6. Thermal history of the Universe

Cosmic expansion reduces particle momentum by a factor $\propto a^{-1}$ and particle density by another a^{-3}. In early times, the Universe was therefore a hot dense state, in which particles could exchange energy and momentum quite efficiently, in other words, some kind of hot temperature plasma in thermal equilibrium (or close to some kind of thermal equilibrium).

CMB is probably the most direct evidence for this. We have seen that the Universe is filled with a gas of microwave photons following a blackbody spectrum:

$$I_\nu = \frac{8\pi\nu^3}{\exp(\nu/T_0) - 1} \tag{6.0.1}$$

with $T_0 = 2.726K$. The CMB photons are *not* in thermal equilibrium with anything: thermal equilibrium implies frequent energy-momentum exchanges via particle collisions, while the immense majority of CMB photons have never interacted since their emission. However, this absence of interactions has preserved the original shape of the CMB spectrum, which has been only affected by redshift. A photon detected at frequency ν was originally emitted with frequency $\nu(1+z)$. That is, the original spectrum was:

$$I_\nu \propto \frac{8\pi\nu^3}{\exp(\nu/((1+z)T_0)) - 1} \tag{6.0.2}$$

that is, still a blackbody shape, with temperature $(1+z)T_0$. This suggests that the CMB photons were is thermal equilibrium when they were emitted, and more generally, that the hot primordial plasma was in thermal equilibrium.

In this chapter, we discuss the properties of this early primordial plasma.

6.1. The Universe at $z \sim 10000$

Let's go back well beyond the redshift of the last scattering surface, and consider the Universe at, say, $z_1 \sim 10,000$. What can we say about it ?

First, we know that it is radiation dominated. Indeed, we have seen in homework that the redshift of matter-radiation equality is: $1 + z_{eq} = \Omega_m/\Omega_r \approx 3450$, quite some time after $z = 10000$[1]. Being radiation dominated, the Universe expands like $a \propto t^{1/2}$.

[1]Exercise: how much time ?

Expansion rate The value of the Hubble parameter at z_1 can be derived from the Friedmann equation:

$$H(z) = H_0 \left(\Omega_m(1+z)^3 + \Omega_r(1+z)^4 + \Omega_X(1+z)^{3(1+z)}\right)^{1/2} \tag{6.1.1}$$

Let's take the canonical values $\Omega_m = 0.3$, $\Omega_X = 0.7$ and $w = -1$. For the radiation density, we have seen in class that $\Omega_\gamma = 10^{-5}$ (from the CMB temperature). To this we need to add the contribution of neutrinos (see later in this chapter), which gives a total of $\approx 910^{-5}$ for the radiation density. This gives

$$H(z_1) = 1.1\ 10^6 \times H_0 \tag{6.1.2}$$

The Universe was expanding much faster than today !

Photons Let's focus now on the properties of the photons. We know that $T_\gamma \propto a^{-1}$, so:

$$T_\gamma(z_1) \approx 2.726\ 10^4 \tag{6.1.3}$$

. The mean energy of the CMB photons is:

$$2.7kT_0(z_1) = 2.7kT_0(1+z_1) \approx 6.34\text{eV} \tag{6.1.4}$$

to be compared with the present value of $6.3410^{-4}eV$. Finally, the photon density is $(1+z_1)^3 \approx 10^{12}$ the photon density today,i.e.:

$$n_\gamma \approx 4.11\ 10^{20}\ \gamma/\text{m}^3 \tag{6.1.5}$$

Baryons The baryon density at z_1 can be derived in a similar way. Remember that the baryon density today is $\Omega_b \approx 0.048$. With a critical density of $5.49\text{protons/m}^{-3}$, this gives about $n_b \approx 0.263\text{baryons/m}^{-3}$ and

$$n_b \approx 0.263\ 10^{12}\text{baryons/m}^{-3} \tag{6.1.6}$$

Mean free path of photons Finally, one may wonder about the mean free path of photons. Photons interact preferentially with electrons via Thomson scattering, and a good approximation of the photon mean free path is given by:

$$\frac{1}{\sigma_T n_e c} \tag{6.1.7}$$

where σ_T is the Thomson scattering cross section ($6.6529\ 10^{-29}\text{m}^2$), c, the speed of light (photon velocity). For the electron density, let's consider that since the Universe is neutral, there is one electron for every proton and $n_p \approx 0.2\text{m}^{-3}$. This gives

$$\frac{1}{\sigma_T n_e c} \approx 7.9\ 10^{12}\text{yr} \gg t_H \tag{6.1.8}$$

today and

$$\frac{1}{\sigma_T n_e (1+z_1)^3 c} \approx 7.9\text{yr} \ll t_H \tag{6.1.9}$$

back then.

To summarize: the Universe at $z \approx 10,000$ was much hotter and denser. The photon number density was significantly higher than the density of baryons and electrons, as it is today, but the energy densities of photons and baryons were comparable. Finally, interactions between photons and charged particles were much more frequent (many per Hubble time), so, it make plenty of sense to consider the Universe as a fluid in thermal equilibrium. In the next section, we introduce the tools of equilibrium thermodynamics.

