Structure formation and Baryon Acoustic Oscillations

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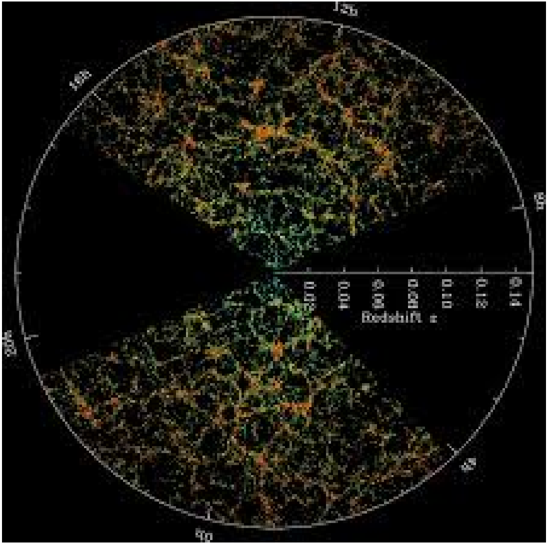
Outline

- Observed inhomogeneities
  - Large-Scale Structure (LSS)
  - Clusters of galaxies
- Spherical collapse model
  - CDM-only
  - CDM+baryons
- Determination of Ω_M and Ω_Λ with BAO
- Description of the inhomogenious universe
  - Power spectra of density and potential fluctuations
- Time evolution of Fourier modes
  - Inflation, Radiation, and Matter epochs
- Hot Dark Matter

The universe is not homogeneous



Slice of the nearby universe ($z < 0.14$) from Sloan Digital Sky Survey (SDSS)

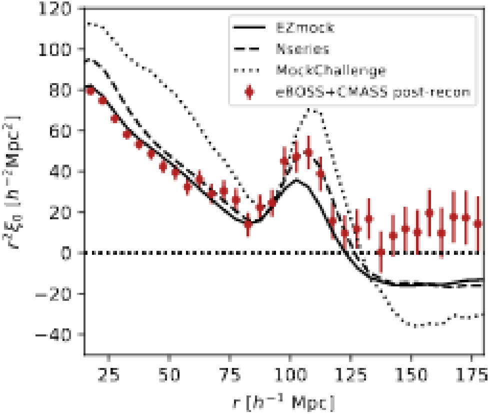
Universe homogeneous when averaged over distances corresponding to
$\Delta z \sim 0.05$
$\Delta r \sim 0.05c/H_0$
$\sim 150h^{-1}Mpc$

$h = H_0/100\mathrm{km\,s^{-1}Mpc^{-1}}$
~ 0.7

$d_H \equiv c/H_0 = 2998h^{-1}\mathrm{Mpc}$

On average $n_{gal} \approx 0.02\mathrm{Mpc}^{-3}$, but the positions are correlated.

Correlation function at $z \approx 0.7$ [arXiv:2007.08993]



$\xi(r) \equiv$ excess probability (over random) to have two galaxies separated by r.

$\xi(20h^{-1}\mathrm{Mpc}) \approx 0.2$

$\xi(100h^{-1}\mathrm{Mpc}) \approx 0.004$ (BAO peak)

$\xi(r > 150h^{-1}\mathrm{Mpc}) \approx 0$

One aim of structure formation theories is to understand $\xi(r, z)$.

Bound Structures

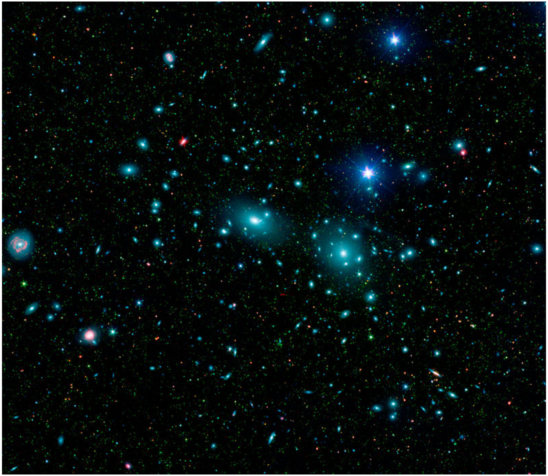
- Galaxies: $10^8 M_\odot < M \leq 10^{13} M_\odot$
  $\rho/\bar{\rho} \approx 10^5 \qquad v_{rot}^2/c^2 \sim (1/2)GM/Rc^2 \approx 10^{-6}$
- Galaxy clusters: $M \leq 10^{15} M_\odot$
  $\rho/\bar{\rho} \approx 10^3 \qquad \langle v^2 \rangle/c^2 \sim (1/2)GM/Rc^2 \approx 10^{-5}$
- Stars: $10^{-1} M_\odot < M < 20 M_\odot$
  $\rho/\bar{\rho} \approx 10^{29} \qquad GM_\odot/R_\odot c^2 = 2 \times 10^{-6}$
- Black holes: $10 M_\odot (?) < M < 10^6 M_\odot$
  $10^{46} > \rho/\bar{\rho} > 10^{28} \qquad GM/Rc^2 = 1$

The density is very inhomogeneous but space-time is very homogeneous

$\text{metric} = (-1, 1, 1, 1) + \text{order } \Phi$

$\Rightarrow$ use of RW metric justified?

Galaxy clusters: largest bound objects



Coma Cluster:

$z = 0.023 \Rightarrow D \approx 99\mathrm{Mpc}$

> 1000 galaxies

$M_{coma} \sim 10^{15} M_{\odot}$

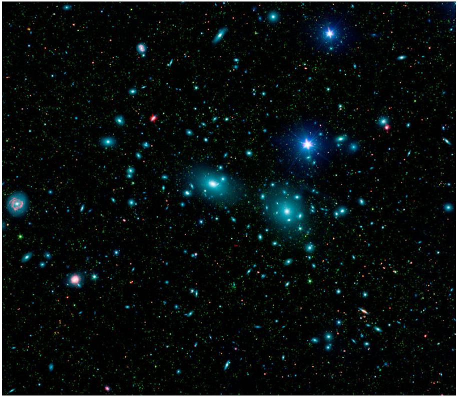
$R_{coma} \sim 1$ Mpc

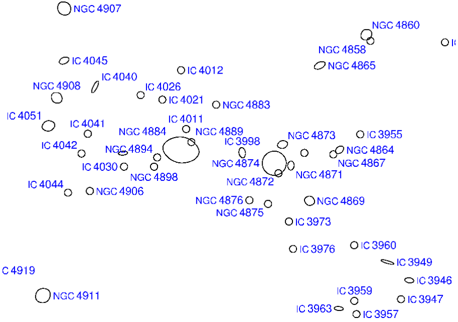
$\rho_{coma}/\rho_0 \sim 10^3$

Gravitational potential:

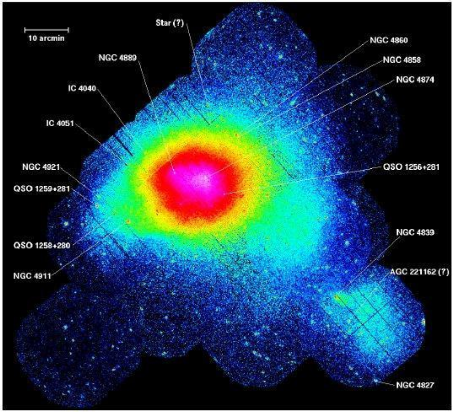
$$\Delta\Phi_g \sim GM_{coma}/R_{coma}c^2 \sim 2\langle v^2\rangle/c^2 \sim 2\times 10^{-5}$$

Coma Cluster: ≈ 1000 galaxies





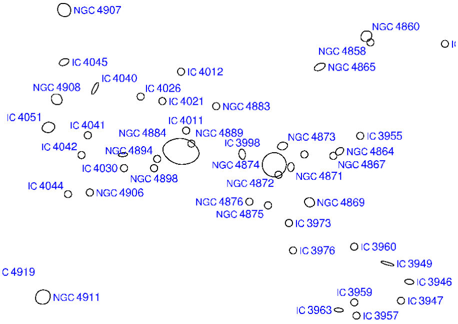
Coma Cluster: hot gas $\Rightarrow$ x-rays



Coma Cluster of galaxies

Image courtesy of U. Briel, MPE Garching, Germany

European Space Agency



$M_{gas} \sim 10\Sigma M_{galaxies}$ (from x-ray observations)

$M_{total} \sim 5M_{gas}$ (from gravitational lensing of background galaxies)

Temperature of a gravitationally-bound cluster

Gas component in hydrostatic equilibrium
$\Rightarrow$ Pressure gradient balances gravitational force on each volume element.
Very approximately

$$\frac{dP}{dr} \approx \frac{mean\, P}{R} \approx \frac{mean\, nkT}{R} \approx \frac{GM_{cluster}\, mean\, \rho}{R^2}$$

$$kT_{cluster} \approx \frac{GM_{cluster}}{c^2 R} m_p c^2 \approx 10^{-5} m_p c^2 \approx 10 keV$$

Exercise: Calculate the mean free path of photons in the cluster. Do the blackbody xrays escape the cluster?

