Quantum ElectroDynamic
QED

Particle Physics
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Overview

1. Introduction to quantum field theory (QFT)
    1. Introduction
    2. Scalar fields
    3. Dirac fields
    4. Electromagnetic field
2. Quantum electrodynamics (QED), an interacting field theory
    1. Interacting field theory
    2. Feynman diagrams
    3. ee to mumu scattering
3. Renormalisation for experimentalists
    1. Preliminary
    2. Vertex correction
    3. Self-energy correction
    4. Vacuum polarisation
    5. First test of QED running coupling constant

Introduction

- Quantum field theory (QFT) developed to solve

Special relativity + quantum mechanics

- Special relativity
  - ✓ $E = m c^2$
  - ✓ Can convert particle into energy and vice-versa
- Classical quantum mechanics
  - ✓ Ψ(x) probability of finding a particle at point x
- QM+ SR
  - ✓ Probability for a single particle can not be conserved since the particle can disappear or appear…
  - ✓ Interpretation of Ψ(x) ???
- QFT : interpret Ψ(x) as an operator

Basics of QFT

- The interpretation of Ψ(x) with the help of the classical quantum oscillator and the ladder operator
- For a real (*i.e.* neutral) scalar field

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi \;=\; \left(\Box + m^2\right) \phi = 0$$

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\,E_p}} \left(a_{\vec{p}}\, e^{-ip.x} + a^\dagger_{\vec{p}}\, e^{+ip.x}\right)$$

$$p^0 = E_p = + \sqrt{m^2 + \vec{p}^{\,2}}$$

$$[a_{\vec{p}}, a^\dagger_{\vec{k}}] = (2\pi)^3 \boxed{2\,E_p}\, \delta^{(3)}(\vec{p} - \vec{k})$$

For Lorentz invariance

Complex scalar fields

- Charged scalar field (ϕ complex)
  - ✓ both ϕ and ϕ* obey KG equation

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\,E_p}} \left(a_{\vec{p}} e^{-ip.x} + b^\dagger_{\vec{p}} e^{+ip.x}\right)$$

$$\phi^\dagger(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\,E_p}} \left(b_{\vec{p}} e^{-ip.x} + a^\dagger_{\vec{p}} e^{+ip.x}\right)$$

- a, b same commutation rules as for a real scalar field
  - ✓ a+ (a) creates (destroys) a particle
  - ✓ b+ (b) creates (annihilates) an anti-particle
- The electric charge is conserved (not the number of particles)

$$Q = \int \frac{d^3\,p}{(2\pi)^3} \left(a^\dagger_{\vec{p}} a_{\vec{p}} - b^\dagger_{\vec{p}} b_{\vec{p}} \right)$$

N particles - M anti-particles = (N+M) q e

Fermion fields

- Fermions field are bi-spinor and obey the Dirac equation

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\,E_{\vec{p}}}}\left(a^s_{\vec{p}}u^s(p)e^{-ip.x} + b^{s\,\dagger}_{\vec{p}}v^s(p)e^{+ip.x}\right)$$

$$\overline{\psi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\,E_{\vec{p}}}}\left(b^s_{\vec{p}}\bar{v}^s(p)e^{-ip.x} + a^{s\,\dagger}_{\vec{p}}\bar{u}^s e^{+ip.x}\right)$$

- u, v are the bi-spinor for fermions (anti-fermions)
  - ✓ note: s index corresponds to the spin (up or down)
- To respect causality; a, a^+ and b, b^+ must anti-commute

$$\{a^s_{\vec{p}}, a^{r\,\dagger}_{\vec{k}}\} = (2\pi)^3\; 2\,E_p\;\delta^{(3)}(\vec{p} - \vec{k})\;\delta_{rs}$$

- Show this implies: $(a^+)^2\,|0> = |0>$. Fermi statistics !

Photon field

- Quantization done on the potential vector A^μ
- 4 scalar field obeying independently KG equations
- Quantization to be done respecting gauge invariance…
- A^μ a boson vector with spin 1
  - ✓ massless: only 2 state of helicity
- Euler Lagrange motion equation = Maxwell equations

$$A^\mu \equiv (V, \vec{A})$$

Show that

$$\partial_\mu F^{\mu\nu} = 0$$ **With** $$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A^\mu \equiv \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & -B_z & +B_y \\ -E_y & +B_z & 0 & -B_x \\ -E_z & -B_y & +B_x & 0 \end{pmatrix}$$

