# Master NPAC

**Cosmology – extra homework**

January 20th, 2022

## 1 Quintessence

Some cosmologists speculate that the universe may contain a quantum field called "*quintessence*" (from *quinta essentia*, the fifth element also named *aether* in Aristotle worldview). This quantum field has a positive energy density and a negative equation-of-state parameter $w_Q < 0$.

Let suppose that we live in a spatially flat universe, containing only matter ($\Omega_{m,0} \leq 1$) and quintessence with $w_Q = -1/2$ and $\Omega_{Q,0} = 1 - \Omega_{m,0}$.

1. At what scale $a_{mQ}$ will the energy density of quintessence and matter be equal. Express it as a function of $\Omega_{m,0}$ and $\Omega_{Q,0}$.

The density of a fluid of equation of state $p = w\rho$ ($w$ constant) varies like

$$\rho \propto \left(\frac{a_0}{a}\right)^{3(1+w)} \tag{1}$$

The equality between matter and quintessence (with $w = -1/2$) is therefore reached for:

$$\rho_{m,0}\left(\frac{a_0}{a}\right)^3 = \rho_{Q,0}\left(\frac{a_0}{a}\right)^{3/2} \tag{2}$$

hence,

$$a = a_0\left(\frac{\Omega_m}{\Omega_Q}\right)^{2/3} \tag{3}$$

For a flat universe:

$$a = a_0\left(\frac{\Omega_m}{1-\Omega_m}\right)^{2/3} \tag{4}$$

2. What is $a(t)$ when $a \ll a_{mQ}$? and when $a \gg a_{mQ}$?

The evolution of $a(t)$ can be obtained from the Friedmann equation. Let's rewrite it as:

$$\left(\frac{1}{x}\frac{dx}{dt}\right)^2 = H_0^2\left(\Omega_m x^{-3} + \Omega_Q x^{-3/2} + \underbrace{(1-\Omega_m-\Omega_Q)}_{=0}\right) \tag{5}$$

with $x = a/a_0$. We have:

$$\int_0^t dt = \frac{1}{H_0}\int_0^{a/a_0} \frac{dx}{(\Omega_m x^{-1} + \Omega_Q x^{1/2})^{1/2}} \tag{6}$$

When $a \ll a_{mQ}$ matter dominates, and the integration of the equation above gives:

$$a(t) \propto t^{2/3} \tag{7}$$

more exactly, we have:

$$\frac{a}{a_0} = \left(\frac{3}{2}H_0\sqrt{\Omega_m}t\right)^{2/3} \tag{8}$$

When $a \gg a_{mQ}$ quintessence dominates, and we have, from the same integral:

$$a(t) \propto t^{4/3} \tag{9}$$

3. Solve the Friedmann equation to find $a(t)$ for this universe (without approximation).

   *Hints:*

$$\int \frac{x^{1/2}dx}{\sqrt{1+\beta x^{3/2}}} = \frac{4}{3\beta}\sqrt{1+\beta x^{3/2}}$$

This is just working out the integral above. We have

$$t = \frac{1}{H_0}\frac{1}{\sqrt{\Omega_m}}\int_0^{a/a_0} \frac{x^{1/2}dx}{(1+\frac{\Omega_Q}{\Omega_m}x^{3/2})^{1/2}} \tag{10}$$

Using the hint above, we get:

$$t = \frac{4}{3}\frac{1}{H_0}\frac{\sqrt{\Omega_m}}{\Omega_Q}\left(\sqrt{1+\frac{\Omega_Q}{\Omega_m}\left(\frac{a}{a_0}\right)^{3/2}} - 1\right) \tag{11}$$

Inverting the equation gives:

$$a(t) = a_0\left(\frac{\Omega_m}{\Omega_Q}\left(\left(\frac{3}{4}H_0\frac{\Omega_Q}{\sqrt{\Omega_m}}t + 1\right)^2 - 1\right)\right)^{2/3} \tag{12}$$

4. What is the current age of this universe, as a function of $H_0$ and $\Omega_{m,0}$?

The current age of the Universe is again given by the equation above:

$$t = \frac{4}{3}\frac{1}{H_0}\frac{\sqrt{\Omega_m}}{1-\Omega_m}\left(\sqrt{1+\frac{1-\Omega_m}{\Omega_m}}-1\right) \tag{13}$$

5. Describe the properties of a universe entirely made of quintessence ($\Omega_m \approx 0$). What would be the current age of this universe, and the current particle horizon distance ?

We have $w < -1/3$. The second Friedmann equation tells us the expansion is accelerating ($\ddot{a} > 0$).
Let's check that by integrating the Friedmann equation, we find:

$$t = \frac{1}{H_0}\int_0^{a/a_0} \frac{x^{-1/4}}{\Omega_Q^{1/2}}dx = \frac{4}{3}\frac{1}{H_0}\frac{1}{\sqrt{\Omega_Q}}\left(\frac{a}{a_0}\right)^{3/4} \tag{14}$$

This gives:

$$a \propto t^{4/3} \tag{15}$$

We find indeed an accelerated expansion.
We find a current age of the Universe of $\frac{4}{3}\frac{1}{H_0}$ (taking the limit $\Omega_m \to 0$).
Finally, we find the horizon distance diverges. The horizon encompasses all the universe.

## 2 Closed Universe

Consider a closed universe which contains only matter : $\Omega_0 = \Omega_{m,0} > 1$.

1. Describe briefly the properties of such a universe, its curvature, and its dynamics (a drawing may help).

A universe that contains only non-relativistic matter with a density above the critical density is closed and has positive curvature ($k = +1$). Its expansion reaches a maximum ($a_{max}$) and is followed by a contraction phase that ends in a Big Crunch event.

