2. The smooth expanding Universe

At the starting point of modern observational cosmology is the extraordinary realization that we live in an expanding Universe. This discovery profoundly changed our vision of the Universe, primarily by re-introducing a beginning of times, an arrow of time, and the notion that the properties of the Universe had to change over the course of its history. The astrophysicist Edwin Hubble gets most of the credit for the discovery of expansion, and this is certainly well deserved, although, like always, the work of Hubble was built on the contributions of many others (e.g. Slipher, Lemaître).

To describe the expansion, we need a theory of gravity: without any evidence that the Universe is globally or locally charged, the only known long range force, which can govern the dynamics of cosmic expansion is gravity. We have seen in class that Newtonian gravity allows us to obtain an unexpectedly good description of the expansion. However, Newtonian gravity describes an instantaneous action at a distance, which is incompatible with a system were mass overdensities (galaxies, clusters) are millions of light years apart, and interactions typically propagate at the speed of light. Fortunately, we have a relativistic theory of gravity, General Relativity, which was elaborated shortly before the discovery of cosmic expansion.

In this lesson, we seek a general relativistic description of the expanding Universe. Despite its elegance and simplicity, General Relativity can lead to intimidating computations. Quite fortunately for us, the Universe, beyond scales of ~ 100 Mpc can be described as a smooth, homogeneous, isotropic fluid in expansion, which simplifies its mathematical description tremendously.

An outline of the chapter follows. We first examine §2.1 the symmetries that apply well to the Universe on the largest scales, and show that they imply that the expansion law has a very simple form. We then define §2.2 a coordinate system particularly well suited to the description of the expanding Universe, and derive in these coordinates, the Friedmann-Lemaître-Robertson-Walker metric, (only) metric that follows the symmetries described above (§2.3). With the metric in hand, we can study the trajectories of free particles (also known as geodesics), in particular, the trajectories of photons (§2.4). Knowing the photon trajectories, we can predict (§2.8) how distant sources appear to us, and derive observables, in particular, the flux and apparent angular size of objects of known luminosity or size.

2.1. Isotropy, homogeneity, expansion

Our description of the Universe on the largest scales starts with three fundamental observational facts: from our point of view, the Universe appears to be isotropic (very), homogeneous (quite) and of course, in expansion.

The most striking evidence for isotropy comes from the spectral properties of the Cosmic Microwave Background (CMB). The CMB temperature uniform on the sky to about 1 part in 10^5. Galaxy counts, performed in the visible or the infrared support this evidence. They also show that the Universe does not seem to display structures larger than a few dozen Mpc – i.e. beyond a typical scale of ~ 100 Mpc the Universe appears quite homegeneous. Note that homogeneity does not imply isotropy: for example, we can consider a homogeneous Universe, with a non-isotropic expansion field or matter following a global rotation motion. Nor does isotropy imply homogeneity: it is easy to imagine a Universe in expansion around us, with a density field $\rho(r)$ and velocity field $v(r)$ that depends uniquely on the radial distance to our galaxy. This latter model would imply that we live at a very special place.

Most scientists dislike this idea, and to be frank, there is nothing observationally special with our star or with our Galaxy. As a consequence, we add to our description the requirement that the Universe should look the same to all observers. This is called the Copernican Principle.

The Copernican principle is a very strong symmetry. We have seen in class that the combination of isotropy with the Copernican principle implies homogeneity: if two observers A and B located in different galaxies measure a radially dependent (isotropic) distribution of matter $\rho_A(r)$ and $\rho_B(r)$, it is easy to see that ρ must be constant everywhere in the Universe.

We have also seen that the Copernican principle contraints very strongly the form of the expansion law. If we describe the velocity field around an observer by:

$$\mathbf{v} = \mathbf{H}(\mathbf{r}, t) \tag{2.1.1}$$

then, it is easy to see that $\mathbf{h}$ is a linear operator for $\mathbf{r}$:

$$\mathbf{v} = \mathbf{H}(t) \cdot \mathbf{r} \tag{2.1.2}$$

If we decompose H into its symmetric and anti-symmetric parts:

$$\mathbf{H}(t) = \mathbf{\Sigma}(t) \cdot \mathbf{r} + \mathbf{\Omega}(t) \wedge \mathbf{r} \tag{2.1.3}$$

Isotropy requires that $\mathbf{\Omega}$ be zero (otherwise, we would see a prefered direction on the sky. It also implies that eigenvalues of $\mathbf{\Sigma}$ are equal (otherwise, we would see a non-isotropic expansion in prefered directions).