And find back Maxwell equations for the interacting field

$$\partial_\mu F^{\mu\nu} = j^\nu \equiv (j^0 = \rho_q, \vec{j} = \vec{j}_q).$$

A word on Lagrangians

- Euler-Lagrange equations from the QED Lagrangian

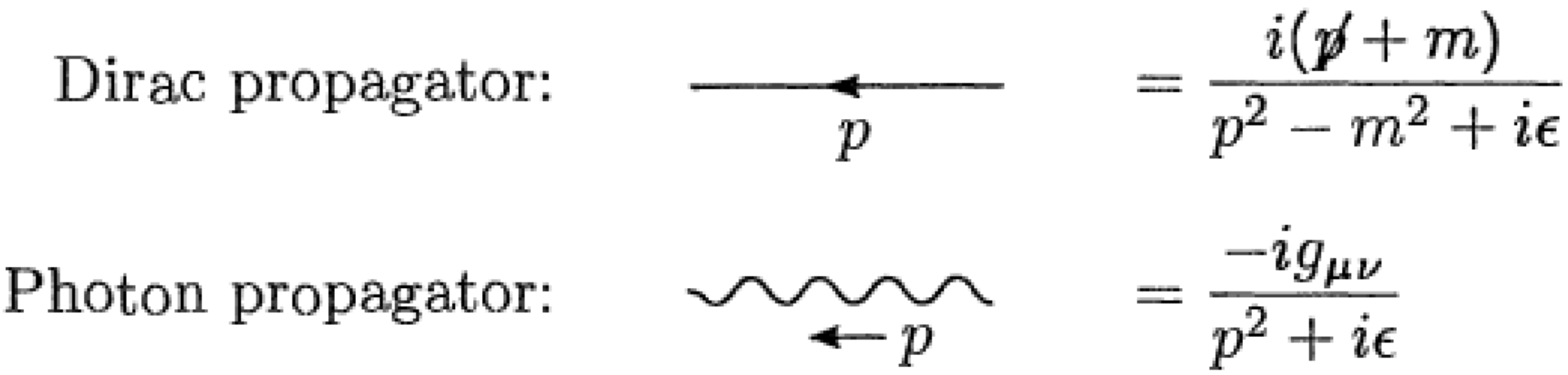
$$\begin{aligned}\mathcal{L}_{QED} &= \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int} \\ &= \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + Q\,e\,\overline{\psi}\gamma^\mu\,\psi\,A_\mu\end{aligned}$$

- The modified Euler-Lagrange equations when fields interact

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\psi &= Q\,e\,A_\mu\gamma^\mu\psi \\ \partial_\mu F^{\mu\nu} &= Q\,e\,\overline{\psi}\gamma^\nu\psi\end{aligned}$$

QED Feynman rules (1) - propagator

- Propagator are actually « propagating » a particle from time-space point x to y
- Are derived from the free theory and used for internal lines in a diagram



Dirac propagator: $$= \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

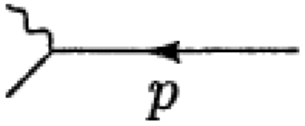
Photon propagator: $$= \frac{-ig_{\mu\nu}}{p^2+i\epsilon}$$

QED vertex:

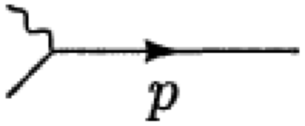
$= iQe\gamma^\mu$

($Q = -1$ for an electron)

External fermions:

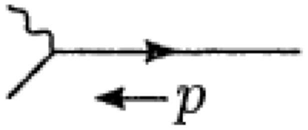


$= u^s(p)$ (initial)

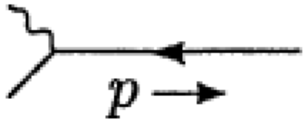


$= \bar{u}^s(p)$ (final)

External antifermions:

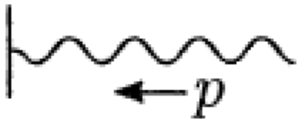


$= \bar{v}^s(p)$ (initial)

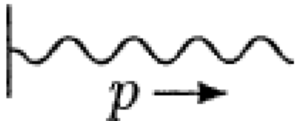


$= v^s(p)$ (final)

External photons:

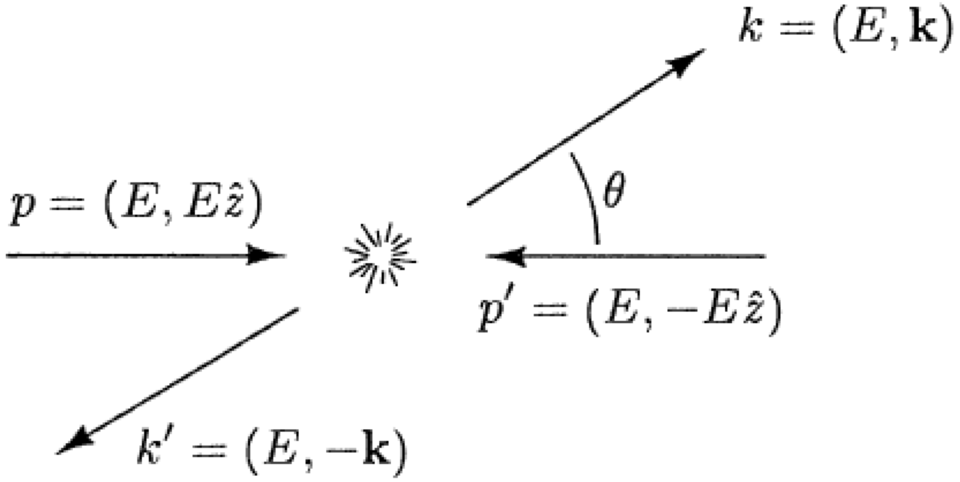


$= \epsilon_\mu(p)$ (initial)



$= \epsilon^*_\mu(p)$ (final)