Solution

Exercise: Calculate the mean free path of photons in the cluster.

$$mfp = \frac{1}{n_e \sigma_T} \qquad n_e \approx \frac{1}{5}\frac{M_{cl}}{m_p}\frac{1}{R_{cl}^3} \qquad \sigma_T \approx \frac{\alpha^2(\hbar c)^2}{m_e^2 c^2}$$

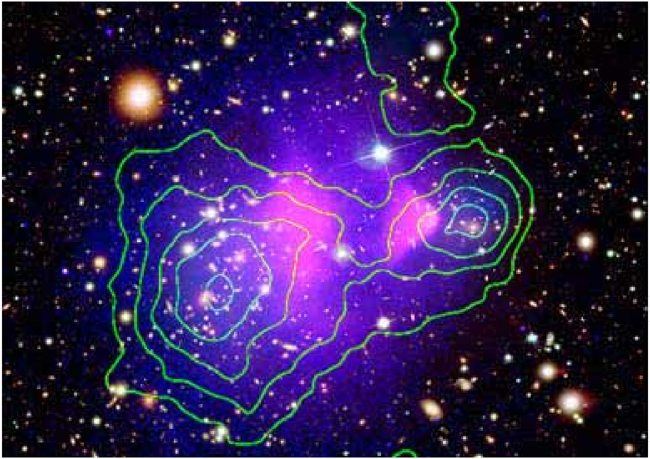
$$R_{cl} \approx \frac{GM_{cl}}{10^{-5}} \approx \frac{M_{cl}}{m_{pl}^2 10^{-5}} \quad (\hbar = c = 1) \qquad m_{pl}c^2 = \sqrt{\frac{\hbar c^3}{G}} \approx 10^{19}\, GeV$$

$$\frac{mfp}{R_{cl}} \approx \frac{m_p}{M_{cl}}\left(\frac{M_{cl}}{m_{pl}^2 10^{-5}}\right)^2 \frac{m_e^2}{\alpha^2} \approx \frac{m_p^2 m_e^2}{10^{-10}\alpha^2 m_{pl}^4}\frac{M_{cl}}{M_\odot}\frac{M_\odot}{m_p}$$

$$\frac{M_\odot}{m_p} \approx 10^{57} \approx \left(\frac{m_{pl}}{m_p}\right)^3 \qquad \Rightarrow \frac{mfp}{R_{cl}} \approx 10^{10}\frac{m_e^2}{m_p m_{pl}\alpha^2}\frac{M_{cl}}{M_\odot} \approx 10^{-7}\frac{M_{cl}}{M_\odot}$$

$\Rightarrow$ Large clusters ($M_{cl} > 10^7 M_\odot$) are transparent

Bullet Cluster(s): dark matter, hot gas, galaxies



Hot gas (colored regions) separated from dark matter and galaxies in collision between two clusters.

Contours show mass distribution as derived from lensing of background galaxies

General idea of structure formation

Today's bound objects started as primordial expanding regions with slight overdensities.

The excess gravitation due to the excess mass in the overdense regions slowed their expansion, eventually reversing it, leading to collapse and the formation of a bound structure.

We can follow this process $\approx$analytically for a spherical overdense region in an otherwise homogeneous universe $\Rightarrow$ "spherical collapse model".

Spherical collapse model for $\Omega_M = 1$

Relevance: $\Omega_M \approx 0.3$ today but was near unity in the past:

$$\Omega_M(a) = \frac{\rho_M(a)}{3H(a)^2/8\pi G} = \frac{\rho_M(a=a_0)\hat{a}^{-3}}{\frac{3H_0^2}{8\pi G}[\Omega_\Lambda(a_0) + \Omega_M(a_0)\hat{a}^{-3} + \Omega_R(a_0)\hat{a}^{-4}]}$$

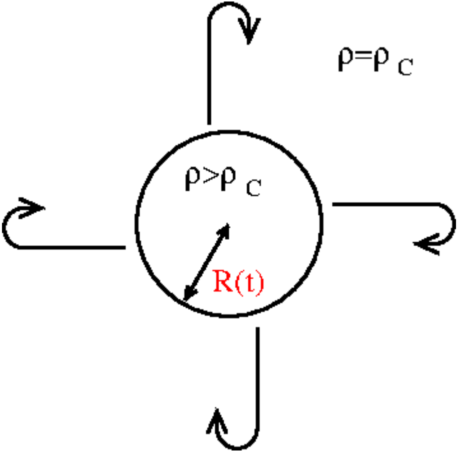
where $\hat{a} = a(t)/a_0$ and we assume no curvature.

$$\Omega_M(a) \approx 1 \quad \Rightarrow \left(\frac{\Omega_M(a_0)}{\Omega_\Lambda(a_0)}\right)^{1/3} > \frac{a}{a_0} > \frac{\Omega_R(a_0)}{\Omega_M(a_0)}$$

During the time of matter domination, the Friedman equation is

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho_M(a) \qquad \Rightarrow \quad a(t) \propto t^{2/3}$$

Spherical collapse model



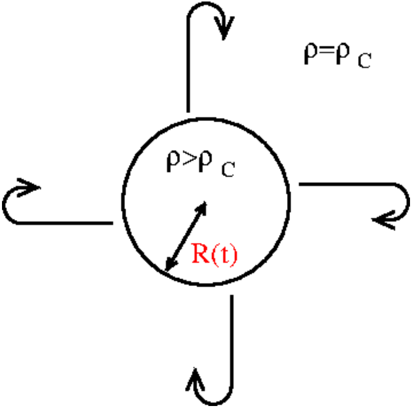
A critical matter-only universe with a small spherical expanding region with $\rho > \rho_c$.

Radius of overdense region $= R(t)$

Overdense region acts like a mini-closed universe:
Gravity excess stops its expansion starting a contraction phase

Note: $R(t)$ is "physical radius", not "comoving radius"

Spherical collapse model



Spherical symmetry ⇒ dynamics of $R(t)$ independent of rest of universe. Conservation of energy of test particle a boundary:

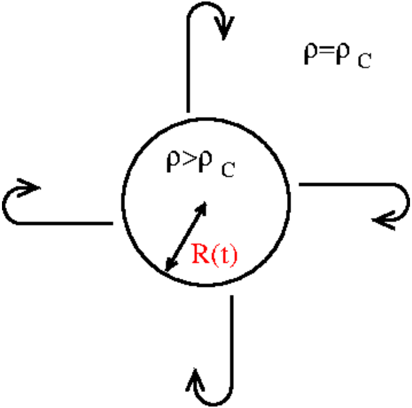
$$(1/2)\dot{R}^2 - \frac{GM}{R} = -\frac{GM}{R_{max}}$$

M = mass contained in spherical region (time-independent)

$$dt = \frac{dR}{\sqrt{2\Phi_g}\sqrt{R_{max}/R - 1}}$$

Model characterized by R_{max} and $\Phi_g = GM/R_{max} <\sim 10^{-5}$)

Spherical collapse model: small time behavior



$$dt = \frac{dR}{\sqrt{2\Phi_g}\sqrt{R_{max}/R - 1}}$$

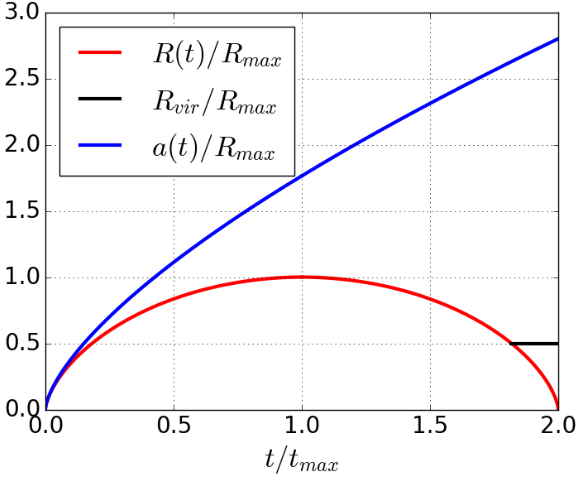
$$t \to 0 \ \Rightarrow R \ll R_{max}$$

$$R(t) \approx \left(\frac{9\Phi_g R_{max}}{2}\right)^{1/3} t^{2/3}$$

Convenient to normalize $a(t)$ so that $a(t) = R(t)$ for small time:

$$a(t) = \left(\frac{9\Phi_g R_{max}}{2}\right)^{1/3} t^{2/3}$$

$R(t)$: expansion, turn-around, collapse, virialization



Expansion: $R(t) \approx a(t)$

Turn-around: $t_{max} =$

$$\frac{R_{max}}{\sqrt{2\Phi_g}} \int_0^1 \frac{dx}{\sqrt{x^{-1} - 1}}$$

$$t_{max} = \frac{R_{max}}{\sqrt{2\Phi_g}} \times \pi/2$$

Virialization: random motions prevent collapse to black hole

$t \ll t_{max}$ (1): linear behavior

$$dt = \frac{dR}{\sqrt{2\Phi_g}\sqrt{R_m/R-1}} \quad \Rightarrow t = \frac{R_m}{\sqrt{2\Phi_g}}\int_0^{R(t)/R_m} \frac{x^{1/2}}{(1-x)^{1/2}}dx$$

$x \ll 1 \Rightarrow (1-x)^{-1/2} \approx 1 + x/2$ which gives

$$t = \frac{R_m}{\sqrt{2\Phi_g}}\frac{2}{3}x^{3/2}\left[1+\frac{3}{10}x\right] \qquad x = \frac{R(t)}{R_m}$$

$$a(t) = (9\Phi_g R_m/2)^{1/3} t^{2/3} \qquad \Rightarrow t^2 = a(t)^3/(9\Phi_g R_m/2)$$

$$a(t) = R(t)\left[1+\frac{1}{5}\frac{R(t)}{R_m}\right] \qquad \Rightarrow \; R(t) = a(t)\left[1-\frac{1}{5}\frac{a(t)}{R_m}\right]$$