As a consequence, the expansion law must take the very simple form, for all observers:

$$\mathbf{v} = H(t) \cdot \mathbf{r} \tag{2.1.4}$$

2.2. Comoving coordinates

It is important to note that not all observers see the Universe as isotropic, only the so-called *comoving observers*, which are locally at rest with the bulk of matter in their vicinity. We, for example, are not comoving observers: when we look at the CMB temperature, the first feature we see is a large dipole, which is the result of the peculiar motion of our Galaxy (and our galaxy group). However, it is easy enough to figure out the appropriate boost which would turn us into comoving observers.

A metric depends on the choice of a coordinate system. Although we are quite free to choose whichever coordinate system we like, is it quite obvious that the metric has strong chances to be simpler in coordinates attached to the comoving observers. These classes of coordinates are called *co-moving coordinates*. There is a lot of freedom in the choice of the comoving coordinates: in particular, euclidean coordinates (t, x, y, z) and spherical (polar) coordinates (t, r, θ, ϕ), with the observer (ourselves) at the origin.

Time deserves a special mention. In our ideal Universe, with no matter over/underdensities, all clocks following the expansion (i.e. with no peculiar motion) tick at the same rate. With an infinite amount of time at our disposal, we can propagate a common convention to synchronize our clocks: for example, when the CMB temperature reaches a given value. Hence, it is possible to define a cosmic time, common to all free falling observers.

2.3. The FRLW metric

Let's start from the general expression for the metric:

$$ds^2 = -c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{2.3.1}$$

where the dx^μ denote comoving coordinates. All observers use the same time, and we have $g_{0i} = g_{i0}$ otherwise, there would we a preferred direction in space, which would violate isotropy, therefore, the general form for the metric is:

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix} \tag{2.3.2}$$

So, the problem is now to obtain the metric for the 3-D spatial slices. Homogeneity and isotropy require that these spaces are *maximally symmetric*, i.e. of constant curvature (everywhere). In the most general case, the curvature tensor of a 3D space R_{ijkl} has 6 independent components, each of which being a function of the coordinates. A maximally symmetric space is characterized only by one constant number, its curvature. It is possible to show (see Weinberg, chap 13 for example) that we have 3 possible maximally symmetric spaces:

1. $ds_3^2 = |d\mathbf{x}|^2 = \delta_{ij} dx^i dx^j$, i.e. the 3D Euclidean space. The scalar curvature is zero (flat space).
2. $ds_3^2 = |d\mathbf{x}|^2 + dw^2$ with the constraint that $\mathbf{x}^2 + w^2 = a^2$. This is a 3-sphere of radius a, embedded in a 4-dimensional Euclidean space.
3. $ds_3^2 = |d\mathbf{x}|^2 - dw^2$ with the constraint $w^2 - \mathbf{x}^2 = a^2$. This is a 3-hyperboloid, embedded in a 4-dimensional pseudo-Euclidean space.

Let's examine in more detail the two last cases. First, we can rescale $\mathbf{x} \leftarrow a\mathbf{x}$ and $w \leftarrow aw$, and rewrite the line element in a more compact form:

$$ds_3^2 = |d\mathbf{x}|^2 \pm dw^2, \quad \text{with} \quad |\mathbf{x}|^2 \pm w^2 = 1 \tag{2.3.3}$$

differentiating the second equation and subtituting w and dw, we obtain:

$$ds_3^2 = a^2 \left(|d\mathbf{x}|^2 \pm \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 \mp |\mathbf{x}|^2} \right) \tag{2.3.4}$$

We can write write this in an even more compact form, that comprises the Euclidean case:

$$ds_3^2 = a^2 \left(|d\mathbf{x}|^2 + K \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{1 - K|\mathbf{x}|^2} \right) \tag{2.3.5}$$

with

$$K = \begin{cases} +1 & 3 - \text{sphere} \\ 0 & 3\text{D} - \text{euclidean} \\ -1 & 3\text{D} - \text{hyperboloid} \end{cases} \tag{2.3.6}$$