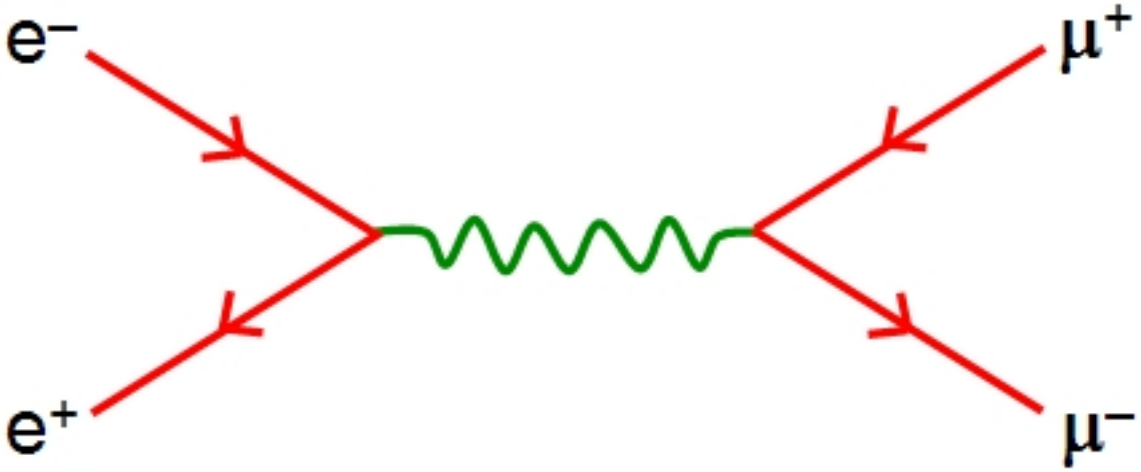
A first example $e^+e^-\rightarrow\mu^-\mu^+$



$$|\mathbf{k}| = \sqrt{E^2 - m_\mu^2}$$

$$\mathbf{k}\cdot\hat{z} = |\mathbf{k}|\cos\theta$$

$e^+ + e^- \rightarrow \mu^+ + \mu^-$



$$\mathcal{M} = \bar{v}^{s'}(p')\,i\,Q\,e\gamma^\mu u^s(p)\ \frac{-i\eta_{\mu\nu}}{(p+p')^2}\ \bar{u}^r(k)\,i\,Q\,e\gamma^\nu v^{r'}(k') \propto Q^2e^2$$

A first example $e^+e^- \rightarrow \mu^-\mu^+$

In the high energy limit, can easily demonstrate

$$\frac{d\sigma}{d\Omega} \propto \alpha^2 \left(1 + \cos^2\theta \right)$$

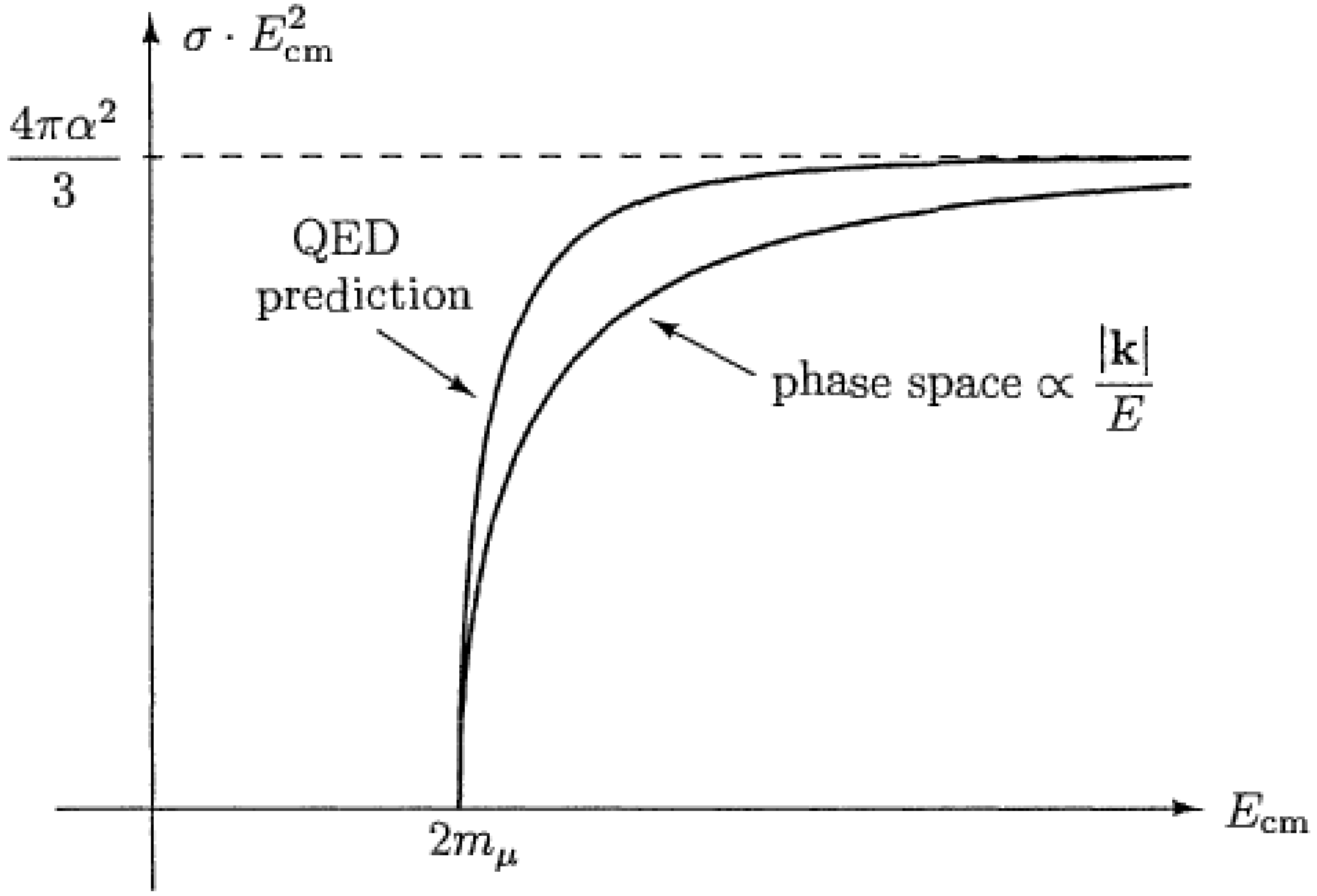
Full computation gives (assuming $m_e = 0$). See the QFT course

