3. The Friedmann equations

Up to now, we have studied the geometric and kinematic properties of the FRLW metric. But we still have to determine the evolution of $a(t)$ (the only unknown left in the metric), i.e. we need to solve the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{3.0.1}$$

which connects the metric and its first and second derivatives with the energy-momentum tensor of the matter and radiation that fills the Universe.

In §3.1 we compute the non-zero components of the Einstein tensor. We then (§3.2) discuss the structure of the energy-momentum tensor, on the right hand side of the field equation. Putting all together (§3.3), we obtain the Friedmann equations. We give an explicit relation for the history of expansion, $H(z)$ as a function of the cosmological parameters 3.4 and observable quantities 3.5. We present a few textbook solutions of the Friedmann equation (§3.6).

3.1. The Einstein tensor

Our first step is to compute the non-zero components of the Einstein tensor. This is a little tedious, but really there is no trap and subtleties.

We start with the components of the Ricci tensor:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\beta}_{\mu\nu}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\beta\nu} \tag{3.1.1}$$

Computing R_{00} turns out to be relatively easy:

$$\begin{aligned}
R_{00} &= \cancel{\Gamma^{\alpha}_{00,\alpha}} - \Gamma^{\alpha}_{0\alpha,0} + \cancel{\Gamma^{\beta}_{00}\Gamma^{\alpha}_{\beta\alpha}} - \Gamma^{\beta}_{0\alpha}\Gamma^{\alpha}_{\beta 0} \\
&= -\Gamma^{\alpha}_{0\alpha,0} - \Gamma^{\beta}_{0\alpha}\Gamma^{\alpha}_{\beta 0} \\
&= -\Gamma^{i}_{0i,0} - \Gamma^{j}_{0i}\Gamma j0^{i} \\
&= -\frac{\partial}{\partial t}\left[\frac{\dot{a}}{a}\delta^{i}_{i}\right] - \left[\left(\frac{\dot{a}}{a}\right)^{2}\delta^{j}_{i}\delta^{i}_{j}\right] \\
&= -3\frac{\ddot{a}}{a}
\end{aligned} \tag{3.1.2}$$

The R_{0i}'s vanish:

$$\begin{aligned}
R_{0i} = R_{i0} &= \Gamma^{\alpha}_{i0,\alpha} - \Gamma^{\alpha}_{i\alpha,0} + \Gamma^{\beta}_{i0}\Gamma^{\alpha}_{\beta\alpha} - \Gamma^{\beta}_{i\alpha}\Gamma^{\alpha}_{\beta j} \\
&= \Gamma^{\alpha}_{i0,\alpha} - \Gamma^{j}_{ij,0} + \Gamma^{j}_{i0}\Gamma^{k}_{jk} - \Gamma^{j}_{ik}\Gamma^{k}_{j0} \\
&= \cancel{\frac{\partial}{\partial x^{j}}\left(\frac{\dot{a}}{a}\right)}\delta^{j}_{i} - \cancel{\frac{\partial}{\partial t}\Gamma^{j}_{ij}} + \left(\frac{\dot{a}}{a}\right)\left[\delta^{j}_{i}\Gamma^{k}_{jk} - \delta^{k}_{j}\Gamma^{j}_{ik}\right] \\
&= \left(\frac{\dot{a}}{a}\right)\left[\Gamma^{k}_{ik} - \Gamma^{k}_{ik}\right] \\
&= 0
\end{aligned} \tag{3.1.3}$$

The only difficult part are the R_{ij}. It is possible to compute them from the connections given in 2.3.17. The computation is long and I don't have the time to type it here today. So, here is the result:

$$\begin{aligned}
R_{ii} &= \left(\ddot{a}a + 2\dot{a}^{2} + 2k\right)\tilde{g}_{ii} \\
&= \left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^{2} + \frac{2k}{a^{2}}\right)g_{ii}
\end{aligned} \tag{3.1.4}$$