$t \ll t_{max}$: (2): linear behavior

$$R(t) = a(t)\left[1 - \frac{1}{5}\frac{a(t)}{R_m}\right]$$

As expected, $R(t) < a(t)$.

$$\frac{\rho(r<R)}{\overline{\rho}} \sim \left(\frac{a(t)}{R(t)}\right)^3 \approx \left[1 + \frac{3}{5}\frac{a(t)}{R_m}\right]$$

Density contrast grows in time, proportional to $a(t)$.

$$\frac{\Delta\rho}{\rho} \equiv \frac{\rho(r<R) - \overline{\rho}}{\overline{\rho}} \approx \frac{3}{5}\frac{a(t)}{R_m}$$

$t \ll t_{max}$: (3): potential fluctuation

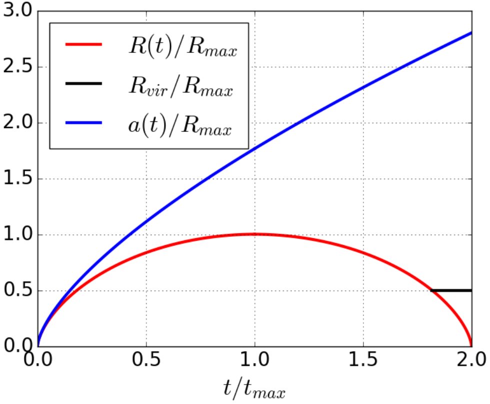
Potential fluctuation:

$$\Delta\Phi = \frac{G \times \text{excess mass}}{R(t)} = \frac{G}{R(t)} \frac{\Delta\rho \times 4\pi R(t)^3/3}{\overline{\rho}} \overline{\rho}$$

$R(t) \propto a(t)$, $\Delta\rho/\overline{\rho} \propto a(t)$ and $\overline{\rho} \propto a^{-3}$ gives $\Delta\Phi$ independent of t.

$$\Delta\Phi = \frac{G}{a(t)} \frac{\Delta\rho}{\rho} M = \frac{3}{5} \frac{GM}{R_m} \qquad t \ll t_{max}$$

Density contrast at t_{max}



Turn-around:

$$t_{max} = \frac{R_{max}}{\sqrt{2\Phi_g}} \int_0^1 \frac{dx}{\sqrt{x^{-1} - 1}}$$

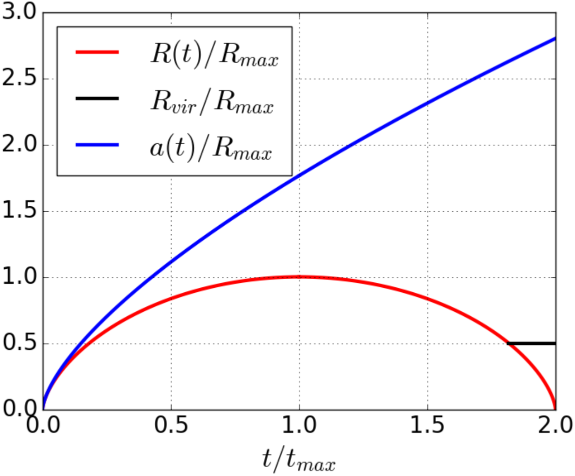
$$= \frac{R_{max}}{\sqrt{2\Phi_g}} \times \pi/2$$

$$a(t) = (9\Phi_g R_m/2)^{1/3} t^{2/3}$$

$$\frac{a(t_{max})}{R(t_{max})} = \left(\frac{3\pi}{4}\right)^{2/3} \approx 1.8$$

$$\frac{\rho}{\bar{\rho}}(t_{max}) = \frac{9\pi^2}{16} \sim 5.5$$

$R(t_{max})$ is small compared to Hubble distance



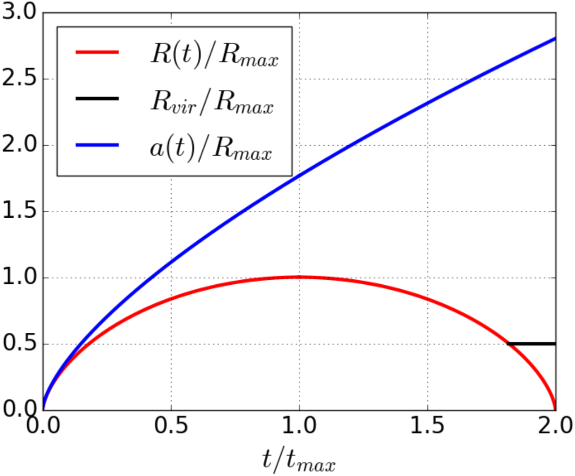
$$a(t) \propto t^{2/3}$$
$$\Rightarrow \; \dot{a}/a = 2/3t$$

$$\frac{c/H(t_{max})}{R(t_{max})} = \frac{3t_{max}/2}{R_{max}}$$

$$= \frac{1}{\sqrt{\Phi_g}} \frac{3\pi}{4\sqrt{2}}$$

Collapse and virialization

E = Total energy (potential + kinetic) of overdense region



At turnaround:
$K.E. \approx 0$

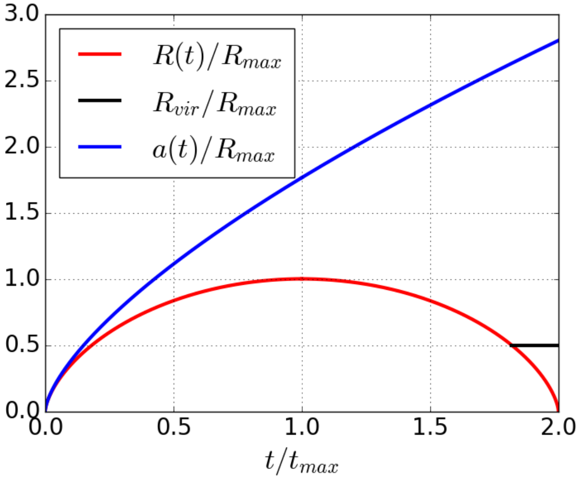
$$E \sim -\frac{GM}{R_m}$$

After virialization:
$K.E. = -P.E/2$

$$E = -\frac{GM}{R_{vir}} + \frac{1}{2}\frac{GM}{R_{vir}}$$

$$\Rightarrow \quad R_{vir} = R_m/2$$

Density contrast at t_{max} and at $2t_{max}$



$$\frac{\rho}{\bar{\rho}}(t_{max}) = \frac{9\pi^2}{16} \sim 5.5$$

$$\frac{\rho}{\bar{\rho}}(2t_{max}) \sim 5.5\frac{2^3}{2^{-(2/3)3}}$$

$$\sim 160$$

Exercise

Consider a mass-dominated critical universe, $a(t) \propto t^{2/3}$, with:

$$\overline{\rho_0} = \frac{3H_0^2}{8\pi G} \approx (3h^2) \times 10^{11} M_\odot Mpc^{-3} \quad c/H_0 = 3000 h^{-1} Mpc$$

with small scale-invariant potential fluctuations:

$$\sqrt{\overline{\Phi_g^2}} = 2 \times 10^{-5}$$

1. In the spherical collapse model, estimate the radius, density and mass of the objects that are now at their maximum (pre-collapse) size. In roughly how much time will these objects be virialized.

2. Consider objects that are half the size of those calculated in 1. At what time were those objects at their maximum size? What redshift does this correspond to?

Solution

$$a \propto t^{2/3} \Rightarrow \dot{a}/a = \frac{2}{3}t^{-1} \Rightarrow H_0 = \frac{2}{3}t_0^{-1} \Rightarrow t_0 = \frac{2}{3}H_0^{-1}$$

$$R_{max} = t_{max}\sqrt{2\phi}\frac{2}{\pi} = \frac{2}{3}3000h^{-1}Mpc\sqrt{4 \times 10^{-5}}\frac{2}{\pi} \approx 8h^{-1}Mpc$$

$$\rho \approx 5.5 \times 3h^2 10^{11} = 1.6h^2 \times 10^{12} M_\odot/Mpc^3$$

$$M \approx \rho \times (4\pi/3)R_{max}^3 \approx 3.67h^{-1}10^{15}M_\odot$$

Virialization in a time $\approx t_0 = 2/3H_0 \approx 10^{10}yr/h$

$$a(t) \sim t^{2/3} \Rightarrow a_1/a_0 = 2^{-2/3} = 1.59 = 1 + z \Rightarrow z = 0.59$$

Some strange things involving dR/dt

$$\left(\frac{dR}{dt}\right)^2 = 2\Phi_g \left(\frac{R_{max}}{R} - 1\right)$$

- $R \rightarrow 0 \;\Rightarrow\; dR/dt > 1$
  $\dot{R} > 1 \;\Rightarrow\; R/(R/\dot{R}) \approx R/(a/\dot{a})) > 1$
  $\Rightarrow$ Initial conditions involve correlations on scales larger than Hubble distance
  $\Rightarrow$ inflation

- $dR/dt < 1$ for $R > 2\Phi_g R_{max}$
  Any freely propogating relativistic "thing" can leave the overdense region during this time
  (neutrinos and relativistic sound waves)

Summary of spherical-collapse model

- Small fluctuations lead to bound objects only if $\Omega_M \sim 1$
  If $\Omega_M < 1$ small fluctuations insufficient to give $\rho > \rho_c$
  If $\Omega_M > 1$ the whole universe collapses.
  $\Omega_M \sim 1$ for $3 \times 10^{-4} < a/a_0 < 0.5$ (our universe).
- At early times, structure growth is linear:
  $\Delta\rho/\rho \propto a(t)$
  $\Delta\Phi_g \propto \bar{\rho} \times (\Delta\rho/\rho) \times R^2$ is time independent
- Virialized structures are formed:
  rather small: $R_{vir} \approx \sqrt{\Phi_g}(c/H)$
  contrast at virialization: $\Delta\rho/\rho \sim 200$.