6.2. Equilibrium thermodynamics

We model the Universe fluids as a gas of weakly interacting particles. We use the formalism of statistical physics and describe the gas by the positions and momenta of all its particles. To keep things practical, we use distribution functions defined on the $\{\vec{x}, \vec{p}\}$ phase-space.

Quantum mechanics tells us that the density of states in the phase space is bounded. Let's consider a box of edge-size L, with periodic conditions and solve the Schrodinger equation, we obtain that the possible momentum values are:

$$\vec{p} = \frac{h}{L}\left(n_x \vec{x} + n_y \vec{y} + n_z \vec{z}\right), \quad n_i = 0, \pm 1, \pm 2, \ldots \tag{6.2.1}$$

where $\vec{x}, \vec{y}$ and $\vec{z}$ are the unit vectors and h is the Planck constant. As a consequence, the state density in the momentum space is:

$$\frac{L^3}{h^3} = \frac{V}{h^3} \tag{6.2.2}$$

and the state density in the phase space is:

$$\frac{1}{h^3} = \frac{1}{(2\pi)^3 \hbar^3} \tag{6.2.3}$$

or just $1/(2\pi)^3$ in a unit system where $\hbar = 1$. As we can see, the state density is independent of the volume. It stays the same for arbitrarily large system. If the particles have g internal degrees of freedom (e.g. spin), the density of states is:

$$\frac{g}{(2\pi)^3 \hbar^3} \tag{6.2.4}$$

The properties (number density, energy density, pressure) of a given gas depend on the distribution function $f(\vec{x}, \vec{p}, t)$, which describes how the particles are distributed in the phase space. Because of homogeneity f cannot vary as a function of $\vec{x}$. Because of isotropy, f can only depend on the norm of the momentum $p \equiv |\vec{p}|$. The number of particles of momentum p is given by:

$$\frac{g}{(2\pi)^3} f(p,t) \tag{6.2.5}$$

Knowing the distribution function, we can compute the gas macroscopic properties, in particular, the number density:

$$\boxed{n = \frac{g}{(2\pi)^3} \int d^3p f(p)} \tag{6.2.6}$$

For the energy density, we just have to sum the particle energies weighted by 6.2.5.

$$\boxed{\rho = \frac{g}{(2\pi)^3} \int d^3p f(p) \sqrt{p^2 + m^2}} \tag{6.2.7}$$

Here, we have assumed that we can ignore the interaction energies between particles (i.e. we are dealing with a gas of weakly interacting particles). In that case, the energy is given by: $E(p) = \sqrt{p^2 + m^2}$ and the available states are indeed the free particle states described above.

We can similarly obtain the pressure of the gas:

$$\boxed{P = \frac{g}{(2\pi)^3} \int d^3p f(p) \frac{p^2}{3E}} \tag{6.2.8}$$

One may wonder where this $p^2/3E$ comes from. This is explained in the box below.

Why $p^2/3E$?

Let's consider a surface element δA. We note $\hat{n}$ its unit vector. The particles of velocity v that hit δA between t and $t + \delta t$ are located in a spherical shell around δ, between radii vt and $v(t + \delta t)$.

$$dN = \frac{g}{(2\pi)^3} f(E) R^2 v \; dt \; d\Omega \tag{6.2.9}$$

Not all of these particles will the surface. Only those whose velocity is aimed at δA, i.e. those whose velocity vector is in the solid angle subtended by δA. So:

$$\begin{aligned} dN_{hit} &= dN \times \frac{|\hat{v} \cdot \hat{n}|}{4\pi R^2} \\ &= \frac{g}{(2\pi)^3} f(E) \frac{\hat{v} \cdot \hat{n}}{4\pi} \; dA \; dt \; d\Omega \end{aligned} \tag{6.2.10}$$

Let's assume that the interactions are elastic and that each particle transfers momentum $2|\vec{p}\vec{n}|$ to the surface. The resulting pressure is:

$$\begin{aligned} dP(v) &= \int \frac{2|\vec{p}\cdot\vec{n}|}{dA\ dt} dN_A \\ &= \frac{g}{(2\pi)^3} f(E) \frac{p^2}{2\pi E} \int \cos^2\theta \sin\theta d\theta d\phi \\ &= \frac{g}{(2\pi)^3} f(E) \frac{p^2}{3E} \end{aligned} \tag{6.2.11}$$

6.2.1. Kinetic equilibrium

When particles can exchange energy and momentum often, the gas reaches a state of maximum entropy, called kinetic equilibrium. It is a well known result of statistical physics that the maximum entropy distribution functions for fermion and bosons are given by the *Fermi-Dirac* and *Bose-Einstein* distributions respectively:

$$\boxed{f(p) = \frac{1}{\exp\left(\frac{E(p)-\mu}{T}\right) \pm 1}} \tag{6.2.12}$$

(+) being for Fermi-Dirac and (-) for Bose-Einstein. These functions can be derived by evaluating the entropy of the gaz ($S = \ln\Gamma$) as a function of energy, and maximizing it, for a given total energy and a given total number of particles.