The last bit is the Ricci scalar:

$$\begin{aligned}
R &= g^{\mu\nu}R_{\mu\nu} \\
&= -R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} \\
&= 3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^{2} + \frac{2k}{a^{2}}\right) \\
&= 6\frac{\ddot{a}}{a} + 6\left(\frac{\dot{a}}{a}\right)^{2} + 6\frac{k}{a^{2}}
\end{aligned} \tag{3.1.5}$$

With all of this in hand, it is now straightforward to compute the Einstein tensor:

$$\boxed{\begin{aligned} G_{00} &= R_{00} - \frac{1}{2} R g_{00} \\ &= 3\left(\frac{\dot{a}}{a}\right)^2 + \frac{3k}{a^2} \end{aligned}} \tag{3.1.6}$$

$$\boxed{\begin{aligned} G_{ii} &= R_{ii} - \frac{1}{2} R g_{ii} \\ &= -\left(2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right) g_{ii} \end{aligned}} \tag{3.1.7}$$

3.2. The Energy-Momentum tensor

We have seen in class that the energy density of a fluid is not a scalar. A covariant description requires a rank-2 tensor. Comoving observers, in the frame of the fluid, see it as an isotropic continuum. Isotropy and homogeneity dictate that:

$$T^{\mu\nu} = \begin{pmatrix} \rho(t)c^2 & 0 & 0 & 0 \\ 0 & p(t) & 0 & 0 \\ 0 & 0 & p(t) & 0 \\ 0 & 0 & 0 & p(t) \end{pmatrix} \tag{3.2.1}$$

or equivalently:

$$T^{\mu\nu} = (\rho c^2 + p) u^\mu u^\nu + p\eta^{\mu\nu} \tag{3.2.2}$$

$T^{\mu\nu}$ is a tensor, since u^μ is a tensor, and ρ and p are defined in the fluid restframe. In any coordinate system, $T^{\mu\nu}$ can be written as:

$$T^{\mu\nu} = (\rho c^2 + p) U^\mu U^\nu + p g^{\mu\nu} \tag{3.2.3}$$

The energy-momentum tensor obeys a conservation equation of the form $T^{\mu\nu}_{;\nu} = 0$, where the semi-colon denotes the covariant derivative:

$$T^{\mu\nu}_{;\nu} = T^{\mu\nu}_{,\nu} + \Gamma^\mu_{\alpha\beta} T^{\alpha\beta} + \Gamma^\beta_{\alpha\beta} T^{\mu\alpha} \tag{3.2.4}$$

The spatial terms $T^{i\nu}_{;\nu}$ do not contain useful information. On the other hand, the $T^{0\nu}_{;\nu}$ term yields the so-called continuity equation:

$$\boxed{\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p/c^2) = 0} \tag{3.2.5}$$

Exercise 1: Compute $T^{0\nu}_{;\nu}$ and show that it yields the continuity equation.

The density and pressure of a fluid are related by an equation of state. We assume that all cosmological fluids have an equation of state which is simple, and can be written as:

$$p = w\varepsilon = w\rho c^2 \tag{3.2.6}$$

Combining the continuity equation (3.2.5) and the equation of state of the fluid (3.2.6), we obtain:

$$\boxed{\rho \propto a^{-3(1+w)}} \tag{3.2.7}$$

giving the evolution of the fluid density (and pressure) as a function of the scale factor. As we see, the evolution of the density depends on the physics of the fluid, encoded in the equation of state. For example, for example, for pressureless non-relativistic matter ($w = 0$), radiation $w = 1/3$ and a cosmological constant ($w = -1$), we have respetively:

$$\rho \propto \begin{cases} a^{-3} & \text{non-relativistic matter}(w = 0) \\ a^{-4} & \text{radiation}(w = 1/3) \\ a^{0} & \text{cosmological constant}(w = -1) \end{cases} \tag{3.2.8}$$

The cosmological fluid may have multiple components (matter, radiation, ...) present at the same time. If we assume that they do not interact, then, we may write $T^{\mu\nu}$ as:

$$T^{\mu\nu} = \sum_a T^{\mu\nu}_{(a)} \tag{3.2.9}$$

If in addition, they are all perfect fluid, and share the same velocity field, then, using 3.2.3, we see immediately that:

$$\rho_{\text{tot}} = \sum_a \rho_a \quad p = \sum_a p_a \tag{3.2.10}$$

Finally, if again, the fluids do not interact together, the continuity equation holds for each fluid separately, which means that each fluid will obey an evolution equation of the form: $\rho_a \propto a^{-3(1+w_a)}$ and evolve independentely of the other components.