Allowing a to be a function of time, and putting everything together, we get the general expression for the metric of a isotropic and homogeneous 3-space in expansion:

$$ds^2 = -c^2 dt^2 + a^2(t)\left(d\mathbf{x}^2 + K\frac{(\mathbf{x}\cdot d\mathbf{x})^2}{1 - K\mathbf{x}^2}\right) \tag{2.3.7}$$

So, the components of the metric, this these coordinates are:

$$g_{00} = -c^2, \quad g_{i0} = 0, \quad g_{ij} = a^2(t)\left(\delta_{ij} + K\frac{x^i x^j}{1 - K\mathbf{x}^2}\right) \tag{2.3.8}$$

In spherical polar coordinates, the metric has a slightly different form:

$$ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{2.3.9}$$

In these coordinates, the components of the metric are:

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & a^2(t)/(1 - Kr^2) & 0 & 0 \\ 0 & 0 & a^2(t)r^2 & 0 \\ 0 & 0 & 0 & a^2(t)r^2\sin^2\theta \end{pmatrix} \tag{2.3.10}$$

A last useful coordinate system we have seen in class, consists in using, instead of r the angular coordinate χ, defined as:

$$\chi = \begin{cases} \sin^{-1}\ r & \text{if } K = 1 \\ r & \text{if } K = 0 \\ \sinh^{-1} r & \text{if } K = -1 \end{cases} \tag{2.3.11}$$

In these coordinates, the same metrics can be written as:

$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + a^2(t)\left(d\chi^2 + S_K^2(\chi)\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right) \tag{2.3.12}$$

with

$$S_K(\chi) = \begin{cases} \sin\ \chi & \text{if } K = 1 \\ \chi & \text{if } K = 0 \\ \sinh\chi & \text{if } K = -1 \end{cases} \tag{2.3.13}$$

and again, the metric is diagonal, with:

$$g_{\mu\nu} = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & a^2(t) & 0 & 0 \\ 0 & 0 & a^2(t)S_K^2(\chi) & 0 \\ 0 & 0 & 0 & a^2(t)S_K^2(\chi)\sin^2\theta \end{pmatrix} \tag{2.3.14}$$

In conclusion: the situation is much simpler than what we could have expected. We are left with only one unknown component: $a(t)$, the scale factor that describes the expansion as a function of cosmic time. And, there are coordinates systems in which the metric is diagonal, which will spare us a lot of effort.

2.3.1. Christoffel connections

Now that we have a metric, we can compute the Christoffel symbols, for example in the (t, r, θ, ϕ) coordinates. In these coordinates, the metric is quite simple (in particular it is diagonal) and the computation turns out to be easy. We have seen in class that all connections with two spatial indices vanish:

$$\boxed{\Gamma^0_{00} = 0, \quad \Gamma^0_{0i} = \Gamma^0_{i0} = 0, \quad \Gamma^i_{00} = 0} \tag{2.3.15}$$

and we have computed the two non-vanishing connections with timelike indices:

$$\boxed{\Gamma^0_{ij} = \frac{a\dot{a}}{c^2}\tilde{g}_{ij} \quad \text{and} \quad \Gamma^i_{j0} = \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j} \tag{2.3.16}$$

Exercise 2.3.1. *Compute all the Γ's and show that:*

$$\begin{aligned}
\Gamma^0_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \dot{a}a/[c^2(1-kr^2)] & 0 & 0 \\ 0 & 0 & \dot{a}ar^2/c^2 & 0 \\ 0 & 0 & 0 & \dot{a}ar^2\sin^2\theta/c^2 \end{pmatrix} \\
\Gamma^1_{\alpha\beta} &= \begin{pmatrix} 0 & \dot{a}/a & 0 & 0 \\ \dot{a}/a & kr/[1-kr^2] & 0 & 0 \\ 0 & 0 & -r/[1-kr^2] & 0 \\ 0 & 0 & 0 & -r(1-kr^2)\sin^2\theta \end{pmatrix} \\
\Gamma^2_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & \dot{a}/a & 0 \\ 0 & 0 & 1/r & 0 \\ \dot{a}/a & 1/r & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{pmatrix} \\
\Gamma^3_{\alpha\beta} &= \begin{pmatrix} 0 & 0 & 0 & \dot{a}/a \\ 0 & 0 & 0 & 1/r \\ 0 & 0 & 0 & \cot\theta \\ \dot{a}/a & 1/r & \cot\theta & 0 \end{pmatrix}
\end{aligned} \tag{2.3.17}$$

2.4. Geodesics of the FRLW metric

Now that we know the metrics and its connections, we can compute the geodesics of massive and massless particles. We are particularly interested in the geodesics of photons, which (so far) are our only messengers.