At low temperatures, we recover that the well-known Maxwell-Boltzmann distribution:

$$\boxed{f(p) = \exp\left(-\frac{E(p)-\mu}{T}\right)} \tag{6.2.13}$$

is valid for both Fermions and Bosons.

The Fermi-Dirac and Bose-Einstein distribution functions depend on two parameters: the temperature of the gas, T, and the chemical potential of the species μ, which characterizes the change in entropy or energy as the number of particles varies (see details in box below).

Chemical potential

The energy variations of a system can be expressed as a function of its entropy, volume and temperature as:

$$dE = TdS - PdV + \mu dN \tag{6.2.14}$$

or alternatively, the entropy variations of the same system may be written as:

$$dS = \frac{dE}{T} + \frac{P}{T}dV - \frac{\mu}{T}dN \tag{6.2.15}$$

Let's consider two systems S_1 and S_2 at temperatures T_1 and T_2 brought into contact. If both systems are isolated (1) the total energy of $(S1+S2)$ is constant; $dE = dE_1 + dE_2 = 0 \Rightarrow dE_1 = -dE_2$ (2) the entropy of $(S_1 + S_2)$ reaches a maximum: $dS = dS_1 + dS_2 = 0 \Rightarrow dE_1/T_1 + dE_2/T_2 = 0$, which gives $T_1 = T_2$ at equilibrium.
Now let's consider that S_1 and S_2 can exchange particles (keeping the total number of particles constant). We have (1) $dN = dN_1 + dN_2 = 0 \Rightarrow dN_1 = -dN_2$ and (2) the entropy of $(S_1 + S_2)$ reaches a maximum, which gives: $-\frac{\mu_1}{T}dN_1 - \frac{\mu_2}{T}dN_2 = 0$ hence $\mu_1 = \mu_2$. At equilibrium, both chemical potentials are equal.
Now, let's consider the case of a chemical reaction: $1 + 2 \rightleftharpoons 3 + 4$, i.e. the case of four systems S_1, S_2, S_3 and S_4 brought into contact. Following the same reasoning, we can show that (1) they reach a single equilibrium temperature T and (2) at equilibrium, we have $\mu_1 + \mu_2 = \mu_3 + \mu_4$.
More generally, when we have a conserved charge $Q = \sum_i q_i N_i$, we get a constraint: $dQ = 0 = \sum_i q_i dN_i$, along with the maximum entropy constraint: $\sum_i \mu_i dN_i = 0$. This means that there is a constant μ such that $\mu_i = \mu q_i$.

If the gas contains several species in interaction, each species i is described by its own distribution function, its own chemical potential μ_i, and possibly (if decoupled) its own temperature T_i. From this, we can derive each species' number density, energy density and temperature.

If all species are in kinetic equilibrium and share the same temperature: $T_i = T$, the system has reach *thermal equilibrium*.

6.2.2. Chemical equilibrium

We have seen in the box above, that if several species interact via a reaction, for example:

$$\nu_1 X_1 + \nu_2 X_2 + \ldots \leftrightharpoons \nu_1' Y_1 + \nu_2' Y_2 + \ldots \tag{6.2.16}$$

and reach chemical equilibrium (i.e. maximum entropy state), the chemical potentials satisfy:

$$\sum_i \nu_i \mu_i = \sum_j \nu_j' \mu_j \tag{6.2.17}$$

plus any conservation equation imposed by a conserved charge (number of particles, electric charge, baryon number etc.)

For photons, we have no conserved charge. Even the number of photons is not conserved. For example, we have double Compton scattering $e^- + \gamma \leftrightharpoons e^- + \gamma + \gamma$ or Bremstrahlung

$e^- + p \to e^- + p + \gamma$. Hence:

$$\mu_\gamma = 0 \tag{6.2.18}$$

For particles and anti-particles: they are of opposite charges, hence, *at equilibrium*:

$$\mu_X = -\mu_{\bar{X}} \tag{6.2.19}$$

(we can also use the reaction $X + \bar{X} \leftrightharpoons \gamma + \gamma$ to reach the same conclusion).

To summarize A system species has reached kinetic equilibrium if it has reached a maximum entropy state described by either a Fermi-Dirac or a Bose-Einstein distribution function. A system composed of several species interacting via one or several chemical reactions has reached chemical equilibrium if it has reached a maximum entropy state, where the sum of the chemical potentials of the reactants is equal to the sum of the chemical potentials of the products. A system has achived thermal equilibrium if it has reach chemical equilibrium and if all species share the same temperature.

6.3. Density and pressure of fermions and bosons

We now have everything we need to compute the number density, energy density and pressure of the constituents of the universe. As shown in the box below, the chemical potentials can be safely neglected, and equations 6.2.6, 6.2.7 and 6.2.8 can be rewritten:

$$\begin{aligned} n &= \frac{g}{2\pi^2} \int dp \frac{p^2}{\exp\left(\sqrt{p^2+m^2/T}\right) \pm 1} \\ \rho &= \frac{g}{2\pi^2} \int dp \frac{p^2\sqrt{p^2+m^2}}{\exp\left(\sqrt{p^2+m^2/T}\right) \pm 1} \\ P &= \frac{g}{2\pi^2} \frac{1}{3} \int dp \frac{p^4}{\sqrt{p^2+m^2}\left[\exp\left(\sqrt{p^2+m^2/T}\right) \pm 1\right]} \end{aligned} \tag{6.3.1}$$

In the general case, the integrals above must be computed numerically. There are two interesting limits however, which allow to understand the physical processes under way: the case where the particles are relativistic, i.e. $T \gg m$ and the opposite case of non-relativistic species: $T \ll m$.