3.3. Friedmann equations

Putting together the lhs and rhs computed in the two previous sections, we obtain 4 equations. We notice that the 3 spatial components are redundant, and we end up with two equations, relating the derivatives of a and the pressures and densities of the various fluids:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} &= \frac{8\pi G}{3c^2}\rho c^2 && \text{Friedmann-I} \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3c^2}(\rho c^2 + 3p) && \text{Friedmann-II} \end{aligned} \tag{3.3.1}$$

So, in the end, with one fluid, we have 3 unknown functions: $a(t)$ the density ρ and pressure p of the fluid, and we can think of four constraints: the two Friedmann equations, the continuity equation and the equation of state. In fact, these four constraints are only 3, because the continuity equation is already contained in the Friedmann equations, as we have seen in class. So, the two Friedmann equations are actually redundant.

Exercise 2: Derive the continuity equation from the Friedmann equation. *Hint:* take the time derivative of the first Friedmann equation. After some re-arrangement, you'll be able to substitude the $\ddot{a}/a$ and k/a^2 terms by injecting the Friedmann equations.

Similarly, with n fluids, we have $1+2n$ unknowns, and $1+2n$ equations to constrain these unknowns (1 continuity equation and one equation of state per fluid), and the Friedmann equations are:

$$\boxed{\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} &= \frac{8\pi G}{3c^2}c^2\sum_i \rho_i \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3c^2}(c^2\sum_i \rho_i + 3\sum_i p_i) \end{aligned}} \tag{3.3.2}$$

Enters the cosmological constant Finally, as we have seen in class, there is an additional ingredient we need to talk about. One of the original forms of the Einstein equation is:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{3.3.3}$$

where Λ is the *cosmological constant*, introduced by Einstein to obtain a static solution for a homegeneous universe, before the discovery of the expansion. The Λ term did the job, without breaking the zero-divergence of the lhs of the equation. The static solution obtained was unstable though, and the Λ term was rejected after the discovery of cosmic expansion. The corresponding Friedmann equations are:

$$\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} - \frac{\Lambda c^2}{3} &= \frac{8\pi G}{3c^2}c^2\sum_i \rho_i \\ \frac{\ddot{a}}{a} - \frac{\Lambda c^2}{3} &= -\frac{4\pi G}{3c^2}(c^2\sum_i \rho_i + 3\sum_i p_i)\end{aligned} \tag{3.3.4}$$

It is possible to interpret the Λ term as another energy density. If we define:

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G}, \quad p_\Lambda = -\rho_\Lambda c^2 \tag{3.3.5}$$

then, the Friedmann equations above can be rewritten as:

$$\boxed{\begin{aligned}\left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} &= \frac{8\pi G}{3c^2}\left(\sum_i \rho_i c^2 + \rho_\Lambda c^2\right) \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3c^2}(c^2\sum_i \rho_i + 3\sum_i p_i - 2\rho_\Lambda c^2) \\ &= -\frac{4\pi G}{3c^2}(\sum_i \rho_i c^2 + \rho_\Lambda c^2 + 3\sum_i p_i + 3p_\Lambda)\end{aligned}} \tag{3.3.6}$$

So Λ may be interpret as a fluid of density $\Lambda c^2/8\pi G$ and of negative pressure, with an equation of state:

$$\boxed{p_\Lambda = -\rho_\Lambda} \tag{3.3.7}$$

The continuity equation gives:

$$\rho_\Lambda \propto a^0, \quad \text{and so} \rho_\Lambda(t) = \rho_\Lambda(0) = \frac{\Lambda c^2}{8\pi G} \tag{3.3.8}$$

so, such a fluid has an energy density which is constant as a function of time, and is generally identified with a non-zero *vacuum energy*.