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4\ s}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 + \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$\boxed{\sigma_{tot} = \frac{4\pi\,\alpha^2}{3\ s}\underbrace{\sqrt{1 - \frac{m_\mu^2}{E^2}}}_{\textbf{Phase space}}\underbrace{\left(1 + \frac{m_\mu^2}{E^2}\right)}_{\textbf{Matrix element}}}$$

With $2E = \sqrt{s}$

A first example $e^+e^-\rightarrow\mu^-\mu^+$



R ratio in e^+e^- collisions

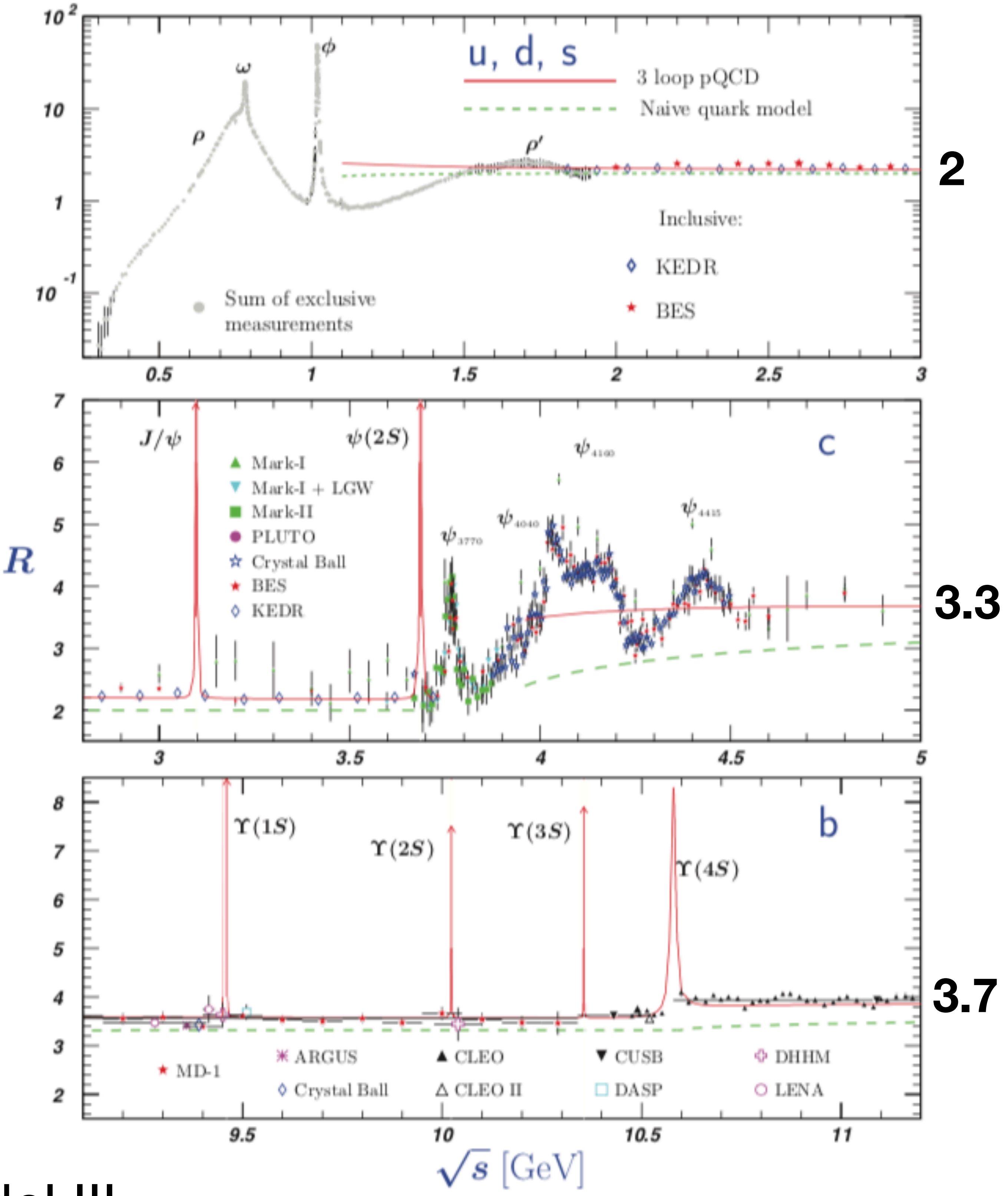
$$R_{had} = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

