Master NPAC

Cosmology homework #4

Jan 5, 2023

For this homework, just write your answers clearly and legibly on a normal sheet of paper. If unsure, you may send me a scan of your solutions. Answers will be posted on Thursday January 12th.

1 Quintessence

Some cosmologists speculate that the universe may contain a quantum field called "*quintessence*" (from *quinta essentia*, the fifth element also named *aether* in Aristotle world-view). This quantum field has a positive energy density and a negative equation-of-state parameter $w_Q < 0$.

Let suppose that we live in a spatially flat universe, containing only matter ($\Omega_{m,0} \leq 1$) and quintessence with $w_Q = -1/2$ and $\Omega_{Q,0} = 1 - \Omega_{m,0}$.

1. At what scale a_{mQ} will the energy density of quintessence and matter be equal. Express it as a function of $\Omega_{m,0}$ and $\Omega_{Q,0}$.
2. What is $a(t)$ when $a \ll a_{mQ}$? and when $a \gg a_{mQ}$?
3. Solve the Friedmann equation to find $a(t)$ for this universe (without approximation).

   Hints:

   $$\int \frac{x^{1/2}\mathrm{d}x}{\sqrt{1+\beta x^{3/2}}} = \frac{4}{3\beta}\sqrt{1+\beta x^{3/2}}$$

4. What is the current age of this universe, as a function of H_0 and $\Omega_{m,0}$?
5. Describe the properties of a universe entirely made of quintessence ($\Omega_m \approx 0$). What would be the current age of this universe, and the current particle horizon distance ?

2 Closed Universe

Consider a closed universe which contains only matter : $\Omega_0 = \Omega_{m,0} > 1$.

1. Describe briefly the properties of such a universe, its curvature, and its dynamics (a drawing may help).

2. Write the Friedmann equation for this universe. Compute the value $a_{\max}$ of the scaling factor a at maximum expansion.

3. Show that H_0, Ω_0 and the current universe curvature radius a_0 are linked by:

$$a_0 = \frac{1}{H_0} \frac{1}{\sqrt{\Omega_0 - 1}}$$

4. Calculate the horizon distance $d_{\text{hor}}(t) = a_0 \chi_{\text{hor}}(t)$ as a function of time. Remember that the comoving coordinate χ_{hor} of the horizon at time t is defined by:

$$\chi_{\text{hor}}(t) = \int_0^{a(t)} \frac{dt'}{a_0(t')}$$

Show that at the moment of maximum expansion ($a = a_{\max}$, $t = t(a_{\max})$), the horizon includes the entire universe, *i.e.*:

$$\chi_{\text{hor}}(a_{\max}) = \pi.$$

Hints:

$$\int_A^B \frac{\mathrm{d}x}{\sqrt{x(1-x)}} = \Big[\arcsin(2x-1) \Big]_A^B \qquad x = \frac{\Omega_0 - 1}{\Omega_0} \times a$$

5. Verify that the evolution of the universe may be described by the following parametric equations:

$$a(\eta) = A\,(1 - \cos\eta)$$
$$t(\eta) = B\,(\eta - \sin\eta)$$

Give the expression of A and B as functions of H_0 and Ω_0. What is the value of η at maximum expansion? Describe briefly the resulting dynamics.

6. Show that the age of the universe at maximal expansion is:

$$t(a_{\max}) = \frac{\pi}{2} \frac{1}{H_0} \frac{\Omega_0}{(\Omega_0 - 1)^{3/2}}$$

7. At some time $t_1 > t(a_{\max})$ during the contraction phase of this universe, an astronomer named Edwin Elbbuh discovers that all nearby galaxies have blueshifts ($-1 \leq z < 0$) proportional to their distance ; he measures as well $H_1 < 0$ and $\Omega_1 > 1$. Knowing $H_1 < 0$ and Ω_1, how much time remains between t_1 and the final Big Crunch at $t = t_{\text{crunch}}$? What is the minimum blueshift our astronomer may observe?