Exercice: From the second Friedmann equation, write a necessary condition on ρ_Λ to have a static Universe. Is the corresponding solution stable ? Why ?

As we said above, the cosmological constant was rejected as unnecessary after the discovery of cosmic expansion. However, at the turn of the century, Λ came back as two groups of cosmologists announced to have discovered an acceleration of cosmic expansion. The fact is that Λ has the power to do that also. Indeed, let's consider for example a Universe with non relativistic matter (density ρ_m) and radiation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\rho_m + \rho_\Lambda + \cancel{3p} + 3p_\Lambda\right) = -\frac{4\pi G}{3c^2}\left(\rho_m - 2\rho_\Lambda\right) \tag{3.3.9}$$

and we see that cosmic expansion is accelerated if $\rho_\Lambda > \rho_m/2$.

This result has been confirmed by many independent surveys using many different techniques (BAO's, SNe Ia's, ...). However, there are however many theoretical problems with a vacuum energy, the first being that the vaccum energy measured by cosmologists is 120 orders of magnitudes below the estimates that can be drawn from field theory. Given the uncertainty on the nature of this repulsive term, it is customary to extend the model, and consider a "Dark Energy" fluid, X, with an unknown equation of state $p_X = w_X \rho_X$.

In the remainder of this section, we will consider a Universe, filled with (1) non relativisitc matter ρ_m ($w = 0$) (2) radiation, i.e. photons and/or relativistic neutrinos, ρ_r, $w = 1/3$) and Dark Energy ρ_X, (w_X).

3.4. Standard form of the Friedmann equation

We are now in a position to determine the dynamical evolution of the Universe. As we have seen, we need to use one of the Friedmann equations, plus the equations of state and continuity equation for each of the fluids in the Universe. Let's start from the first Friedmann equation:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 + \frac{Kc^2}{a^2} &= \frac{8\pi G}{3c^2}\rho c^2 \\ H^2 + \frac{Kc^2}{a^2} &= \frac{8\pi G}{3c^2}\rho c^2 \\ H^2\left(1 - \frac{8\pi G}{3H^2}\rho\right) &= -\frac{Kc^2}{a^2} \end{aligned} \tag{3.4.1}$$

$8\pi G/3H^2$ has the dimensions of an energy density, and is called the "critical density". As we have seen in class, it customary to express the energy densities in units of the critical density:

$$\boxed{\rho_c(t) \equiv \frac{8\pi G}{3H^2}, \quad \Omega_i(t) = \frac{\rho_i(t)}{\rho_c(t)}} \tag{3.4.2}$$

and so, we have:

$$\boxed{H^2\left(1 - \Omega(t)\right) = -\frac{Kc^2}{a^2}} \tag{3.4.3}$$

We see that there is a direct connection between the critical energy density and the geometry of the Universe, as the curvature will be respectively positive, zero or negative if $\Omega > 1$, $\Omega = 1$ or $\Omega < 1$ respectively.

Today, we have:

$$H_0^2(1-\Omega_0) = -\frac{Kc^2}{a_0^2} \tag{3.4.4}$$

so, we have an expression of the curvature as a function of H_0 and the total density today.