Let $x^\mu(\lambda)$ be the wordline of the particle, where λ is an affine parameter, and $u^\mu = dx^\mu/d\lambda$ the 4-velocity of the particle. The geodesics equation is:

$$\dot{u}^\mu + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta = 0 \tag{2.4.1}$$

There is another form for the geodesic equation, which is:

$$\dot{u}_\mu = \frac{1}{2}(\partial_\mu g_{\alpha\beta}) u^\alpha u^\beta \tag{2.4.2}$$

This second form is quite useful, because it tells us that if the metric is independant of a particular coordinate, x^μ, then u_μ is constant over the geodesics.

Since the metric is isotropic and homegeneous, we can choose our origin anywhere. It is quite convenient to pick it somewhere on the geodesics itself. Let's first consider the ϕ-component of the 4-velocity, u^3. The metric does not depend on ϕ, we immediately see that u_3 is constant. $u_3 = g_{33}u^3 = a^2 S_K^2(\chi 2)\sin^2\theta u^3$. u_3 does vanish at the origin, where $\chi = 0$, and therefore is zero everywhere on the geodesics. We can then consider the θ component. The only metric component that depends on θ is $g_{33} = a^2 r^2 \sin^2\theta$, but since u^3 is zero, $\dot{u}_2$ also vanishes. Again, $u_2 = g_{22}u^2 = a^2 S_K^2(\chi) u^2$ is zero at the origin and therefore everywhere on the geodesics. As a consequence, the geodesics that pass though the origin satisfy:

$$\boxed{\theta = \text{constant}, \quad \phi = \text{constant}} \tag{2.4.3}$$

Similarly, we find that, for the first component $u_1 = g_{11}u^1 = a^2(t)\dot{\chi} = \text{constant}$, and so:

$$\boxed{a^2(t)\dot{\chi} = \text{constant}} \tag{2.4.4}$$

Finally, we can get the first component $u^0 = dt/d\lambda$ from the normalization of the 4-velocity:

$$g_{\mu\nu}u^\mu u^\nu = \begin{cases} 0 & \text{for massless particles} \\ -c^2 & \text{for massive particles} \end{cases} \tag{2.4.5}$$

and we

$$\boxed{\dot{t}^2 = \begin{cases} 1 + a^2\dot{\chi}/c^2 \\ a^2\dot{\chi}/c^2 \end{cases}} \tag{2.4.6}$$

Are fundamental observers in free fall ?

Let's consider an observer, that is at rest at t_0 in this coordinate system. Her 4-velocity at t_0 is therefore $u^\mu = dx^\mu/d\lambda = (1, 0, 0, 0)$. Since Γ^μ_{00} always vanishes, it is easy to see from 2.4.1 that u^μ stays constant. The comoving observers experience no acceleration. They are free falling observers. Furthermore, the proper time of such observers is just dt, the time measured by a comoving clock.

2.5. Cosmological redshift

Let's look in more detail at the photon geodesics. We chose the origin of the coordinate system so that it coincides with our our observatory. We are interested only in the photons we can detect (hence, which intercept the origin). These photons follow the null geodesics:

$$ds^2 = 0 = -c^2dt^2 + a^2(t)d\chi \tag{2.5.1}$$

since $\theta =$ constant and $\phi =$ constant. Therefore:

$$d\chi = \pm\frac{cdt}{a(t)} \tag{2.5.2}$$

and since we are only interested in the photons that are coming towards us ($\dot{\chi} < 0$):

$$d\chi = -\frac{cdt}{a(t)} \tag{2.5.3}$$

If we consider a photon that was emitted by a distance galaxy at t_1 and detected by us at $t_0 > t_1$, we have:

$$\boxed{\chi = \int_{t_1}^{t_0} \frac{cdt}{a(t)}} \tag{2.5.4}$$

Now, let's note λ_1 (resp λ_0) the wavelength of the photon at emission (resp reception), δt_1 (resp δt_0) the time interval between two crests of the electromagnetic wave at emission (resp reception). Then, since the comoving coordinates of the galaxy do not change:

$$\chi = \int_{t_1}^{t_0} \frac{cdt}{a(t)} = \int_{t_1+\delta t_1}^{t_0+\delta t_0} \frac{cdt}{a(t)} \tag{2.5.5}$$