More accurate models:

- Expand density field in modes and apply perturbation theory.
- When the density contrast becomes big, use N-body simulations.

Baryon Acoustic Oscillations (BAO)

The Spherical collapse model assumes that during the expansion phase, the particles are initially "comoving": $R(t) \propto a(t)$.

Note: $\dot{R} < c$ for $R < c/H$

While $R(t) \propto a(t)$ for non-relativistic matter, if the matter forms a fluid, the initial over-density can be dispersed by initiating a sound wave.

This is especially true during the radiation epoch when the sound speed was $c_s \sim c/\sqrt{3}$.

Acoustic waves $\Leftrightarrow$ $\sim$perfect fluids

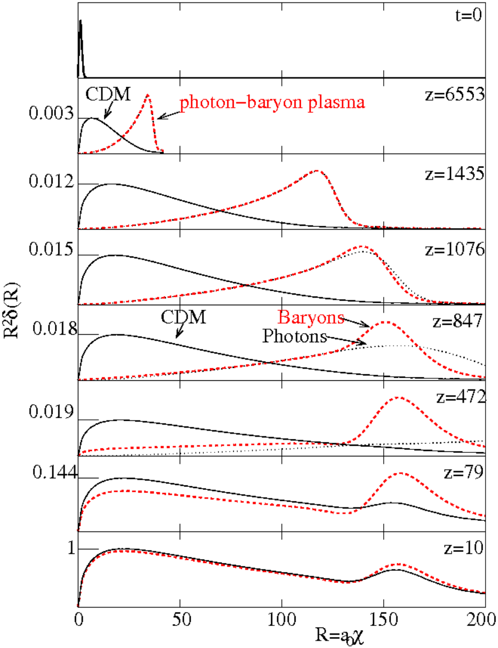
"perfect fluid": mean-free-path of particles much less than spatial extent of perturbation.

Early universe:
WIMPS: $\lambda_{mfp} \gg c/H(z) \Rightarrow$no waves
photon-electron-proton plasma: $\lambda_{mfp} \ll c/H(z)$
electron-photon (Compton) scattering + electron-proton (Coulomb) scattering.

$\Rightarrow$ plasma supports acoustic waves until recombination.

$$c_s^2 = \left(\frac{dP}{d\rho}\right)_{\mathrm{adiabatic}} \sim \frac{c^2}{3}$$

(since ρ is dominated by photons with $P = \rho/3$.)



An initial over-density:

$t = 0$

$c_s \sim c/\sqrt{3}$
(γ,p,e plasma)

$c_s \to 0$ at recombination
($r \sim 150\mathrm{kpc}$)

Today: Enhanced correlation
at $r = 147.5$Mpc

Reminder on photon propagation

Photon radial trajectory $\chi(t)$ where $\chi =$ radial comoving coordinate.

$$d\tau = 0 \qquad \Rightarrow d\chi = \frac{dt}{a(t)} = \frac{da}{a^2(\dot{a}/a)}$$

$$z(t) = \frac{a_0}{a(t)} - 1 \qquad \Rightarrow a_0[\chi(t_1) - \chi(t_2)] = \int_{z_1}^{z_2} \frac{dz}{H(z)}$$

$\int_0^z$ gives present distance to object seen today at redshift z.

$\int_z^\infty$ gives present distance a photon can travel between $t = 0$ and time t where $z(t) = a_0/a(t) - 1$.

$\int_0^\infty$ is present distance to the (present) horizon

Calculation of the sound horizon

Same as "particle horizon" except $c_s < c$

$c_s = (c/\sqrt{3})f(\rho_{\rm B}/\rho_\gamma)$ (baryon inertia slows sound)

$$r_d = \int_{z_d}^{\infty} \frac{c_s(z)dz}{H(z)} \approx (c/H_0) \int_{z_d}^{\infty} \frac{c_s(z)dz}{\sqrt{\Omega_M(1+z)^3 + \Omega_R(1+z)^4}}$$

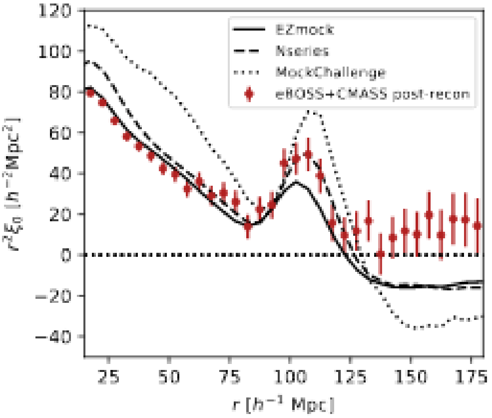
$(\Omega_M, \Omega_\Lambda) = (0.3, 0.7)$ plus photons and neutrinos give

$$\Rightarrow r_d = (147.3 \pm 0.5)\mathrm{Mpc}$$

BAO Peak in galaxy-galaxy correlation function

On average $n_{gal} \approx 0.02\mathrm{Mpc}^{-3}$, but the positions are correlated.

Correlation function at $z \approx 0.7$ [arXiv:2007.08993]



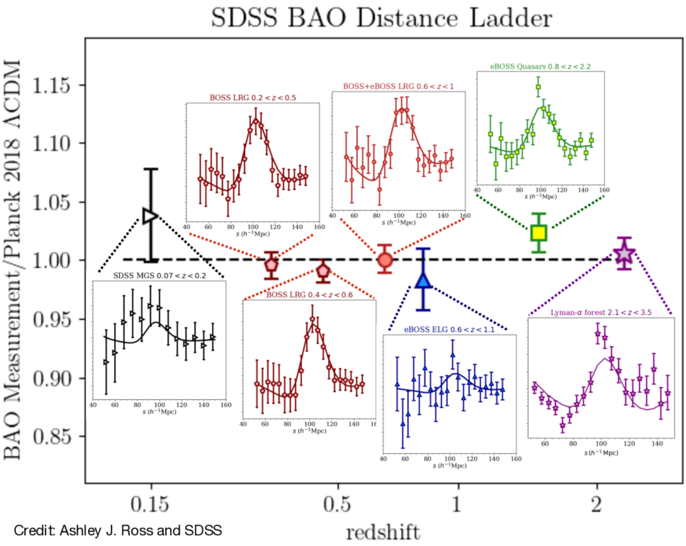
$\xi(r) \equiv$ excess probability (over random) to have two galaxies separated by r.

$\xi(20h^{-1}\mathrm{Mpc}) \approx 0.2$

$\xi(100h^{-1}\mathrm{Mpc}) \approx 0.004$ (BAO peak)

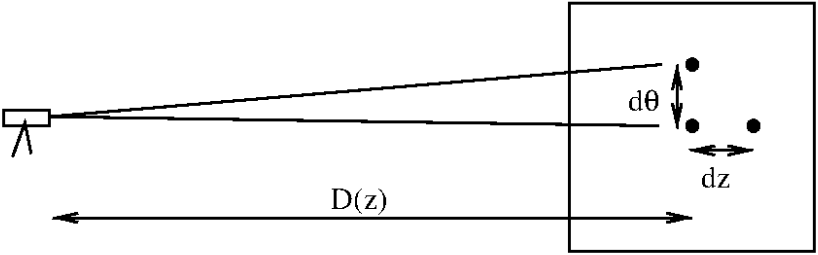
$\xi(r > 150h^{-1}\mathrm{Mpc}) \approx 0$

BAO Peak at various redshifts



Credit: Ashley J. Ross and SDSS

BAO Peak $\Rightarrow D_M(z)/r_d$ and $(c/H(z))/r_d$



Galaxy positions are found in (z, θ, ϕ) space, For an ensemble of galaxies near redshift z, galaxies separated by r_d in the radial direction or transverse direction are separated by redshifts or angles given by

$$\Delta z_{BAO} = \frac{r_d}{c/H(z)} \qquad |\Delta\vec{\theta}_{BAO}| = \frac{r_d}{D_M(z)}$$

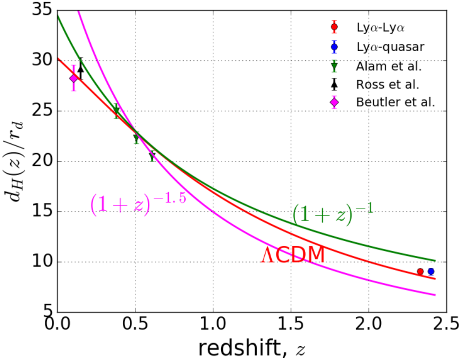
The measured values of Δz_{BAO} and $|\Delta\vec{\theta}_{BAO}|$ determines $D_M(z)/r_d$ and $(c/H(z))/r_d$.