Starting from:

$$H^2 = \frac{8\pi G}{3}\left(\rho_m + \rho_r + \rho_X\right) - \frac{Kc^2}{a^2} \tag{3.4.5}$$

we can substitute the value of K and the evolutions of the densities of the various fluids:

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\left(\rho_m + \rho_r + \rho_X\right) - \frac{Kc^2}{a_0^2}\frac{a_0^2}{a^2} \\ &= \frac{8\pi G}{3}\left(\rho_{m,0}\left(\frac{a_0}{a}\right)^3 + \rho_{r,0}\left(\frac{a_0}{a}\right)^4 + \rho_X\left(\frac{a_0}{a}\right)^{3(1+w_X)}\right) + H_0^2(1-\Omega_0)\frac{a_0^2}{a^2} \\ &= H_0^2\left(\Omega_{m,0}(1+z)^3 + \Omega_{r,0}(1+z)^4 + \Omega_{X,0}(1+z)^{3(1+w_X)} + (1-\Omega_0)(1+z)^2\right) \end{aligned} \tag{3.4.6}$$

This is the standard form of the Friedmann equation, which connects the evolution of the Hubble parameter $H(z)$ with the densities *today*. We generally drop the '0' subscripts, and define $\Omega_K = 1 - \Omega_0$, which gives:

$$\boxed{H(z) = H_0\left[\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_X(1+z)^{3(1+w)} + \Omega_K(1+z)^2\right]^{1/2}} \tag{3.4.7}$$

Have we made any progress ? Yes, of course ! We have a prediction for $H(z)$ as a function of the matter, radiation and dark energy densities today, we have a good idea of how these densities evolve with z. And above all, we can use the expression of $H(z)$ above to compute distances, in other terms, we can connect the evolution of cosmic expansion to quantities which can all be observed.

3.5. Distance-redshift relations

Remember equation 2.7.3. We can write it as:

$$\begin{aligned} \chi &= \frac{1}{a_0}\int_0^z \frac{cdz}{H(z)} \\ &= \frac{c}{H_0}\sqrt{-K/\Omega_K}\int_0^z \frac{cdz}{H(z)} \end{aligned}$$

If we know the Ω's, we have an expression for $H(z)$, hence, we know how to map redshifts to radial coordinates:

For example, the angular distance:

$$d_A = \frac{a_0 S_K(\chi)}{(1+z)} \tag{3.5.1}$$

can be rewritten as:

$$d_A = \begin{cases} \frac{c}{H_0}\frac{1}{1+z}\sqrt{-K/\Omega_K}\sin\left(H_0\sqrt{-K\Omega_K}\int_0^z \frac{dz}{H(z)}\right) & \text{if } K > 0 \\ \frac{1}{1+z}\int_0^z \frac{cdz}{H(z)} & \text{if } K = 0 \\ \frac{c}{H_0}\frac{1}{1+z}\sqrt{-K/\Omega_K}\sinh\left(H_0\sqrt{-K\Omega_K}\int_0^z \frac{dz}{H(z)}\right) & \text{if } K < 0 \end{cases} \tag{3.5.2}$$

We get a similar expression for $d_L(z) = (1+z)^2 d_A(z)$.

3.6. Integrating the Friedmann equation for simple cases

When the energy densities are dominated by one single component, it is quite easy to integrate the Friedmann equation and get the evolution of a as a function of cosmic time.

Let's define:

$$x \equiv \frac{a}{a_0} = \frac{1}{1+z} \tag{3.6.1}$$

Then, the Friedmann equation can be rewritten:

$$\frac{dx}{dt} = H_0\left(\Omega_m/x + \Omega_r/x^2 + \Omega_X x^{2+3(1+w)} + \Omega_K\right)^{1/2} \tag{3.6.2}$$

3.6.1. The age of the Universe

From the expression above, computing the age of the Universe is straightforward:

$$t = \int_0^t dt' = \frac{1}{H_0}\int_0^1 \frac{dx}{\left(\Omega_m/x + \Omega_r/x^2 + \Omega_X x^{2+3(1+w)} + \Omega_K\right)^{1/2}} \tag{3.6.3}$$

Exercice: what is the age of a flat, matter dominated Universe $\Omega_m = 1, \Omega_r \ll 1, \Omega_X = 0, H_0 = 70$km/s/Mpc ? Compare it with the age of our Universe.