Before we proceed, let's define: $x \equiv m/T$ and $\xi \equiv p/T$, we can then rewrite n and ρ above as:

$$\begin{aligned} n &= \frac{g}{2\pi^2} T^3 I_\pm(x) \quad \text{with} \quad I_\pm(x) = \int_0^\infty d\xi \frac{\xi^2}{\exp\left(\sqrt{\xi^2+x^2}\right) \pm 1} \\ \rho &= \frac{g}{2\pi^2} T^4 J_\pm(x) \quad \text{with} \quad J_\pm(x) = \int_0^\infty d\xi \frac{\xi^2\sqrt{\xi^2+x^2}}{\exp\left(\sqrt{\xi^2+x^2}\right) \pm 1} \end{aligned} \tag{6.3.2}$$

6.3.1. Relativistic limit

In the relativistic limit, we can neglect x and the integrals $I_\pm(0)$ and $J_\pm(0)$ can be computed exactly, as long as we are familiar with the Γ and ζ functions (see box below). We find:

	Bosons	Fermions
$n =$	$\frac{\zeta(3)}{\pi^2} g T^3$	$\frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3$
$\rho =$	$\frac{\pi^2}{30} g T^4$	$\frac{7}{8}\frac{\pi^2}{30} g T^4$

(6.3.3)

For the pressure, we have $p^2/E \sim p$ for relativistic particles. We find that:

$$P = \frac{1}{3}\frac{g}{2\pi^2}T^4 \int d\xi \frac{\xi^3}{\exp \xi \pm 1} = \frac{1}{3}\frac{g}{2\pi^2}T^4 J_\pm(0) = \frac{\rho}{3} \tag{6.3.4}$$

We recover the equation of state of radiation (we used it when computing the expansion rate for a radiation dominated Universe).

Another known fact we can recover from the integrals above is the CMB photon density:

Exercise Using $T_0 = 2.726K$, compute the photon number density (today) and the photon energy density (today). Show that:

$$n_\gamma = 411 cm^{-3}$$
$$\rho_\gamma = 4.6 \times 10^{34} \text{ gcm}^{-3}$$

and recover the CMB photon density today (in units of the critical density):

$$\Omega_\gamma = 2.5 h^{-2} 10^{-5}$$

To get the correct numerical answer, you will need to do a little bit of dimensional analysis. Where did we drop the physical constant(s) you had to retrieve ?

Computing $I_\pm(0)$ and $J_\pm(0)$

To compute $I_-(0)$ it is useful to know the definition of the Riemann-zeta function:

$$\zeta(s) = \sum_{i=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^s}{e^x - 1} dx \quad \text{where} \quad \Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx$$

For bosons, we get immediately:

$$I_-(0) = 2\zeta(3) \approx$$

For fermions, we have:

$$\begin{aligned} I_+(0) &= \int_0^\infty \xi^2 \left(\frac{1}{e^\xi - 1} - \frac{2}{e^{2\xi} - 1} \right) d\xi \\ &= I_-(0) - 2 \int_0^\infty \frac{\xi^2}{e^{2\xi} - 1} d\xi \\ &= \frac{3}{4} I_-(0) \end{aligned} \tag{6.3.5}$$

$J_\pm(0)$ can also be expressed as a function of ζ. For bosons, we obtain immediately:

$$J_-(0) = \underbrace{\Gamma(4)}_{3!} \underbrace{\zeta(4)}_{\pi^4/90} = \frac{\pi^4}{15}$$

For fermion, we use the same trick as above, and we get:

$$J_+(0) = \frac{7}{8} J_-(0)$$

6.3.2. Non relativistic limit

In the non-relativistic limit, the energy of the particles is equal to their rest-mass: $m \gg T$, i.e. $x \gg 1$. The I and J integrals defined above are the same for fermions and bosons and we find:

$$\boxed{\begin{aligned} & n \approx g \left(\frac{mT}{2\pi} \right)^{3/2} \exp\left(-\frac{m}{T} \right) \\ & \rho \approx nm \\ & P = nT \ll \rho = mn \end{aligned}} \tag{6.3.6}$$

When the temperature drops below the particle rest-mass, the particle number density drops exponentially: massive particles and their anti-particles anihilate while the photon bath energy is no longer sufficient to balance anihilations by particle-anti-particle pair production. The energy density and pressure, are (at first order) proportional to n and drop accordingly. Non relativistic species therefore behave like a pressureless gas, of energy density its mass density. This is the description of non-relativistic matter we have used to compute the Universe expansion in the so-called “matter-dominated” regime.