Decomposing and re-arranging the integrals we obtain:

$$\chi = \int_{t_1}^{t_1+\delta t_1} \frac{cdt}{a(t)} = \int_{t_0}^{t_0+\delta t_0} \frac{cdt}{a(t)} \tag{2.5.6}$$

and neglecting the variations of $a(t)$ over the (short) integration interval:

$$\frac{a_0}{a_1} = \frac{\lambda_0}{\lambda_1} \equiv 1 + z \tag{2.5.7}$$

In an expanding Universe, the photons get redder as they travel. This phenomenon is what we call *cosmological redshift*. It can be measured with a spectrograph, taking advantage of the source emission or absorption lines. A very nice thing, it that it is directly connected to the relative variations of the scale factor a between absorption and emission. For example, at a redshift $z = 1$, the typical distances between galaxies were 50% smaller than they are today.

2.5.1. Time dilation

In a similar way, we can consider two events taking place in a distance galaxy at times t_1 and $t_1 + \delta t_1$. These events are luminous and each emits a flash than we can detect later in our galaxy, at t_0 and $t_0 + \delta t_0$. With a similar reasoning, we find that:

$$\frac{\delta t_0}{\delta t_1} = \frac{a_0}{a_1} \tag{2.5.8}$$

So, again, events taking place in a distant galaxy with appear slower to us. This phenomenon is called "time dilation".

2.6. Cosmography

In what follows, we will label the present epoch with a $_0$ subscript, and coordinates of distant emitting objects with a $_1$ subscript. So, we observe (today, at time t_0) a galaxy located at a radial coordinate χ_1. At what time $t_1 < t_0$ were these photons emitted ? Or equivalently, how long did it take to the photons to reach us ? This quantity: $\delta t \equiv t_0 - t_1$ is often called the "lookback time" in cosmological texts.

Equation 2.5.4 gives the link between events connected by photon geodesics, provided that we know the history of the expansion (i.e. the variations of $a(t)$, which in turn, depend on the densities and pressures of the various fluids which populate the Universe).

We do not know yet how to compute that (this will be done in the next lesson). However, if the galaxy is not too distant, or, equivalently, if the photon travel time is small compare to the age of the Universe, if $\delta t \equiv t_0 - t_1 \ll t_0$, we can develop $a(t)$ as a function of δt.

$$a(t) = a(t_0) - \dot{a}(t_0)(t_0 - t_1) + \frac{1}{2}\ddot{a}(t_0)(t_0 - t_1)^2 - \ldots \quad (2.6.1)$$

which can be rewritten:

$$a(t) = a_0 \left[1 - H_0(t_0 - t_1) - H_0^2 q_0 (t_0 - t_1)^2 - \ldots\right] \quad (2.6.2)$$

where we have defined the Hubble parameter:

$$H(t) = \frac{\dot{a}}{a}, \qquad H_0 = H(t_0) \quad (2.6.3)$$

and the *deceleration parameter*:

$$q(t) = -\frac{1}{2}\frac{\ddot{a}(t)a(t)}{\dot{a}^2(t)}, \qquad q_0 = q(t_0) \quad (2.6.4)$$

Note that we could have developed $a(t)$ at higher orders and defined the corresponding parameters. We do not need them here, but there is a lot of (mostly historical) litterature about higher order developments of $a(t)$.

So, we have a relation between the redshift (which we can observe) and the lookback time:

$$1 + z = \frac{a_0}{a} \approx \left[1 - H_0(t_0 - t_1) - H_0^2 q_0 (t_0 - t_1)^2 - \ldots\right]^{-1} \quad (2.6.5)$$

This is all very nice, but it would be more interesting to derive the lookback time (which we cannot directly measure) from the redshift. Do do this, let's invert the power series above. How do we do that ? All we need are the values of dt/dz and d^2t/dz^2, to invert this series up to order 2, we just need the first and second derivatives $t(z)$ at $t = t_0$, which we can derive from $\frac{dz}{dt}(t_0) = H_0^{-1}$ and $d^2z/dt^2 = 2(1 + q_0)H_0^2$. – remember that that $f^{-1\prime} = 1/f$ and $f^{-1\prime\prime} = -f''/f'^3$. We this, we get:

$$t_0 - t_1 \approx \frac{z}{H_0}\left[1 - \left(1 + \frac{1}{2}q_0\right) z + \ldots\right] \quad (2.6.6)$$

Exercise 1: we observe a nearby galaxy and determine its redshift: $z = 0.12$. When were the galaxy photons we observe today emitted ?