3.6.2. Lookback time

We have defined the lookback time in §2.6 and given already a general expression for it in equation 2.7.2. Pluggin in the Friedmann equation, we get:

$$t_0 - t_1 = \int_{t_1}^{t_0} dt' = \int_{a_1/a_0=1/1+z}^{1} \frac{dx}{(\Omega_m/x + \Omega_r/x^2 + \Omega_X x^{2+3(1+w)} + \Omega_K)^{1/2}} \quad (3.6.4)$$

Exercise: For a flat, matter-dominated Universe, with $H_0 = 70$km/s/Mpc, what is the lookback time between $z = 3$ and now ? Compare it with the age of the same Universe.

3.6.3. Integrating the Friedmann equation

In most cases, the Friedmann equation cannot be integrated analytically – however, it is straightforward to solve numerically, for example, in a python notebook. However, in some cases (that do not correspond to our Universe, unfortunately), simple analytical solutions can be derived.

Empty Universe (Milne)

$$t = \frac{1}{H_0}\int_0^{a/a_0} \frac{dx}{\Omega_K^{1/2}} = \frac{1}{H_0}\frac{a}{a_0} \Rightarrow \boxed{\frac{a}{a_0} = H_0 t} \quad (3.6.5)$$

The age of the Milne Universe is exactly the Hubble time, $1/H_0$.

Radiation-dominated Universe

i.e. $\Omega_r \gg \Omega_i$, $\Omega_K \ll 1$.

$$t = \frac{1}{H_0}\int_0^{a/a_0} \frac{x dx}{\sqrt{\Omega_r}} = \frac{1}{2H_0\sqrt{\Omega_r}}\left(\frac{a}{a_0}\right)^2 \Rightarrow \boxed{\frac{a}{a_0} = (2H_0\sqrt{\Omega_r} t)^{1/2}} \quad (3.6.6)$$

The age of a radiation dominated Universe is $1/2H_0\sqrt{\Omega_r}$

Matter-dominated Universe

i.e. $\Omega_m \gg \Omega_i$, $\Omega_K \ll 1$.

$$t = \frac{1}{H_0}\int_{a_{eq}}^{a/a_0} \frac{x^{1/2}dx}{\sqrt{\Omega_m}} = \frac{2}{3H_0\sqrt{\Omega_m}}\left(\frac{a}{a_0}\right)^{3/2} \Rightarrow \boxed{\frac{a}{a_0} \approx \left(\frac{3}{2}H_0\sqrt{\Omega_m}t\right)^{2/3}} \tag{3.6.7}$$

Age $\approx 2/3H_0\sqrt{\Omega_m}$

Λ-dominated Universe

$$t = \int_{t_\star}^{t} dt = \frac{1}{H_0}\int_{a_\star/a_0}^{1} \frac{dx}{\sqrt{\Omega_\Lambda}x} = \frac{1}{H_0}\frac{1}{\sqrt{\Omega_\Lambda}}\log\left(\frac{a}{a_\star}\right) \Rightarrow \boxed{\frac{a}{a_\star} = e^{H_0\sqrt{\Omega_\Lambda}(t-t_\star)}} \tag{3.6.8}$$

3.7. The evolution of Ω with time

We have not examined yet the evolution of curvature as a function of time (or redshift).

$$\Omega_K = 1 - \Omega = -\frac{c^2K}{H^2a^2} \tag{3.7.1}$$

Substituting c^2K as we have done before, we obtain:

$$\Omega_K(z)\left(\frac{H_0(1+z)}{H(z)}\right)^2 \Omega_K \tag{3.7.2}$$

Substituting $H(z)$ from 3.4.7, we get:

$$\Omega_K(z) = \frac{\Omega_K}{\Omega_m(1+z) + \Omega_r(1+z)^2 + \Omega_X(1+z)^{1+3w} + \Omega_K} \tag{3.7.3}$$

We see that, if we let aside the models with only vacuum energy (or say a fluid with $w < -1/3$), in all other models $\Omega_K(z)$ becomes very close to 1 at very high redshifts. This fine tuning problem is one of the motivation for inflationary mechanisms in the early Universe.