ΛCDM parameters from BAO

Expansion rate and Hubble distance:

$$H(z) \ = c/d_H(z) \ = H_0 \left[\Omega_\Lambda \ + \Omega_M(1+z)^3 \ + \Omega_k(1+z)^2 + ...\right]^{1/2}$$

where $\Omega_k = 1 - \Omega_M - \Omega_\Lambda$. BAO measures $r_d H(z)$



z dependence of $r_d H(z)$ determined by $(\Omega_\Lambda, \Omega_M)$

Models:
standard ΛCDM
$(\Omega_M, \Omega_\Lambda) = (1, 0)$
$(\Omega_M, \Omega_\Lambda) = (0, 0)$

ΛCDM parameters from BAO

Expansion rate and Hubble distance:

$$H(z) \ = c/d_H(z) \ = H_0 \left[\Omega_\Lambda \ + \Omega_M(1+z)^3 \ + \Omega_k(1+z)^2 + ...\right]^{1/2}$$

where $\Omega_k = 1 - \Omega_M - \Omega_\Lambda$.

Distance to z:

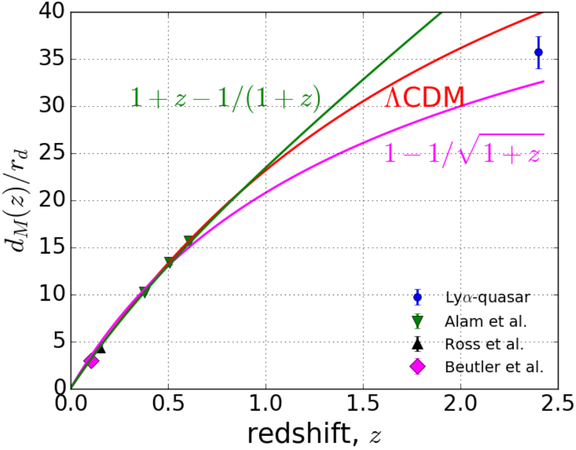
$$d(z) \ = \ \int_0^z d_H(z)dz$$

Angular diameter distance to z:

$$d_M(z) = d_c S(d(z)/d_c) \qquad d_c = \frac{c/H_0}{\sqrt{|\Omega_k|}} \qquad S = \begin{matrix} \sin \\ \sinh \end{matrix} \ \mathrm{for} \ \begin{matrix} \Omega_k < 0 \\ \Omega_k > 0 \end{matrix}$$

$\Rightarrow d_M(z)/r_d$ and $d_H(z)/r_d$ are functions of z and $(\Omega_M, \Omega_\Lambda, r_d H_0)$

$d_M(z)/r_d$ vs. z



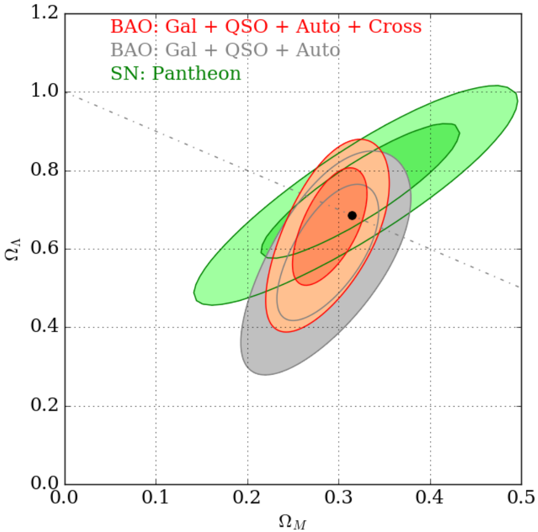
Models:

standard ΛCDM

$(\Omega_M, \Omega_\Lambda) = (1, 0)$

$(\Omega_M, \Omega_\Lambda) = (0, 0)$

BAO and SNIa constraints



BAO results:

$\Omega_M = 0.288 \pm 0.033$

$\Omega_\Lambda = 0.695 \pm 0.115$

$\Omega_k = 0.02 \pm 0.14$

$H_0 r_d = 147.33\mathrm{Mpc} \times (68.5 \pm 1.5)\ \mathrm{km\,s^{-1}Mpc^{-1}}$

(de Sainte Agathe et al, 2019)

Outline

- Observed inhomogeneities
  - Large-Scale Structure (LSS)
  - Clusters of galaxies
- Spherical collapse model
  - CDM-only
  - CDM+baryons
- Determination of Ω_M and Ω_Λ with BAO
- Description of the inhomogenious universe
  - Power spectra of density and potential fluctuations
- Time evolution of Fourier modes
  - Inflation, Radiation, and Matter epochs
- Hot Dark Matter

Fourier expansion of density in box, $V = L^3$

$$\rho(\vec{r}) \;=\; \overline{\rho}\left[1 \;+\sum_{\vec{k}}\delta_{\vec{k}}\exp(i\vec{k}\cdot\vec{r})\right] \qquad \vec{k} \;=\; \frac{2\pi\vec{n}}{L} \qquad \delta_{-\vec{k}} \;=\; \delta^*_{\vec{k}}$$

Density dispersion:

$$\frac{\overline{\rho^2}-\overline{\rho}^2}{\overline{\rho}^2} \;=\; \sum_{\vec{k}}|\delta_{\vec{k}}|^2 \qquad \text{(the crossterms average to zero)}$$

Gaussian random fields: $Re(\delta_{\vec{k}})$ and $Im(\delta_{\vec{k}})$ are each Gaussian random variables.
Isotropy $\Rightarrow$ Variance of $\delta_{\vec{k}}$ depends only on $|\vec{k}| = k$.

$$\frac{\overline{\rho^2}-\overline{\rho}^2}{\overline{\rho}^2} \;=\; \int_{k_{min}}^{\infty}\frac{dk}{k}\,\frac{k^3P(k)}{2\pi^2} \qquad P(k) = V\langle|\delta_{\vec{k}}|^2\rangle \qquad k_{min} \approx \frac{2\pi}{L}$$

Exercise

Show that the correlation function is the Fourier transform of the power spectrum:

$$\xi(\vec{r}) \ = \langle \delta(\vec{r}\,')\delta(\vec{r}\,' + \vec{r}\,) \rangle \sim \int e^{i\vec{k}\cdot\vec{r}} P(k)\, d^3k \qquad \delta(\vec{r}) = \rho(\vec{r}\,)/\bar{\rho} - 1$$

Solution:

$$\delta(\vec{r}\,')\delta(\vec{r}\,' + \vec{r}\,) = \sum_{\vec{k}'\vec{k}''} \delta_{\vec{k}'} \exp(i\vec{k}\,'\cdot\vec{r}\,')\delta_{\vec{k}''} \exp(i\vec{k}\,''\cdot(\vec{r}\,' + \vec{r}))$$

$$= \sum_{\vec{k}'\vec{k}''} \delta_{\vec{k}'} \exp(i\vec{k}\,'\cdot\vec{r}\,')\delta^*_{\vec{k}''} \exp(-i\vec{k}\,''\cdot(\vec{r}\,' + \vec{r}))$$

$$\langle \delta(\vec{r}\,')\delta(\vec{r}\,' + \vec{r}\,) \rangle = \sum_{\vec{k}'\vec{k}''} \delta_{\vec{k}'}\delta^*_{\vec{k}''} \exp(-i\vec{k}\,''\cdot\vec{r})\langle \exp(-i(\vec{k}\,'' - \vec{k}')\cdot\vec{r}\,') \rangle$$

The average over $\vec{r}\,'$ vanishes unless $\vec{k}\,' = \vec{k}\,''$

$$\langle \delta(\vec{r}\,')\delta(\vec{r}\,' + \vec{r}\,) \rangle = \sum_{\vec{k}} |\delta_{\vec{k}}|^2 \exp(i\vec{k}\cdot\vec{r})$$

Exercise: A 1d Gaussian Random field

(just for fun)

1. Generate an uncorrelated 1d Gaussian random field. For example:

```
from pylab import *
from matplotlib.pyplot import *
import numpy as np

L=1000.  # length of space in Mpc
nmodes=50000  # number of positive k modes
kfund=2.*np.pi/L  # fundamental mode
k=kfund*(np.arange(nmodes)+1)  # positive k values
sigmak=1.  # standard deviation of real and imaginary parts of delta_k
realdeltak=np.random.normal(0.,sigmak,nmodes)  #real part of delta_k
imagdeltak=np.random.normal(0.,sigmak,nmodes)  #imaginary part of delta_k
npixels=2*nmodes  # number of discrete x values
x=L*(np.arange(npixels)+0.5)/float(npixels)  # xvalues
#delta in real space as a function of pixel i: (~2.5minutes on my lap top)
delta=[sum(cos(k*x[i])*realdeltak - sin(k*x[i])*imagdeltak) for i in range(npixels)]
delta=2.*kfund*np.array(delta)  # normalization
#correlation function as a function of pixel separation j  ~2min on my laptop
cf=[np.mean(np.array([delta[i]*delta[i+j] for i in range(npixels-j) ])) for j in range(4000)]
```

2. Plot $|\delta_k|^2$ vs k and the correlation function vs. $\Delta x = L \times j/npixels$ The first should be statistically k-independent and the second should statistically vanish except for zero separation.

Exercise: 1d Gaussian Random field (cont.)

3. Modify δ_k to generate a correlated random field by cutting off the high-k modes: $\delta_k \rightarrow \delta_k \exp(-(k/4k_0)^2)$ with $k_0 = 0.1 Mpc^{-1}$. Plot $\delta(x)$. Where would you expect galaxies to form? Calculate the correlation function and plot it, showing that it is non-zero for non-zero separation.