Computing the number density in the non-relativistic regine

In the non-relativitic regime, only the particle density is somewhat tricky to compute. With the same definitions as above: $x \equiv m/T$, $\xi \equiv p/T$ and $x \gg 1$, the the integrals I_- and I_+ reduce to one single expression:

$$I_\pm = \int_0^\infty \frac{\xi^2 d\xi}{\exp(\sqrt(x^2+\xi^2))} \tag{6.3.7}$$

$\xi \ll x$ and we can develop: $(x^2+\xi^2)^{1/2} \approx x(1+\frac{1}{2}\frac{\xi^2}{x^2})$, and we can approximate the integral above with:

$$\begin{aligned} I_\pm &\approx e^{-x}\int_0^\infty \xi^2 e^{-\frac{\xi^2}{2x}} d\xi \\ &\approx e^{-x}(2x)^{3/2}\frac{1}{2}\underbrace{\Gamma\left(\frac{3}{2}\right)}_{\sqrt{\pi}/2} \end{aligned} \tag{6.3.8}$$

6.4. Thermal history of the early Universe

We now have (almost) everything we need to discuss the evolution of the primordial plasma. When temperature is high enough, the primordial plasma contains all the particles of the standard model, in relativistic form (plus all the particles that haven't been discovered yet, for example, hypothetical particles that constitute the Cold Dark Matter).

In the early Universe, all the particle species are in thermal (kinetic and chemical equilibrium, same temperature T). As the Universe expands, the temperature decreases ($T \propto a^{-1}$). One after the other, the various massive species become non relativistic, anihilate, and their energy density becomes subdominant compared to the relativistic species.

If the Universe was is perfect thermal equilibrium, and if this equilibrium had persisted until today, the observed abundances of massive particles would be much lower than what they are, since every massive species is exponentially suppressed when it becomes non relativistic. In fact, thermal and chemical equilibrium need frequent collision (and / or reaction) rates to be maintained. As the Universe expands, particles dilute making it more difficult to maintain the reaction rates. A good rule of thumb is that we need several reactions per Hubble time to maintain thermal equilibrium. So, if

$$\Gamma \gg H \tag{6.4.1}$$

the equilibrium is maintained. When the reaction rate drops below H, thermal equilibrium is no longer maintained, particle densities *freeze out* to their pre-decoupling values. *Freeze out* is an essential mechanism to explain today's particle abundances.

6.4.1. Effective Number of relativistic species

We start with a primordial plasma in thermal and chemical equilibrium. All species share the same temperature which we note T. T is in particular the temperature of the photon bath.

The expansion rate is a direct function of the total energy density:

$$H^2 = \frac{8\pi G}{3}\rho(T) \tag{6.4.2}$$

where $\rho(T)$ is the sum of the densities of each species present in the primordial fluid.

$$\rho(T) = \sum_i \rho_i(T) \tag{6.4.3}$$

we have seen in the previous section that $\rho_i \propto T^4$ while the particle stays relativistic, and drops to almost nothing when the temperature drops below the particle mass. More precisely, we can write

$$\rho(T) = \frac{\pi^2}{30} g_\star(T) T^4 \tag{6.4.4}$$

where $g_\star(T)$ is the effective number of *relativistic* degrees of freedom of the plasma at temperature T:

$$g_\star(T) = \sum_{i=b} g_i + \frac{7}{8}\sum_{i=f} g_i \tag{6.4.5}$$

When the temperature drops below the mass of one of the species, m_i, it becomes relativistic and drops from the sum above. In the intervals between the particle masses, $g_\star(T)$ remains nearly constant. Since radiation dominates, we have $p = \rho/3$ and therefore $\rho \propto a^{-4}$. Since $\rho \propto T^4$, we have the usual

$$\boxed{T \propto a^{-1}} \tag{6.4.6}$$

in the primordial plasma.

6.4.2. Expansion of the primordial plasma

The expansion law obeys the first equation of Friedmann, which we know quite well now:

$$H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\frac{\pi^2}{30} g_\star(T) T^4 \tag{6.4.7}$$

and therefore:

$$\boxed{H = \sqrt{\frac{8\pi^3 G}{90}} g_\star^{1/2}(T) T^2} \tag{6.4.8}$$

So, $H \propto T^2$ modulo the variations of the effective number of degrees of freedom in the primordial plasma. Keep this is mind, it will be useful when comparing the expansion rate with the various reaction rates between the various species.

Table 6.1.: Particles of the standard model.

Type		mass	spin	g
quarks	$t, \bar{t}$	173 GeV	$\frac{1}{2}$	$2 \cdot 2 \cdot 3 = 12$
	$b, \bar{b}$	4 GeV		
	$c, \bar{c}$	1 GeV		
	$s, \bar{s}$	100 MeV		
	$d, \bar{d}$	5 MeV		
	$u, \bar{u}$	2 MeV		
gluons	g_i	0	1	$8 \cdot 2 = 16$
leptons	$\tau^\pm$	1777 MeV	$\frac{1}{2}$	$2 \cdot 2 = 4$
	$\mu^\pm$	106 MeV		
	$e^\pm$	511 keV		
	$\nu_\tau, \bar{\nu}_\tau$	< 0.6eV	$\frac{1}{2}$	$2 \cdot 1 = 2$
	$\nu_\mu, \bar{\nu}_\mu$	< 0.6eV		
	$\nu_e, \bar{\nu}_e$	< 0.6eV		
gauge bosons	W^+	80 GeV	1	3
	W^-	80 GeV	1	3
	Z^0	91 GeV	1	3
	γ	0	1	2
Higgs boson	H	125 GeV	0	1