Show that in the quark model

$$\sqrt{s} < 2m_c \Rightarrow R_{had} = 2$$

$$\sqrt{s} < 2m_b \Rightarrow R_{had} = 3.3$$

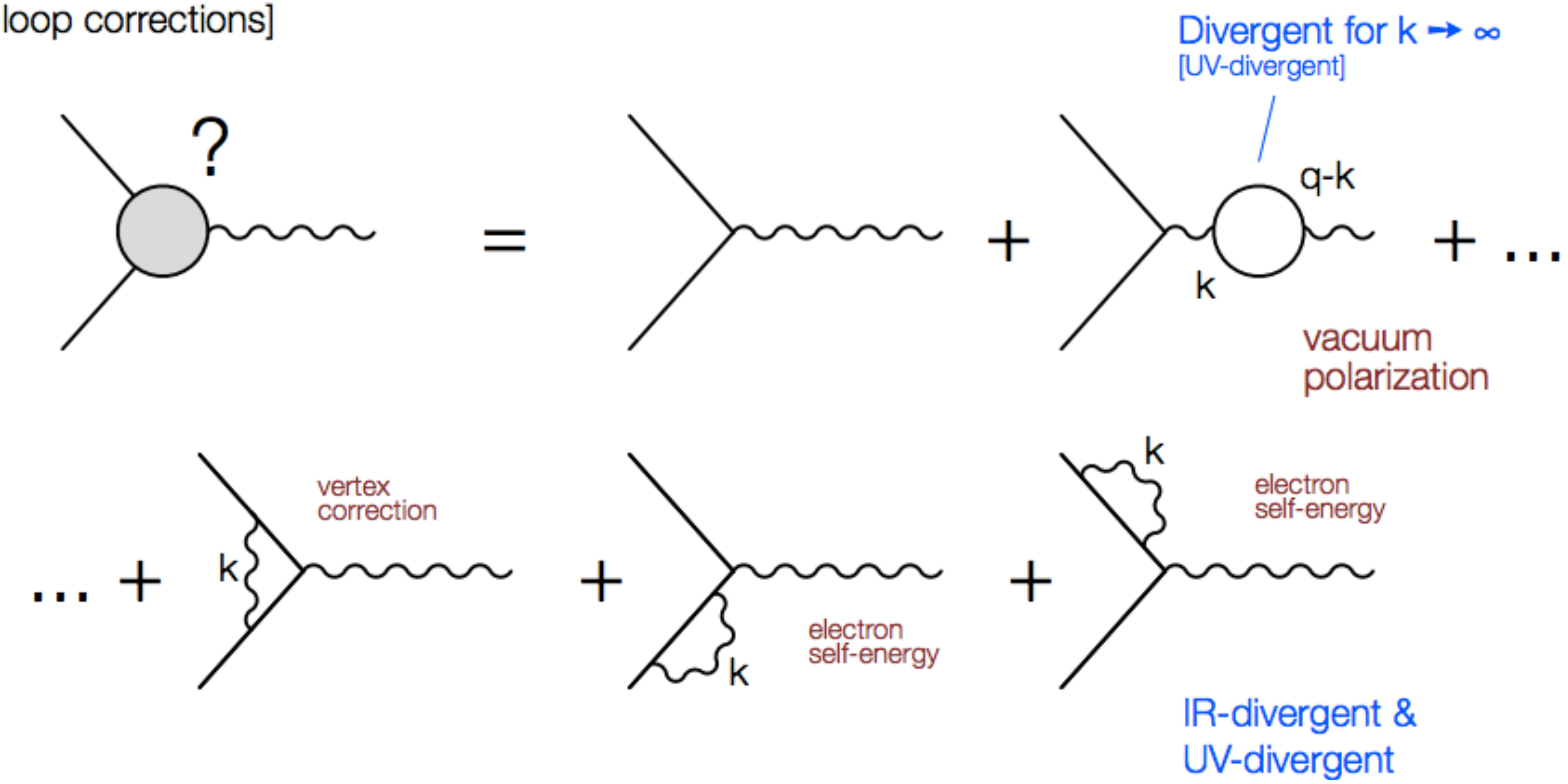
$$\sqrt{s} < 2m_t \Rightarrow R_{had} = 3.7$$



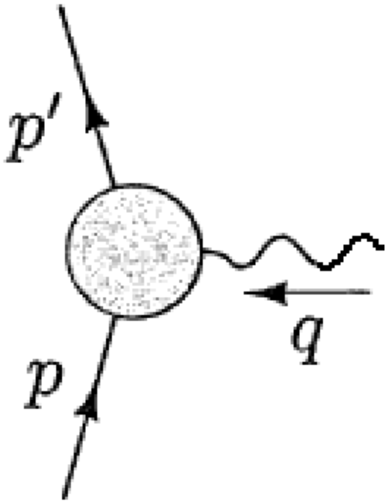
Very big success of the quark model !!!