4. Suppose that the amount of material necessary to form a galaxy is contained in a superpixel of length 2Mpc, i.e. 200 of the original pixels. Regroup the original pixals in superpixels of length 2Mpc with associated δ equal to the mean of the deltas in the superpixel. Place a galaxy in the 1% of the superpixels with the highest mean δ. Count the number of pairs of galaxies separated by less than 10Mpc and compare it to the number of pairs one would have if the galaxies were placed randomly.

Potential Fluctuations

Solution to Poisson equation: $\nabla^2\Phi = -4\pi G(\rho - \overline{\rho})$

$$\Phi(\vec{r}) \;=\; \sum_{\vec{k}} \phi_{\vec{k}} \exp(i\vec{k}\cdot\vec{r}) \qquad\qquad \phi_{\vec{k}} \sim \frac{4\pi G\overline{\rho}}{k^2}\delta_{\vec{k}}$$

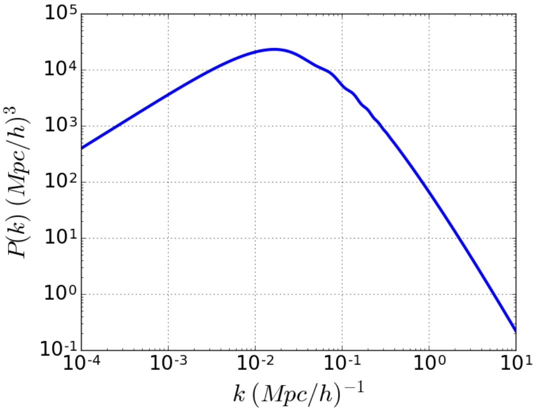
Potential dispersion:

$$\overline{\Phi^2} \;=\; \sum_{\vec{k}} |\phi_{\vec{k}}|^2 \;=\; \int_{k_{min}}^{\infty} \frac{dk}{k}\,\frac{k^3 P_\phi(k)}{2\pi^2} \qquad\qquad P_\phi(k) = V\langle|\phi_{\vec{k}}|^2\rangle$$

Expect $\sqrt{k^3 P_\phi(k)/2\pi^2} \sim 2\times 10^{-5}$ at small k

$\Rightarrow$ Expect $P(k) \propto k$ at small k

Power spectrum in standard ΛCDM

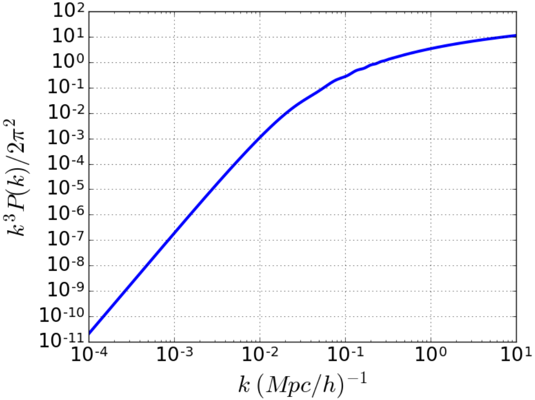


Mean square amplitude maximum at $k \sim 0.02(Mpc/h)^{-1}$

High k modes have small amplitude but there are many of them!

Standard ΛCDM : $(\Omega_{cdm}, \Omega_b) = (0.269,\ 0.0484)$,
$\Omega_\Lambda\ \sim 1 - \Omega_{cdm} - \Omega_b,\ h = 0.674,\ A_s = 2 \times 10^{-9}$

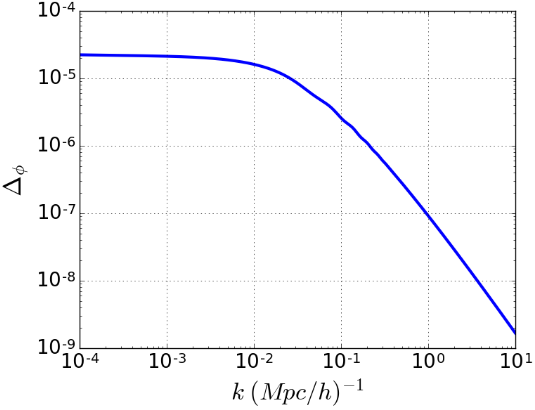
Density fluctuation vs. scale



$$\left(\frac{\Delta\rho}{\rho}\right)^2 \sim \int \frac{dk}{k} \frac{k^3 P(k)}{2\pi^2}$$

Universe is $\sim$ homogeneous on scales
$k < 0.1(Mpc/h)^{-1}$
$(\lambda = 2\pi/k > 60 Mpc/h)$

Gravitational potential fluctuations: $\leq 2 \times 10^{-5}$



Gravitational potential fluctuation:

$$\phi_{\vec{k}} \sim \frac{4\pi G\bar{\rho}}{k^2}\delta_{\vec{k}}$$

$\sqrt{k^3 P_\phi(k)/2\pi^2} \sim 2 \times 10^{-5}$ at small k

(Primordial scale-invariant potential fluctuations from inflation.)

$\delta_{\vec{k}}(t)$ and the shape of $P(k)$

The shape of $P(k)$ in ΛCDM results from inflation combined with post-inflation acoustic waves.

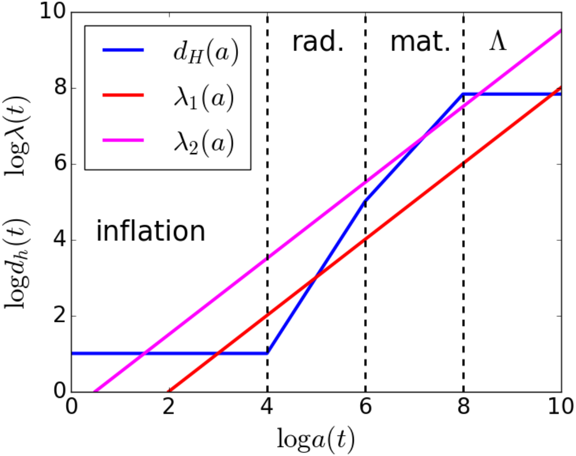
It is convenient to use "co-moving modes":

$$\lambda_k(t) = \frac{2\pi}{k}\frac{a(t)}{a_0}$$

In non-inflationary cosmology, wavelengths are initially larger than $D_H(t) = c/H(t)$ and the become smaller than $D_H(t)$. ("Hubble-entry")

With inflation, we have both Hubble-exit and Hubble-entry.

Hubble exit, then Hubble entry



$d_H(a) = \sqrt{3/8\pi G\rho(a)}$

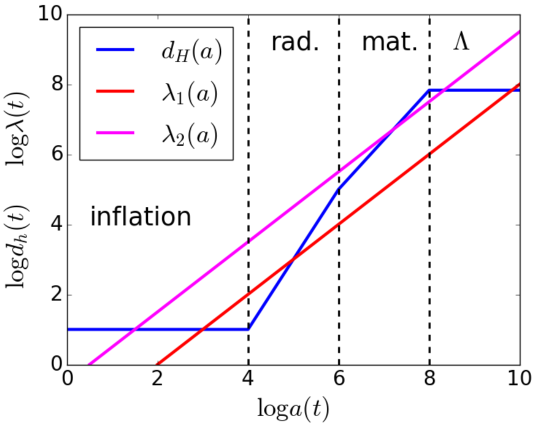
$\lambda_k(a) = (2\pi/k)a(t)/a_0$

λ_1 and λ_2 "leave" the Hubble volume during inflation and then "enter".

λ_1 enters during radiation epoch

λ_2 enters during matter epoch

Inflation ⇒ scale-independent fluctuations



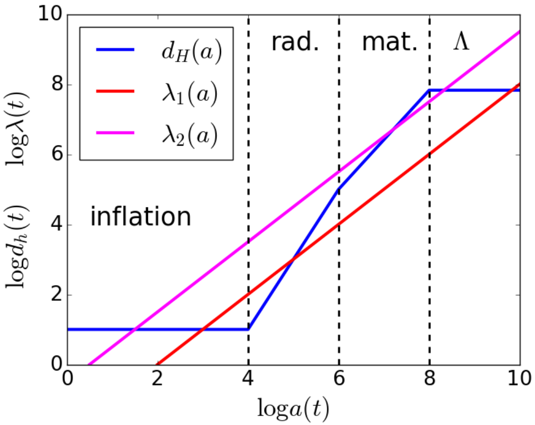
$\rho(a)$ ∼constant during inflation

⇒ fluctuation amplitude scale-independent at Hubble-exit

Super-Hubble dynamics preserves amplitude

⇒ All scales enter Hubble radius with equal amplitude.

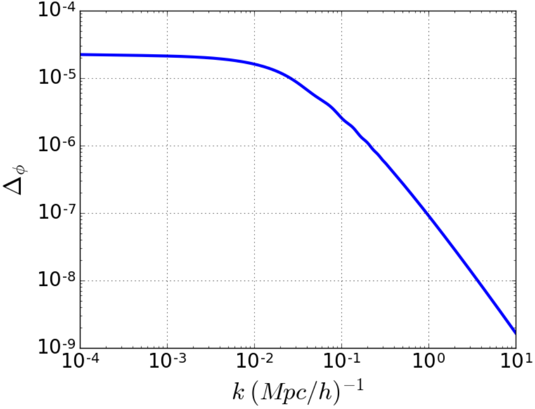
Radiation epoch: potential decay



The gravitational potential of modes that enter during the radiation epoch decays because acoustic oscillation prevents increase of $\Delta\rho/\rho$:

$$\Phi_g \sim G\bar{\rho}\,\frac{\Delta\rho}{\rho}\,\lambda^2 \sim a^{-4} \times a^2$$

Gravitational potential fluctuations: $\leq 2 \times 10^{-5}$



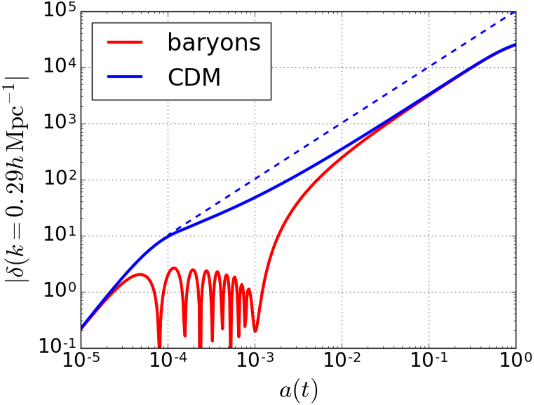
Short wavelength modes have Hubble-entry during radiation epoch resulting in decay of gravitational potential.