Furthermore, since the Universe is radiation dominated, we have:

$$H = \frac{1}{2t} \qquad (a \propto t^{1/2}) \tag{6.4.9}$$

which gives

$$\boxed{T \approx [10^{10}\text{K}] \left(\frac{t}{1\text{ sec}}\right)^{-1/2}} \tag{6.4.10}$$

Or equivalently, we can derive the evolution of the typical particle energy with time.

$$\boxed{E \approx [3\text{ MeV}] \left(\frac{t}{1\text{ sec}}\right)^{-1/2}} \tag{6.4.11}$$

So, when the Universe was 1 second old, the typical particle energy was of the order of 1 MeV.

6.4.3. Evolution of the primordial plasma

The last missing piece, is the evolution of $g_\star$, which is just telling the evolution of the primordial plasma as it cools down with expansion. Let's start around $T \leq 100$GeV.

All standard model particles are relativistic (see table 6.1). When all particles are relativistic, the total number of degrees of freedom is:

$$g_f = \underbrace{6 \times 12}_{\text{quarks}} + \underbrace{3 \times 4}_{\ell^\pm} + \underbrace{3 \times 2}_{\nu' s} = 90 \tag{6.4.12}$$

for fermions, and

$$g_b = \underbrace{8 \times 2}_{g_i' s} + \underbrace{3 \times 3}_{W,Z} + \underbrace{2}_{\gamma} + \underbrace{1}_{H} = 28 \tag{6.4.13}$$

for bosons, which gives

$$g_\star = g_b + \frac{7}{8} g_f = 106.75 \tag{6.4.14}$$

To see what will happen next, we just need a look at the particle masses listed in table 6.1. The top quark anihilates first, reducing the number of degrees of freedom to:

$$g_\star(T < m_{\text{top}}) = 106.75 - \frac{7}{8} \times 12 = 96.25 \tag{6.4.15}$$

then, we have the Higgs, follows by the electroweak bosons $W^\pm$ and Z^0: reducing $g_\star$ to 86.25. Then b and c anihilate, at which point $g_\star$ has been reduced to 61.75.

The next event is the QCD phase transision, which occurs at $T \sim 150 MeV$. The quarks combine into baryons (protons, neutrons and mesons), all of them but the pions being relativistic. At this stage, the only relativistic species left are (1) the photons (2) in the lepton family, the neutrinos, electrons and muons and (3) for the baryons the pions of spin 0, hence: $g_\pi = 3 \cdot 1 = 3$. So:

$$g_\star = \underbrace{2}_{\gamma} + \underbrace{3}_{\pi} + \frac{7}{8} \times (\underbrace{4+4}_{e^\pm, \mu^\pm} + \underbrace{6}_{\nu' s}) = 17.25 \tag{6.4.16}$$

Then, the pions and the muons anihilate, leaving us with

$$g_\star = 2 + \frac{7}{8} \times (4 + 6) = 10.75 \tag{6.4.17}$$

The next two significant events are (1) the neutrino decoupling around 1 MeV and (2) the anihilation of the electrons and positrons ($m_e = 511\text{keV}$). This is the subject of the next section.

6.5. Neutrino decoupling and electron-positron anihilations

6.5.1. Neutrino decoupling

Neutrino decoupling is our first experience of freeze-out. Neutrinos interact only through the weak interaction. Around ~ 1 MeV, they are still thermalized through interactions

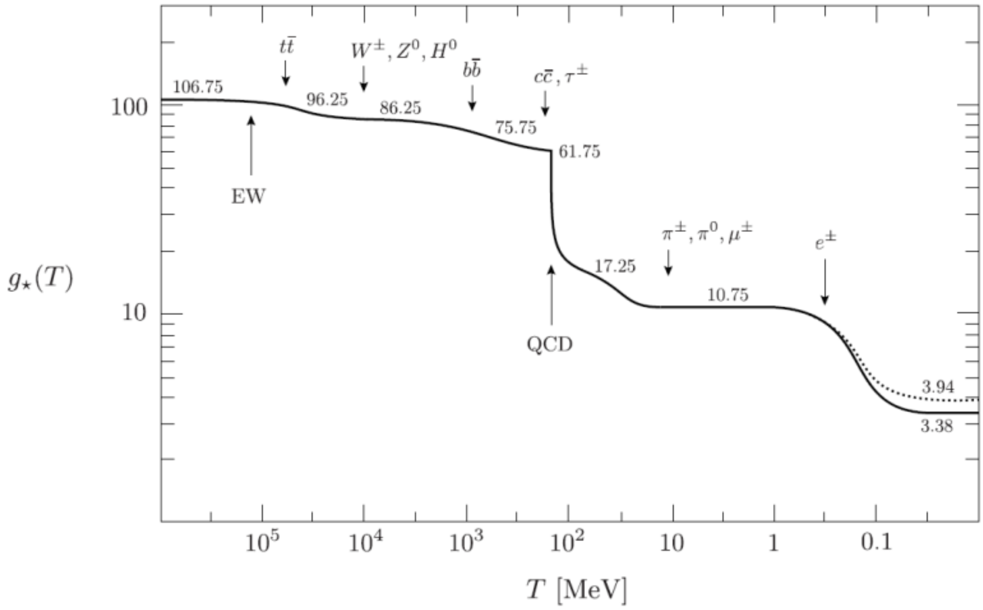

Figure 6.1.: The evolution of the effective number of species as a function of T (taken from Baumann).

