/a_0$

(B) $\frac{dt}{d\tau}_{|t_0} = (1 - \frac{v_0^2}{c^2})^{-1/2} \quad \frac{d\chi}{d\tau} = \frac{v_0}{a_0}\frac{1}{\sqrt{1 - v_0^2/c^2}}$

(C) $\frac{dt}{d\tau}_{|t_0} = (1 - \frac{v_0^2}{c^2})^{-1/2} \quad \frac{d\chi}{d\tau} = \frac{v_0}{a_0}\sqrt{1 - v_0^2/c^2}$

(D) None of the above

3. Derive from the geodesic equation obeyed by the spacecraft trajectory the differential equations followed by $t(\tau)$ and $\chi(\tau)$:

(A) $\frac{d^2t}{d\tau^2} = -a\dot{a}\left(\frac{d\chi}{d\tau}\right)^2 \quad \frac{d}{d\tau}(a^2\frac{d\chi}{d\tau}) = 0$

(B) $\frac{d^2t}{d\tau^2} = 1 \quad \frac{d}{d\tau}(\frac{da}{d\tau}\frac{d\chi}{d\tau}) = 0$

(C) $\frac{d^2t}{d\tau^2} = -a\dot{a}\dot{\chi}^2 \quad \frac{d}{d\tau}(a^2\frac{d\chi}{d\tau}) = 0$

(D) None of the above

4. Derive from the above one single differential equation $d\chi/dt = f(a, \chi, t)$. Write the limits of this equation for a slowly moving spacecraft ($v_0 \to 0$) and a spacecraft moving at relativistic speeds: $v_0 \to c$.

(A) $(v_0 \to 0)\ \dot{\chi} = \frac{a_0 v_0}{a^2(t)} \quad (v_0 \to c)\ \dot{\chi} = \frac{c}{a(t)}$

(B) $(v_0 \to 0)\ \dot\chi = \dfrac{v_0}{a(t)} \quad (v_0 \to c)\ \dot\chi = \dfrac{c}{a(t)}$

(C) $(v_0 \to 0)\ \dot\chi = v_0 \dot a \quad (v_0 \to c)\ \dot\chi = \dfrac{a_0}{a^2(t)}$

(D) None of the above

5. We assume that we live in a matter-dominated Universe: $a(t) = a_0 (t/t_0)^{2/3}$, with $t_0 = (2/3) H_0^{-1}$. The spacecraft reaches a galaxy at $t = t_1$. What is the comoving distance $a_0\chi_1$ to this galaxy in the two cases above ($v_0 \to 0$ and $v_0 \to c$) ?

(A) $(v_0 \to 0)\ a_0\chi_1 = \dfrac{2}{H_0}\left(1 - \left(\dfrac{t_0}{t_1}\right)^{2/3}\right) \quad (v_0 \to c)\ a_0\chi_1 = \dfrac{2c}{H_0}\left(\left(\dfrac{t_0}{t_1}\right)^{2/3} - 1\right)$

(B) $(v_0 \to 0)\ a_0\chi_1 = \dfrac{2a_0}{H_0}\left(1 - \left(\dfrac{t_0}{t_1}\right)^{1/3}\right) \quad (v_0 \to c)\ a_0\chi_1 = \dfrac{2a_0}{H_0}\left(\left(\dfrac{t_1}{t_0}\right)^{2/3} - 1\right)$

(C) $(v_0 \to 0)\ a_0\chi_1 = \dfrac{2v_0}{H_0}\left(1 - \left(\dfrac{t_0}{t_1}\right)^{1/3}\right) \quad (v_0 \to c)\ a_0\chi_1 = \dfrac{2c}{H_0}\left(\left(\dfrac{t_1}{t_0}\right)^{1/3} - 1\right)$

(D) None of the above

6. Same question for a vacuum-energy-dominated Universe: $a(t) = a_0 \exp H_0(t - t_0)$

(A) $(v_0 \to 0)\ a_0\chi_1 = \dfrac{v_0}{2H_0}\left(1 - e^{-2H_0(t_1 - t_0)}\right) \quad (v_0 \to c)\ a_0\chi_1 = \dfrac{c}{H_0}\left(1 - e^{-H_0(t_1 - t_0)}\right)$

(B) $(v_0 \to 0)\ a_0\chi_1 = \dfrac{v_0}{H_0}\left(1 - e^{-H_0(t_1 - t_0)}\right) \quad (v_0 \to c)\ a_0\chi_1 = \dfrac{c}{2H_0}\left(1 - e^{-2H_0(t_1 - t_0)}\right)$

(C) $(v_0 \to 0)\ a_0\chi_1 = \dfrac{c}{H_0}\left(1 - e^{-2H_0(t_1 - t_0)}\right) \quad (v_0 \to c)\ a_0\chi_1 = \dfrac{v_0}{H_0}\left(1 - e^{-H_0(t_1 - t_0)}\right)$

(D) None of the above

7. For $t \to \infty$ and $v_0 \to c$ is there a limit to the distance we can travel in (1) a matter-dominated Universe and (2) a vacuum-dominated Universe ?

(A) Yes Yes

(B) Yes No

(C) No Yes

(D) No No

8. With the expansion of the Universe, the energy density of non relativistic matter falls like a^{-3} and the energy density of radiation falls like a^{-4}. Assuming $\Omega_{m,0} = 0.3$ (you know $\Omega_{r,0}$ already, right ?), plot the evolution of ρ_m and ρ_r as a function of z. Draw on the plot the value of ρ_m at $z = 3$. At which redshift were ρ_m and ρ_r equal ? What was their value ?