Long wavelength modes enter during matter epoch and therefore preserve primordial potential fluctuation from inflation.

k_{eq}: mode with Hubble entry at matter-radiation equality:

$$\frac{\lambda(a_{eq})}{2\pi} = \frac{a_{eq}/a_0}{k_{eq}} = d_H(a_{eq}) \qquad \Rightarrow k_{eq} \sim 0.01\mathrm{Mpc/h}^{-1}$$

Numerical solutions from CAMB: https://camb.info/B



CDM growth $\delta \propto a(t)$ for $a > a_{eq}$

CDM growth slows when Λ begins to dominate ($a > 0.5$)

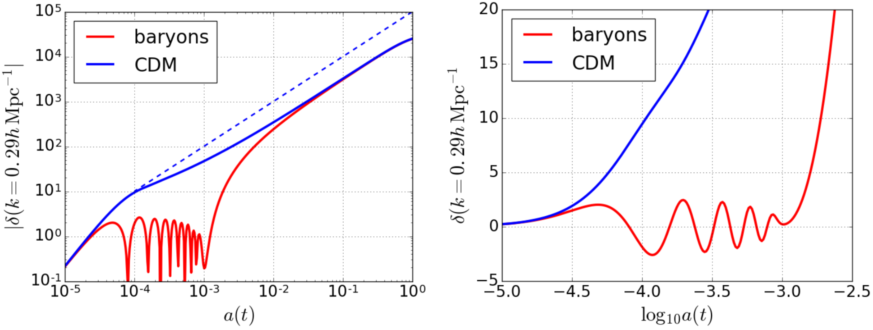
Baryons oscillate until recombination ($a \sim 10^{-3}$)

The Einstein equation for each mode is an ordinary differential equation. Example: for sub-hubble modes during matter domination:

$$\ddot{\delta}_{\vec{k}} + 2H(t)\dot{\delta}_{\vec{k}} + [c_s^2(ka_0/a)^2 - 4\pi G\bar{\rho}]\delta_{\vec{k}} = 0$$

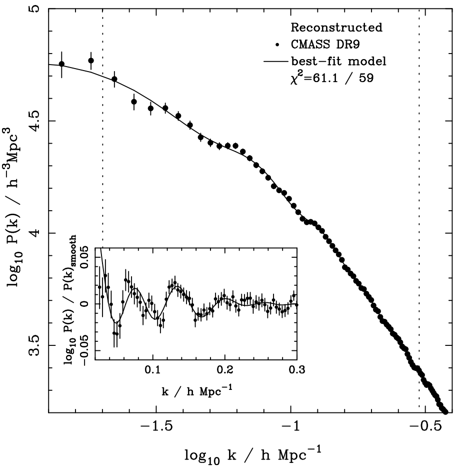
where $c_s \sim 0$ for CDM and $c_s \sim c/\sqrt{3}$ for baryons ($z \gg z_{rec}$).

At early times, each mode develops independently



Numerical solution from CAMB: `https://camb.info/`

Observed power spectrum of galaxies



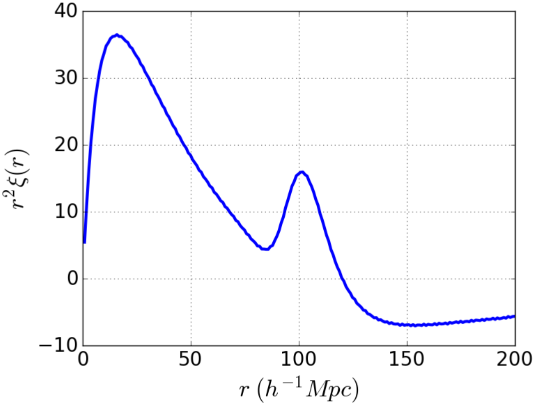
Galaxies are a "biased" tracer of matter at large scale ($k < 0.05$):

$$P(k)_{gal} = b_{gal}^2 P(k)_{matter}$$

At small scale ($k > 0.05$) non-linear growth complicates the galaxy power spectrum.

"Wiggles" due to BAO are more easily seen in the correlation function.

Correlation function



$$\xi(\vec{r}) = \langle \delta(\vec{r}')\delta(\vec{r}' + \vec{r})\rangle$$

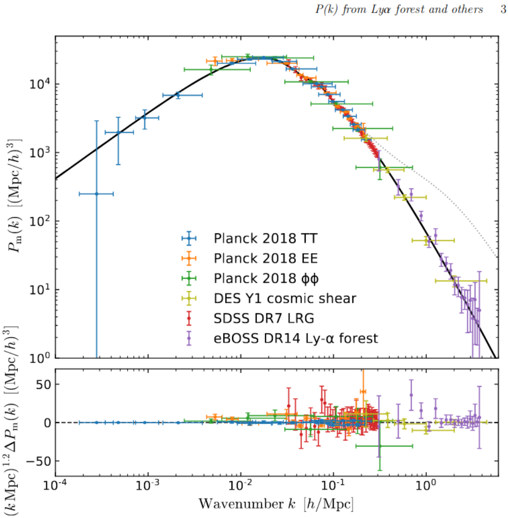
$$\sim \int e^{i\vec{k}\cdot\vec{r}} P(k)\, d^3k$$

$\xi(r)$ has a peak at $r = r_d$

(sound horizon)

Fourier transform of a wiggle is a peak.

Power spectrum from many "tracers"



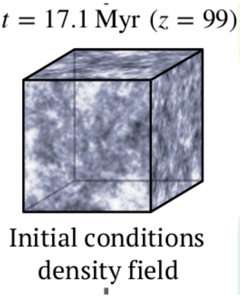
Complilation of Chabanier, Millea, Palanque-Delabrouille, arXiv:1905.08103

line is prediction of ΛCDM

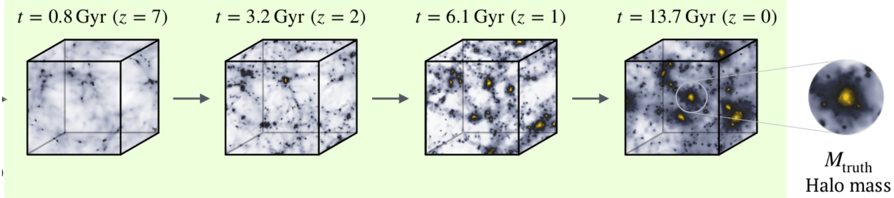
$k > 0.01$ need n-body simulations

$k > 0.1$ need hydrodynamic simulations

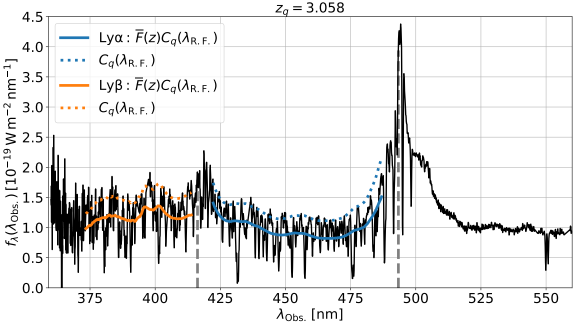
N-body simulations, arXiv:2011.10577



Evolve "primordial" density field using perturbation theory to $z \approx 99$.
Place "particles" ($M \approx 10^7 M_\odot$) in phase-space according to $f(\vec{r}, \vec{v}, z = 99)$.
Move particles with Newtonian gravity.
At $z = 0$ define "galaxies" as bound structures



A quasar spectrum at $z = 3.058$

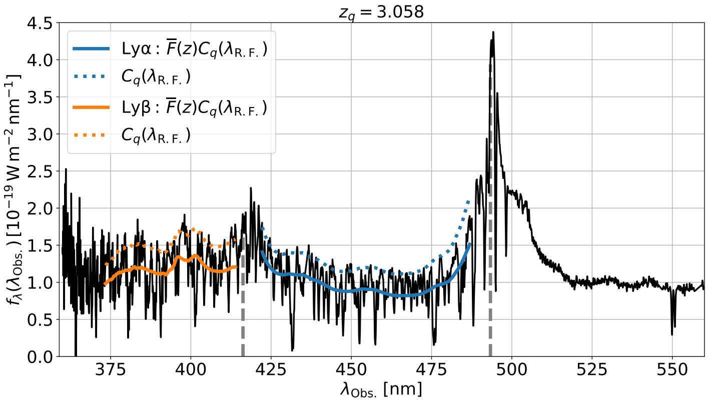


Quasar= Black Hole accreting matter. Broadband radiation from falling matter plus emission lines from atoms in the hot medium around the quasar.

$$\text{Ly}\alpha \text{ emission at } \lambda = 493\,\text{nm} \quad \Rightarrow z_{quasar} = \frac{493}{121.57} - 1 = 3.058$$

Lyα: $(n = 2) \rightarrow (n = 1)$ transition of atomic hydrogen

The Lyα forest



Fluctuating Lyα absorption at $\lambda < 493$ nm

$$\Rightarrow z_{absorber} = \frac{\lambda}{121.57} - 1 < z_{quasar}$$

Absorption due to atomic hydrogen between the quasar and observer.