We can do the same with the radial coordinate χ:

$$\chi = \int_{t_1}^{t_0} \frac{cdt}{a(t)} \quad (2.6.7)$$

From equation 2.6.2 above we get:

$$\frac{1}{a(t)} = \frac{1}{a_0}\left[1 + (t_0 - t_1)H_0 + \ldots\right] \quad (2.6.8)$$

which gives

$$\chi \approx \frac{c}{H_0}\int_{t_1}^{t_0}\left[1+H(t_0-u)+\ldots\right]du \tag{2.6.9}$$

$$\approx \frac{c}{a_0}\left[(t_0-t_1)+\tfrac{1}{2}H_0(t_0-t_1)^2+\ldots\right] \tag{2.6.10}$$

plugging in the expression of the lookback time we got above, and keeping only the order=2 terms:

$$\chi = \frac{1}{a_0}\frac{cz}{H_0}\left[1-\frac{1}{2}(1+q_0)z+\ldots\right] \tag{2.6.11}$$

So, again, knowing the expansion history to some order, we can map the redshift (which we can observe) to the proper distance (i.e. the distance which we would measure with a tape meter, if we could freeze the expansion and take our time to measure it) $a_0\chi$ of a distant object. Conversely, if we know the proper distances and redshifts of a collection of cosmological objects, we can try and infer H_0 and q_0. This is a game cosmologists have tried to play between the 1930's and the late 1980's without much success (using something more measurable than the proper distances), until they learnt how to use type Ia supernovae as precise distance indicators.

2.7. Mapping coordinates with redshift

So far, we have worked with the Taylor series of $a(t)$, which is valid only at low-z (or lookback times that are small compared to the age of the Universe). Of course, there are exact expressions, that connect the lookback time and radial coordinate with the redshift. All we need is equation 2.5.4 and a change of variable:

$$dz = d(1+z) = d\left(\frac{a_0}{a}\right) = -a_0\frac{da}{a^2} = -\frac{a_0}{a}\frac{\dot{a}}{a}dt = -(1+z)H(z)dt \tag{2.7.1}$$

which gives:

$$\boxed{t_0 - t_1 = \int_{t_1}^{t_0} dt = \int_0^z \frac{dz}{(1+z)H(z)}} \tag{2.7.2}$$

And for the radial coordinate of the emitting object:

$$\boxed{\chi_1 = \int_{t_1}^{t_0}\frac{cdt}{a(t)} = \frac{1}{a_0}\int_0^z\frac{cdz}{H(z)}} \tag{2.7.3}$$

If there is one thing you need to remember from this class, equations 2.7.2 and 2.7.3 are on top of the list. Please remember where they come from, and commit them to memory. You'll need them more than once in the future if you ever do observational cosmology. They are important because they connect the expansion history ($\int dz/H(z)dz$) with possible observables (z, the lookback time or some distance to the object). They are at the core of the classical cosmological tests.

2.8. Distances and volumes

Measuring distances in an expanding Universe is tricky. First, because of the expansion, and the finiteness of the speed of light, we can come up with many definition of the distance between to galaxies, which have moved apart between the times of photon emission and detection. Furthermore, it is easy to come up with definitions of distances which are not practical or simply measurable. In this section, we discuss distances and volumes, and focus on two practical ways to define the distance to an object.

2.8.1. Proper distance

Let's imagine that we freeze the expansion, and are allowed to wander the Universe with a measuring tape. Then, what we measure are the "proper distance" between objects. For example, let's consider a galaxy at coordinate χ (we still take the origin of our coordinate system at our observatory). The proper distance to that galaxy is:

$$d = \underbrace{a(t_0)}_{a_0}\ \chi = \int_0^z \frac{cdz}{H(z)} \tag{2.8.1}$$

unfortunately, proper distances cannot be measured in practice. We can infer them once we have a model of the Universe (the function $H(z)$, which is ultimately something we measure).