Corrections to the QED vertex

Renormalization in QED:
[1-loop corrections]



The QED vertex



General form for vertex

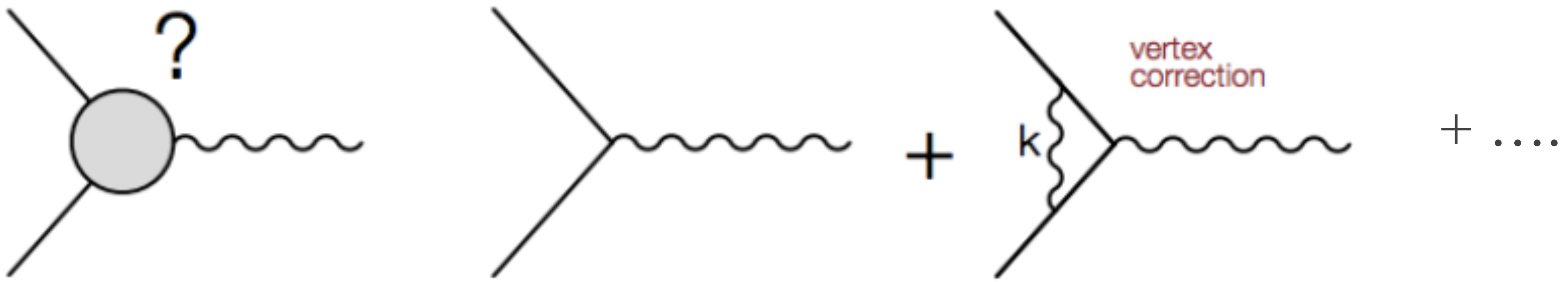
$-ie\gamma^\mu \rightarrow -ie\Gamma^\mu$

$$\Gamma^\mu(q = p' - p) = \gamma^\mu F_1(q^2) + \frac{i\,\sigma^{\mu\nu}\,q_\nu}{2\,m} F_2(q^2)$$

Tree level: $F_1(q^2) = 1$; $F_2(q^2) = 0$

The second term modifies the electron coupling to a classical B field.

QED vertex correction



At NLO, vertex correction

$F_1(q^2) = 1 - \delta F_1(q^2{=}0) + \delta F_1^{IR}(q^2)$
$F_2(q^2) = 0 + \alpha/2\pi \text{ x } [\ 1(q^2 \rightarrow 0)\ ]$

F_1 diverges at low (IR) and high (UV) q^2

F_2 does not diverge!

$a_e = (g-2)/2 = F_2(0) = \alpha/(2\pi) = 0.001161$

Electron anomalous g-2

$$a_e^{QED} = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

C_4 891 diagrams, C_5 12672 diagrams (precision 3%) !!!

$$a_e^{SM} = \overset{\text{dominant}}{a_e^{QED}} \overbrace{+a_e^{QED}(\mu) + a_e^{QED}(\tau) + a_e^{QED}(\mu,\tau) + a_e(\text{hadr}) + a_e(\text{weak})}^{\text{small terms (i.e. } \leq 3\times 10^{-12})}$$

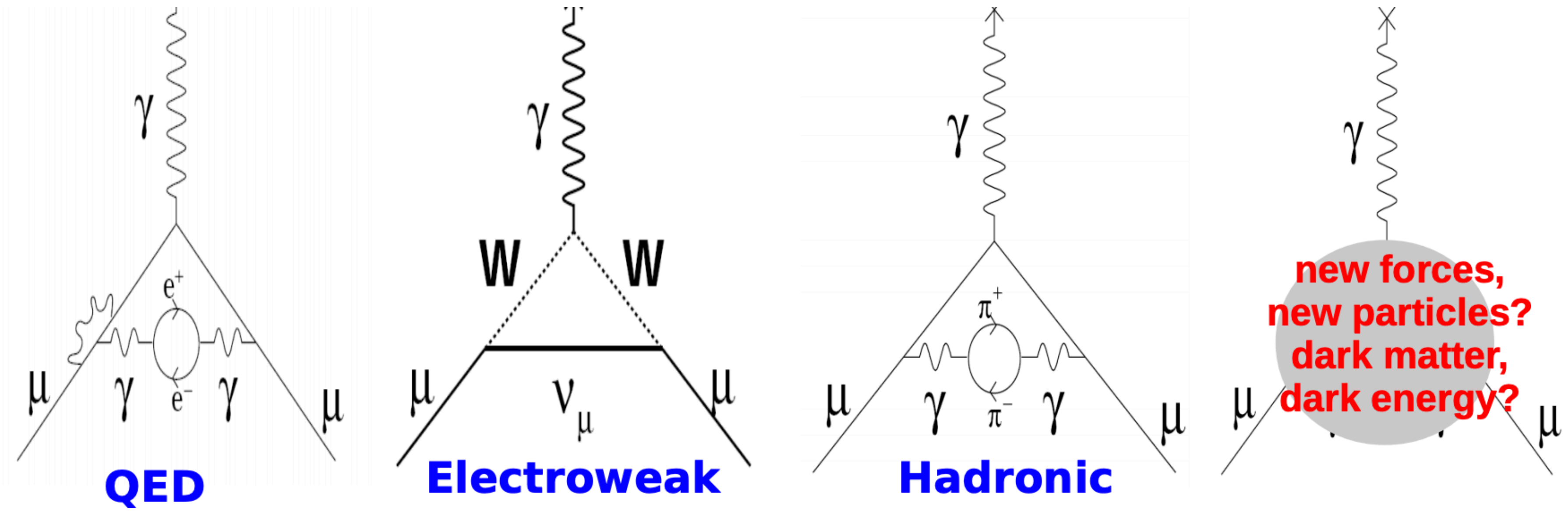
$$a_e^{SM}(\alpha) = \quad 1\ 159\ 652\ 182.031\overset{C_5}{(15)}\overset{had}{(15)}\overset{\alpha}{(720)} \times 10^{-12}$$

$$a_e^{exp} = \quad 1\ 159\ 652\ 180.730(280) \times 10^{-12} \qquad 0.25\ ppb$$

$$a_e^{SM}(\alpha) - a_e^{exp} = \quad 1.30(77) \times 10^{-12}$$ **1.6 σ agreement**