Thermal History of Baryons

- $T > 0.2\text{eV}$, $z > 1000$:
  baryon and electrons ionized and coupled to photons.
  $c_s \approx c/\sqrt{3}$ for $z \gg 1000$ $(\rho_\gamma \gg \rho_b)$
- $z \approx 1000$ recombination
- $10 < z < 1000$: Baryons and photons decoupled
  $T_\gamma \propto a^{-1}$
  $T_{baryons} \propto a^{-2}$
- $z \approx 10$ Baryons reionized by UV photons from first stars.
- $z <\approx 10$ IGM (Intergalactic medium):
  atoms and ions in equilibrium via flux of UV photons
  $atom + \gamma \leftrightarrow e^- + ion$
  $\Rightarrow\approx 1\%$ atomic.

$T_{IGM} \sim E_{UV\gamma} - 13.6eV \approx 1eV \Rightarrow$ non-relativistic ionized gas.

Gravity vs. pressure in the IGM

Density fluctuations ⇒ gravitational collapse but pressure gradients resist collapse. Who wins?

Consider a sphere of radius R containing an excess mass ΔM, excess pressure ΔP, and excess temperature ΔT.
$(\Delta M, \Delta P, \Delta T)$ are related by adiabatic compression.

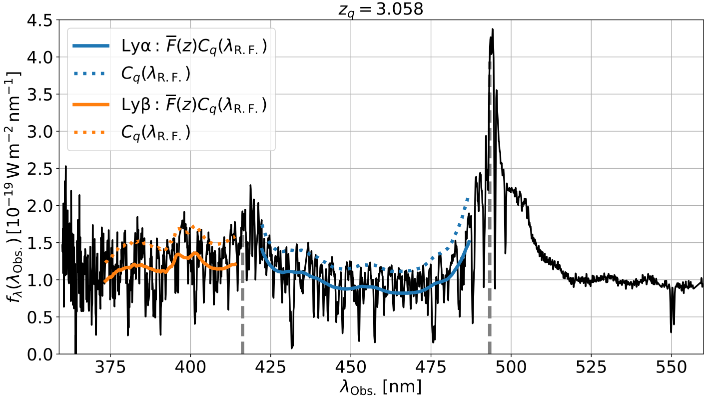
Gravity wins at large mass density ρ, small T, and large R:

$$R > R_J \approx \left(\frac{kT}{m_p G\rho}\right)^{1/2} \approx 100 kpc \qquad \text{“Jeans length”}$$

$(kT \sim eV,\ m_p \sim GeV,\ G\rho \sim H_0^2,\ c/H_0 = 3000 h^{-1} Mpc)$

Objects smaller than 100kpc (galaxies) formed by radiative cooling in CDM potential wells.

Forest absorption flutuations due to IGM density fluctuations



Hydrodynamical simulations can then relate absorption fluctuations to cosmological density fluctuations.

A high-resolution spectrum in the Lyα forest

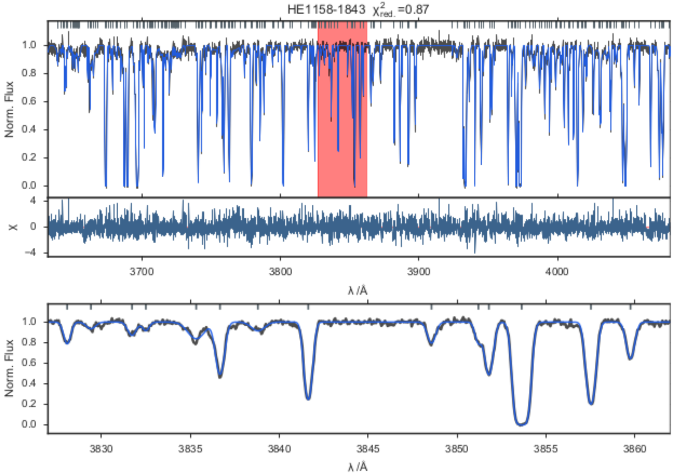


Figure 2. The resulting VP-fit of the Lyα forest of the QSO HE1158-1843 at $z \simeq 2.46$ from the UVES sample. **Upper panel**: The original spectrum (black line) is well described by the superposition of Voigt-profiles fitted by `VPFIT` (blue line). The position of individual lines is shown by gray rugs in the upper part of the panel. Underneath we plot the resulting $\chi = (F_{\rm spec} - F_{\rm fit})/\sigma_{F_{\rm fit}}$ as a measure for the goodness of the fit. **Lower Panel:** Zoom in of the area marked in red in the upper panel.

Hiss et al., arXiv:1710.00700

Exercise

The spectrum on the previous slide shows excess absorption at $\lambda \approx 383.5$nm and at $\lambda \approx 383.6$nm.

Assuming that the absorption is due to the Lyα line $\lambda_\alpha = 121.567$nm,

1. Calculate the redshifts of the atomic hydrogen responsible for the absorption
   $z_1 = 383.5/121.567 - 1 = 2.1546$, $z_2 = 2.1554$
2. Give a formula for the comoving coordinates, χ, of the absorbers

$$\chi(z_1) = \int_{t_0}^{t_1} \frac{dt}{a(t)} = a_0^{-1} \int_0^{z_1} \frac{dz}{H(z)}$$

3. Calculate the coordinate separation, $\Delta\chi$, of the two absorbers. What cosmological parameters does the separtion depend on?

$$\chi(z_2) - \chi(z_1) = a_0^{-1} \frac{z_2 - z_1}{H(z_1)} \qquad H(z_1) \approx H_0 \left[\Omega_\Lambda + \Omega_M(1+z_1)^3 + \Omega_k(1+z)^2\right]^{1/2}$$

4. Calculate the present distance between the two absorbers.
   $\Omega_\Lambda = 0.7$, $\Omega_M = 0.3$, $c/H_0 = 3000h^{-1}$Mpc, $h = H_0/100$km/sec/Mpc:

$$a_0(\chi(z_2) - \chi(z_1)) \approx \frac{3000h^{-1}}{\sqrt{0.3 \times 3.15^3}} \frac{0.1}{121.5} \approx 0.8h^{-1}Mpc$$

Hot Darkmatter: return to spherical collapse model

The Spherical collapse model assumes that during the expansion phase, the particles are initially "comoving": $R(t) \propto a(t)$.

Two ways to disperse an initial overdensity:

- If the matter forms a fluid, the initial over-density can be dispersed by initiating a sound wave.
- Weakly interacting particles can stream out of the overdensity at $v = c$ until the particles become non-relativistic $T \approx m$. ("free-streaming")

Calculation of the free-streaming distance of a weakly-interacting particle of mass m

$$r_{fs} = \int_{z_{fs}}^{\infty} \frac{cdz}{H(z)} \qquad z_{fs} + 1 = \frac{m}{T_0}$$

For $m >\approx 1\text{eV}$, z_{fs} is in the radiation epoch:

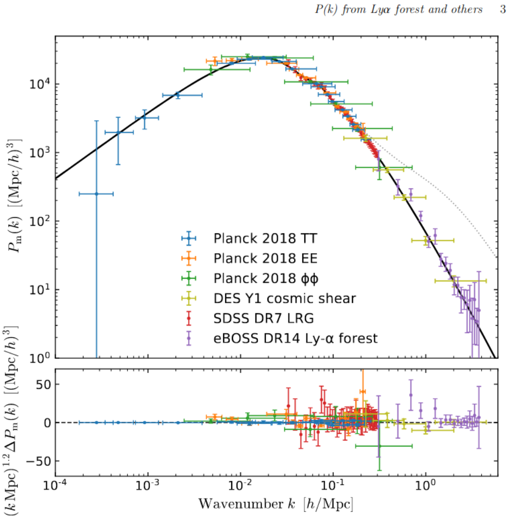
$$r_{fs} \approx \frac{c}{H_0\sqrt{\Omega_{rad}}} \int_{z_{fs}}^{\infty} \frac{dz}{(1+z)^2} \; = \frac{c}{H_0\sqrt{\Omega_{rad}}} \frac{T_0}{m}$$

$\Omega_{rad} \approx 10^{-4},\ T_0 \approx 2 \times 10^{-4}\text{eV},\ c/H_0 = 3000h^{-1}\text{Mpc}$

$$r_{fs} \approx 60h^{-1}Mpc\frac{1eV}{m} \approx 0.06h^{-1}pc\frac{1GeV}{m}$$

$\Rightarrow$ Neutrinos with $m_\nu \approx$eV cannot create galactic cluster-size fluctuations. Fluctuations in Lyα forest require $m > 10$keV.

Power spectrum from many “tracers”



Complilation of Chabanier, Millea, Palanque-Delabrouille, arXiv:1905.08103

line is prediction of ΛCDM

Hot dark matter gives

$$P(k > 2\pi/r_{fs}) \sim 0$$

$$r_{fs} \approx 60h^{-1}Mpc\frac{1eV}{m}$$