2.8.2. Angular distance

Let's turn to something we can measure. Imagine we observe an object of proper transverse size $d\ell$, and placed at radial comoving coordinate χ. And let's consider the radial geodesics connecting the two ends of the object with the observer. The observer will measure an angle $d\theta$ on the sky.

The metric *at the time of emission* allows to connect the proper size of the object, with the angular separation measured on the sky:

$$d\ell = a(t_1)S_K(\chi)d\theta \tag{2.8.2}$$

or equivalently:

$$d\ell = \frac{a}{1+z} S_K(\chi) d\theta \tag{2.8.3}$$

From this, we can define an "angular distance", as we would do in a Euclidean Universe:

$$\boxed{d_A = \frac{d\ell}{d\theta} = \frac{a_0 S_K(\chi)}{1+z}} \tag{2.8.4}$$

Note that the angular distance does not correspond directly to the radial part of the metric (χ) but to the transverse part $S_K(\chi)$. As we will see later in this class, it is used a lot in observational cosmology today, in the framwork of BAO measurements.

2.8.3. Luminosity distance

We now turn to another practical way to measure cosmological distances: we can compare the luminosity ($\mathcal{L}$) of an object with the flux f we measure on earth. Luminosity and flux are not the same thing. We call luminosity is the energy, or the number of photons emitted per unit second in all directions (4π). The flux is the energy (or number of photons) detected per second and per unit surface (i.e. what we measure in practice with a telescope). To fix ideas, let's say that we measure luminosities in $W = J/s$ and fluxes in W/m^2.

So, let's consider a source, at comoving coordinate χ, that emits N photons during a time interval Δt_1. To simplify things, we assume that all these photons have the same energy E_1 (at emission time). The luminosity of the source is:

$$\mathcal{L} = \frac{N E_1}{\Delta t_1} \tag{2.8.5}$$

These photons are detected much later on Earth, with a telescope of primary mirror area δA. The flux is a function of (1) the energy of the incoming photons $E_1/(1+z)$ (remember the redshift?) (2) the time interval during which these photons arrive $(1+z)\Delta t_1$ (time dilation, remember ?) and (3) the fraction of the sphere of diameter $a_0\chi$ covered by our primary mirror.

The surface element of a sphere of radius χ is:

$$dA = d\ell_\theta d\ell_\phi = a_0^2 S_K^2(\chi) \sin\theta d\theta d\phi \tag{2.8.6}$$

integrating on θ and phi, we get:

$$A = 4\pi a_0^2 S_K^2(\chi) \tag{2.8.7}$$

Putting everything together, we can write the source flux (energy per second per unit area):

$$
\begin{aligned}
f &= \frac{1}{4\pi a_0^2 S_K^2(\chi)} \times \frac{N E_0}{\Delta t_0} \\
&= \frac{1}{4\pi a_0^2 S_K^2(\chi)} \times \frac{N E_1/(1+z)}{\Delta t_1 \times (1+z)} \\
&= \frac{\mathcal{L}}{4\pi a_0^2 S_K^2(\chi)} \times \frac{1}{(1+z)^2}
\end{aligned}
\tag{2.8.8}
$$

If we define the luminosity distance d_L, as we would do in a Euclidean space:

$$
f = \frac{\mathcal{L}}{4\pi d_L^2} \tag{2.8.9}
$$

and identify with equation 2.8.8, we obtain:

$$
\boxed{d_L = a_0 S_K(\chi)(1+z) = d_A (1+z)^2} \tag{2.8.10}
$$

This is our second operational definition of a cosmological distance: if we know the luminosity of a source, then measuring it flux today allows us to infer the luminosity distance. Again, the luminosity distance probes the transverse part of the metric (not χ but $S_K(\chi)$). Luminosity distances have played a very important role in observational cosmology when people realized that type Ia supernovae (SNe Ia), i.e. thermonuclear explosions of white dwarfs are excellent standard candles (i.e. have all the same luminosity, with a small dispersion). In the late 1990's SNe Ia have allowed to map the expansion history of the Universe beyond the Hubble law, and to discover the acceleration of cosmic expansion.