Best measurement: D. Hanneke et al, Phys. Rev. Lett. 100, 120801 (2008)

Muon anomalous g-2

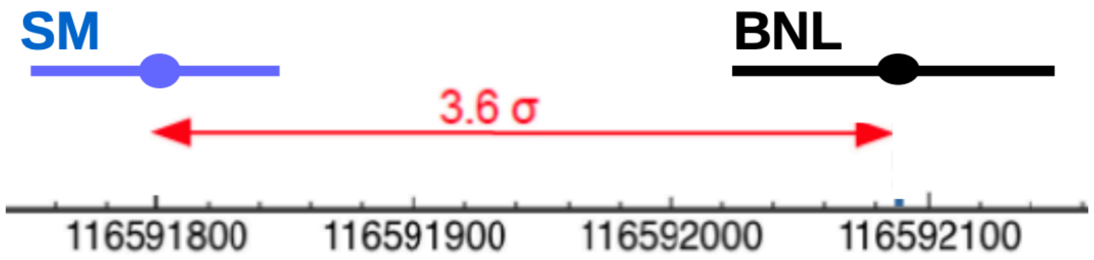


a_μ^{QED} =
116 584 719.0 (0.1)

a_μ^{EW} =
154 (1)

a_μ^{Had} =
6983 (44)

$\times 10^{-11}$



15-year, persistent
discrepancy

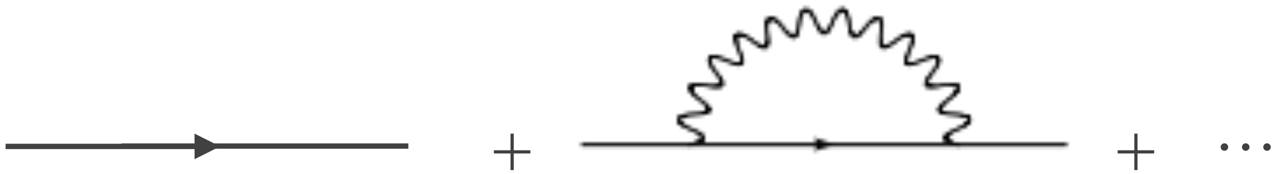
a_μ $\times 10^{-11}$

BNL: G. W. Bennett et al, Phys Rev D 73, 072003 (2006)

For muon the EW and hadronic contribution are not negligible.
Long standing oscillating discrepancy between 2 and 3 σ.
Lively field both on theory side (hadron contribution, LxL diagrams) and experimental side (Fermilab starting to take data, J-PARC approved)

Electron self-energy (1)

Electron self energy $S_F(p) =$



Loop corrections have 2 effects

1. shift the pole in the propagator, *i.e.* change the electron mass
2. change the electron field strength by a factor $\sqrt{Z_2}$

Find the divergence of the loop diagram by dimensional arguments.
In fact pole mass only shifted by +log(∞) (bare mass is infinite),

Z_2 also UV divergent, but
$\delta Z_2(q^2)$ is not diverging

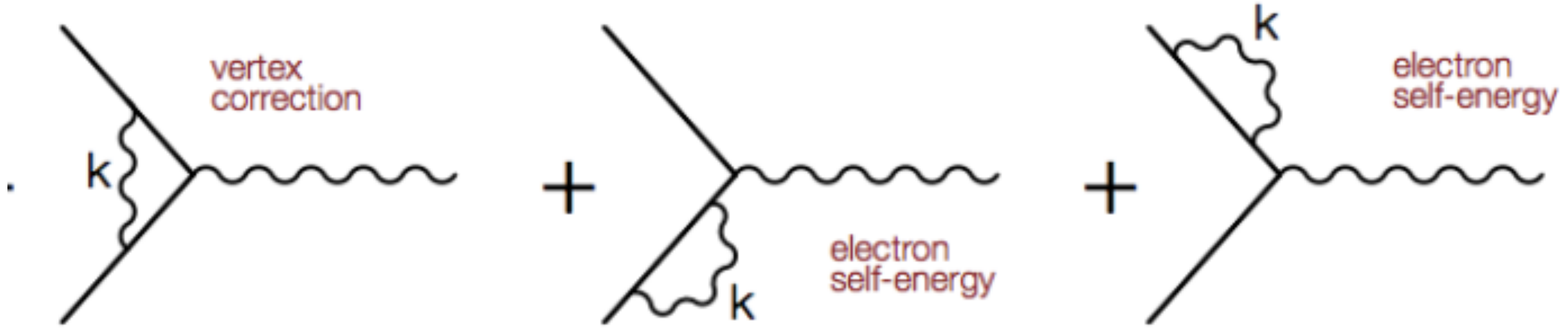
$\delta Z_2(q^2{=}0)$ diverges but fortunately

$$\delta Z_2(q^2{=}0) = \delta F_1(q^2 = 0)$$

$$Z_2 = 1 + \delta Z_2(q^2{=}0) + \delta Z_2(q^2)$$

$$\bar{u}(p')\,\Gamma^\mu(q)\,u(p) \to \left(\sqrt{Z_2}\bar{u}(p')\right)\,\Gamma^\mu(q)\,\left(\sqrt{Z_2}u(p)\right)$$

Electron self-energy (2)



UV divergences cancels out!