such as:

$$\begin{aligned} \nu_e + p &\rightleftharpoons p + e^- \\ \nu_e + \bar{\nu}_e &\rightleftharpoons e^+ + p^- \\ e^- + \nu_e &\rightleftharpoons e^+ + \nu_e \end{aligned} \tag{6.5.1}$$

However, at these energies, the weak interaction cross section is $\sigma_w \sim G_F^2 T^2$, hence, the interaction rate $\Gamma = n_e \sigma_w c \propto G_F^2 T^5$ drops much more rapidly than the Hubble parameter ($\propto T^2$). Around 1 MeV, $\Gamma \sim H$ and interactions between neutrinos and other SM particles becomes highly unlikely. Neutrinos *decouple* and move free along geodesics.

At this stage, neutrinos are still relativistic ($m_\nu \ll 1 MeV$). Even though they do not interact with other particles anymore, they preserve to an excellent approximation their Fermi-Dirac distribution function (see box) with a temperature affected only by redshift. Hence, at this stage:

$$T_\nu = T_\gamma \propto a^{-1} \tag{6.5.2}$$

The spectrum of non-interacting, decoupled species

For ultra-relativistic species, we have $p \sim E$. The number of particles at t_1 in phase space volume $d^3p_1dV_1$ is:

$$dN = \frac{g}{(2\pi)^3}\frac{d^3p_1dV_1}{\exp((E(p_1)-\mu_1)/T_1)\pm 1} \tag{6.5.3}$$

At t_0, a while later, the same particles are in phase space volume $d^3p_0dV_0$. The momenta scale like a^{-1} and the volume scales like a^3. We can therefore write:

$$\begin{aligned} dN &= \frac{g}{(2\pi)^3}\frac{d^3p_1dV_1}{\exp((p_1-\mu_1)/T_1)\pm 1} \\ &= \frac{g}{(2\pi)^3}\frac{d^3p_0\left(\frac{a_0}{a_1}\right)^3 dV_0\left(\frac{a_1}{a_0}\right)^3}{\exp((p_0\left(\frac{a_1}{a_0}\right)-\mu_1)/T_1)\pm 1} \\ &= \frac{g}{(2\pi)^3}\frac{d^3p_0dV_0}{\exp((p_0-\mu_0)/T_0)\pm 1} \end{aligned} \tag{6.5.4}$$

with $\mu_0 \equiv \frac{a_1}{a_0}\mu_1$ and $T_0 \equiv \frac{a_1}{a_0}T_1$.

6.5.2. $e^+ - e^-$ anihilation

Shortly after neutrino decoupling around 1 MeV, electrons and positrons anihilate (511 keV). Naively, one could say that $g_\star(T)$ then becomes:

$$2 + \frac{7}{8}\times 6 = 7.25$$

but Nature is subtler. Indeed, the electron-positron anihilation produces enough energy and entropy to heat the photon bath, and change the photon temperature. Neutrinos are unaffected and their temperature still scales like a^-1. Therefore, after $e^+ - e^-$ anihilation, we have $T_\nu < T_\gamma$.

We are therefore in a new situation where we have several relativistic species with different temperatures. To account for this, we can modify equation 6.4.5, by allowing each species to have its own temperature:

$$g_\star(T) = \sum_{\text{bosons}} g_i\left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{\text{fermions}} g_i\left(\frac{T_i}{T}\right)^4 \tag{6.5.5}$$

To go further, we need to determine T_ν, or more exactly, to relate T_ν and T_γ (we know T_γ pretty well). The simplest way to do this is to use entropy conservation.

6.6. Conservation of entropy

The entropy of the Universe can only increase or stay constant. We know that, at equilibrium, entropy is conserved (see box). The primordial plasma is not *exactly* at equilibrium[2]: expansion makes it only a local equilibrium, we'll see many examples of out-of-equilibrium processes in what follows. However, since entropy is at first order proportional to the number of particles, and since photons are by far the most abundant species in the Universe, we can safely assume that entropy is conserved, and that, to a very high precision, cosmic expansion is an adiabatic process.