2. Write the Friedmann equation for this universe. Compute the value $a_{\max}$ of the scaling factor $a$ at maximum expansion.

The first Friedmann equation is:

$$H^2(t) = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \tag{16}$$

It can be rewritten as:

$$H^2 = H_0^2\left(\Omega_0 a^{-3} + (1-\Omega_0)a^{-2}\right) \tag{17}$$

At maximum expansion, $\dot{a} = 0$, $H = 0$, which gives

$$a_{\max} = \frac{\Omega_0}{\Omega_0 - 1} \tag{18}$$

3. Show that $H_0$, $\Omega_0$ and the current universe curvature radius $a_0$ are linked by:

$$a_0 = \frac{1}{H_0}\frac{1}{\sqrt{\Omega_0 - 1}}$$

From (1) the first Friedmann equation and (2) the definition of the critical density ($\rho_c = \frac{3H^2}{8\pi G}$), we have:

$$H_0^2 = H_0^2\Omega_0 - \frac{k}{a_0^2} \tag{19}$$

Factorizing out $H_0^2$, this gives

$$\Omega_0 - \frac{k}{a_0^2 H_0^2} = 1 \tag{20}$$

from which we get:

$$a_0 = \frac{1}{H_0}\frac{1}{\sqrt{\Omega_0 - 1}} \tag{21}$$

4. Calculate the horizon distance $d_{\text{hor}}(t) = a_0\chi_{\text{hor}}(t)$ as a function of time. Remember that the comoving coordinate $\chi_{\text{hor}}$ of the horizon at time $t$ is defined by:

$$\chi_{\text{hor}}(t) = \int_0^t \frac{dt'}{a(t')}$$

The horizon distance is given by:

$$d_{\text{hor}} = a_0 \int_0^{t_0} \frac{dt}{a(t)} = a_0 \int_0^{a_0} \frac{da}{\dot{a}a} = a_0 \int_0^{a_0} \frac{da}{Ha^2} \tag{22}$$

Injecting the Friedmann equation (H(a) = ...) and changing variables: $x = a/a_0$, we obtain:

$$d_{\text{hor}} = \frac{1}{H_0} \int_0^{a/a_0} \frac{dx}{(\Omega_0 x - (\Omega_0 - 1)x^2)^{1/2}} \tag{23}$$

If we pose $u = \frac{\Omega_0 - 1}{\Omega_0} x$, the integral may be rewritten:

$$d_{\text{hor}} = \frac{1}{\sqrt{\Omega_0 - 1}} \frac{1}{H_0} \times \int_0^{\frac{\Omega_0 - 1}{\Omega_0} \frac{a}{a_0}} \frac{du}{\sqrt{u(1-u)}} \tag{24}$$

which gives:

$$d_{\text{hor}}(t) = \frac{1}{\sqrt{\Omega_0 - 1}} \frac{1}{H_0} \times \arcsin\left(2 \frac{\Omega_0 - 1}{\Omega_0} \frac{a(t)}{a_0} - 1\right) \tag{25}$$

Using the results of the previous question, we can write:

$$\chi_{\text{hor}}(t) = \frac{1}{a_0} d_{\text{hor}}(t) = \arcsin\left(2 \frac{\Omega_0 - 1}{\Omega_0} \frac{a(t)}{a_0} - 1\right) \tag{26}$$

Show that at the moment of maximum expansion ($a = a_{\text{max}}$, $t = t(a_{\text{max}})$), the horizon includes the entire universe, *i.e.*:

$$\chi_{\text{hor}}(a_{\text{max}}) = \pi.$$

At the moment of maximum expansion, $a_{\text{max}} = \frac{\Omega_0}{\Omega_0 - 1}$, hence

$$\chi_{\text{hor}} = \arcsin(2 - 1) = \pi \tag{27}$$

The horizon includes the entire Universe.

*Hints:*

$$\int_A^B \frac{dx}{\sqrt{x(1-x)}} = \Big[\arcsin(2x - 1)\Big]_A^B \qquad x = \frac{\Omega_0 - 1}{\Omega_0} \times a$$

5. Verify that the evolution of the universe may be described by the following parametric equations:

$$a(\eta) = A\,(1 - \cos\eta)$$
$$t(\eta) = B\,(\eta - \sin\eta)$$

Give the expression of $A$ and $B$ as functions of $H_0$ and $\Omega_0$. What is the value of $\eta$ at maximum expansion? Describe briefly the resulting dynamics.

By differentiating $a$ and $t$, forming $\dot{a}/a$ and indentifying with the Friedmann equation terms, we obtain:

$$A = \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1} \qquad B = \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{28}$$

6. Show that the age of the universe at maximal expansion is:

$$t(a_{\mathrm{max}}) = \frac{\pi}{2}\frac{1}{H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}$$

From the previous question, the maximum of $a$ is reached for $\eta = \pi$. Therefore, we have

$$t(a_{\mathrm{max}}) = \frac{\pi}{2H_0}\frac{\Omega_0}{(\Omega_0 - 1)^{3/2}} \tag{29}$$

7. At some time $t_1 > t(a_{\mathrm{max}})$ during the contraction phase of this universe, an astronomer named Edwin Elbbuh discovers that all nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance ; he measures as well $H_1 < 0$ and $\Omega_1 > 1$. Knowing $H_1 < 0$ and $\Omega_1$, how much time remains between $t_1$ and the final Big Crunch at $t = t_{\mathrm{crunch}}$?

The function that describes the expansion is symmetric around $a_{\mathrm{max}}$. The remaining time before the big crunch is equal to the age of the universe when it was expanding, with $H_0 = -H_1$.

$$t = -\frac{2}{3}\frac{1}{H_1}\frac{1}{\sqrt{\Omega_1}} \tag{30}$$