How to compute the total entropy of the Universe ? We can start with:

$$E = TS - PV + \sum_i \mu_i N_i \tag{6.6.1}$$

which gives:

$$S = \frac{E}{T} + \frac{P}{T}V - \sum_i \frac{\mu_i}{T} N_i \tag{6.6.2}$$

neglecting the chemical potentials:

$$S \approx \frac{E}{T} + \frac{P}{T}V \tag{6.6.3}$$

It is useful to consider the entropy *density* instead:

$$s \equiv \frac{S}{V} \approx \frac{\rho + P}{T} \tag{6.6.4}$$

For relativistic species, plugging in the expressions for density and pressure (6.3.3, 6.3.4), we find:

$$\boxed{\begin{aligned} s &= \frac{2\pi^2}{45} g T^3 \quad \text{for bosons} \\ s &= \frac{7}{8}\frac{2\pi^2}{45} g T^3 \quad \text{for fermions} \end{aligned}} \tag{6.6.5}$$

For a collection of species (fermions and bosons), we have:

$$s = \frac{2\pi^2}{45} g_{\star S}(T) T^3 \tag{6.6.6}$$

with

$$g_{\star S}(T) = \sum_{\text{bosons}} g_b \left(\frac{T_b}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_f \left(\frac{T_f}{T}\right)^3 \tag{6.6.7}$$

If the entropy S is conserved, then:

$$dS = 0 \Rightarrow \boxed{d(sa^3) = 0} \tag{6.6.8}$$

which gives:

$$\boxed{g_{\star S}(T) T^3 a^3 = \text{constant}} \tag{6.6.9}$$

[2]that's fortunate, otherwise, we would not be here to think about all that

6.6.1. The temperature of the Cosmic Neutrino Background

Let's go back to determining the relation between the temperature of the photons and the temperature of the cosmic neutrino background. During anihilation, the entropy and energy of the $e^{\pm}$ is transfered to the photon bath. The entropy being conserved, we have, before anihilation, taking apart the entropy of the neutrinos (which is conserved):

$$g^{\gamma}_{\star S,\mathrm{before}}(T) = 2 + \frac{7}{8}(4) = \underbrace{\frac{11}{2}}_{e^{\pm},\gamma} \tag{6.6.10}$$

after anihilation:

$$g^{\gamma}_{\star S,\mathrm{after}}(T) = 2 \tag{6.6.11}$$

Writing down the conservation of entropy, we have:

$$a^3_{\mathrm{before}}\ g^{\gamma}_{\star S\mathrm{before}}\ T^3_{\mathrm{before}} = a^3_{\mathrm{after}}\ g^{\gamma}_{\star S\mathrm{after}}\ T^3_{\mathrm{after}} \tag{6.6.12}$$

which gives:

$$\underbrace{T^3_{\mathrm{after}}}_{T_\gamma} = \frac{g^{\gamma}_{\star S,\mathrm{before}}}{g^{\gamma}_{\star S,\mathrm{after}}} \underbrace{\left(\frac{a_{\mathrm{before}}}{a_{\mathrm{after}}}\right)^3 T^3_{\mathrm{before}}}_{T_\nu} \tag{6.6.13}$$

and therefore:

$$T_\gamma = \left(\frac{11}{4}\right)^{1/3}\ T_\nu \tag{6.6.14}$$

So, we find that, after $e^{\pm}$ anihilation, the temperature of the cosmic neutrino background is indeed lower than the temperature of the CMB. Today, using $T_{CMB} = 2.726K$ we find:

$$T_\nu \approx 1.95K \tag{6.6.15}$$

From this, we can derive the number density of neutrinos n_ν as function of n_γ. Neutrinos are fermions (hence the 3/4 factor):

$$n_\nu = \frac{3}{4} \times 3 \times \frac{4}{11} n_\gamma \tag{6.6.16}$$

which gives $\approx 112\mathrm{cm}^{-3}$ per flavor (336 cm^{-3} total).

For the energy density of the neutrino background, we find:

$$\rho_\nu = \frac{7}{8} \times 3 \times \left(\frac{4}{11}\right)^{4/3} \rho_\gamma \tag{6.6.17}$$

and numerically, we find $\Omega_\nu h^2 \approx 1.710^{-5}$.

In fact, neutrinos have masses, with two important consequences (1) we do not know whether they are still relativistic today (all species) (2) $\Omega_\nu h^2$ is larger than the value quoted above. Cosmology with massive neutrinos will be the subject of an upcoming homework.

Another remark, is that neutrino decoupling overlapped slightly with $e^{\pm}$ anihilation. Since neutrinos were still interacting when anihilation occured the neutrino background was slightly affected by the enormous energy and entropy release from $e^{\pm}$ anihilation. In the literature, this is taken into account by introducing an "effective number of neutrinos", $N_{eff} \approx 3.046$. Accounting for this, the neutrino number and energy density are:

$$\begin{aligned} n_\nu &= \frac{3}{4} N_{\text{eff}} \frac{4}{11} n_\gamma \\ \rho_\nu &= \frac{7}{8} N_{\text{eff}} \frac{4}{11} n_\gamma \end{aligned} \tag{6.6.18}$$

And finally, the correct values $g_\star$ and $g_{\star S}$ after $e^{\pm}$ anihilation are:

$$\begin{aligned} g_\star &= 2 + \frac{7}{8} 2N_{\text{eff}} \left(\frac{4}{11}\right)^{4/3} \approx 3.36 \\ g_{\star S} &= 2 + \frac{7}{8} 2N_{\text{eff}} \left(\frac{4}{11}\right) \approx 3.94 \end{aligned} \tag{6.6.19}$$