In fact : true to all orders thanks to gauge properties

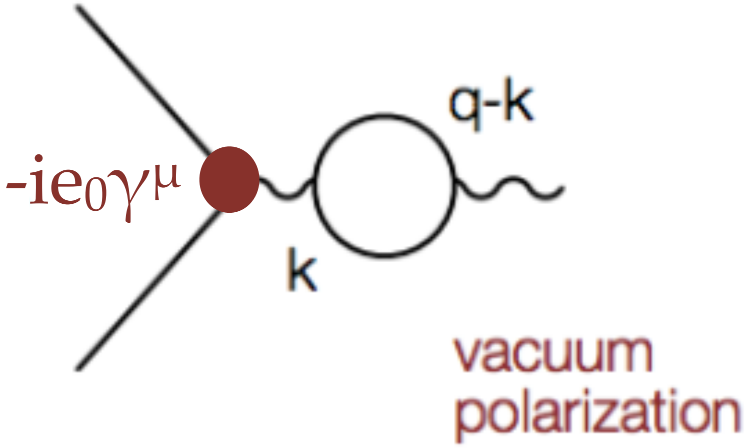
$Z_2 = 1 + \delta Z_2(q^2=0) + \delta Z_2(q^2)$

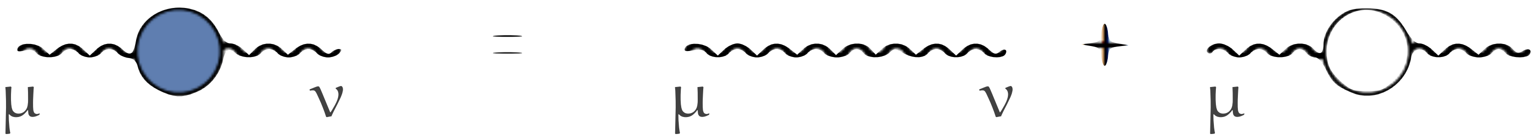
$F_1(q^2) = 1 - \delta F_1(q^2=0) + \delta F_1^{IR}(q^2)$

$\delta Z_2(q^2=0) = \delta F_1(q^2 = 0)$

$$\bar{u}(p')\,\Gamma^\mu(q)\,u(p) \to \left(\sqrt{Z_2}\bar{u}(p')\right)\,\Gamma^\mu(q)\,\left(\sqrt{Z_2}u(p)\right)$$

Vacuum polarisation





$$\frac{-i\eta^{\mu\nu}}{q^2+i\varepsilon} \rightarrow \frac{-i\eta^{\mu\nu}}{q^2+i\varepsilon}\,\frac{1}{1-\Pi(q^2)}$$

Dimension regularization
$\Pi(q^2=0) = -2\alpha/(3\pi\varepsilon)$
UV divergence!

Do not change the propagator, but absorb (1-Π) in the electron charge definition

$$\alpha(q^2) = \frac{e_0{}^2}{4\pi}\,\frac{1}{1-\Pi(q^2)}$$

$$\boxed{\alpha_{\mathrm{eff}}(q^2) = \alpha(q^2=0)\,\frac{1}{1-\Pi(q^2)-\Pi(0)}}$$

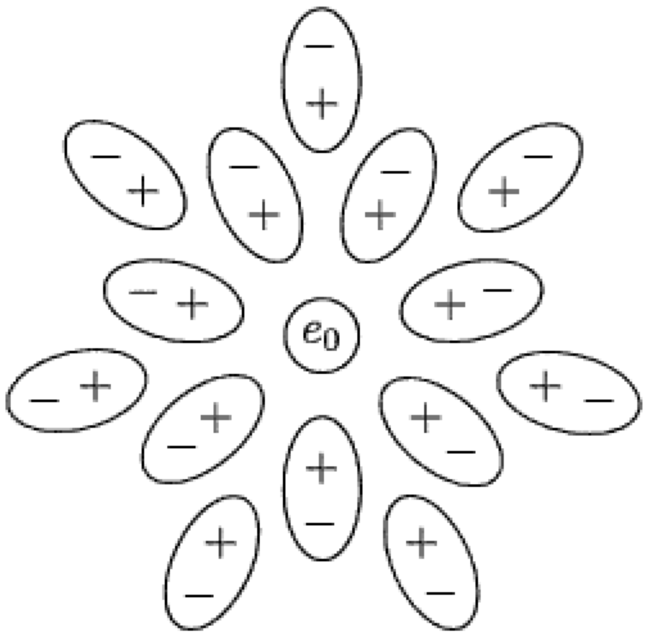
Bare electron charge is infinite but it's absorbed in $\alpha(q^2 = 0)$

Electron charge depends on q^2 of the photon probe!

⟹ coupling constants are running in QFT!

Running of QED

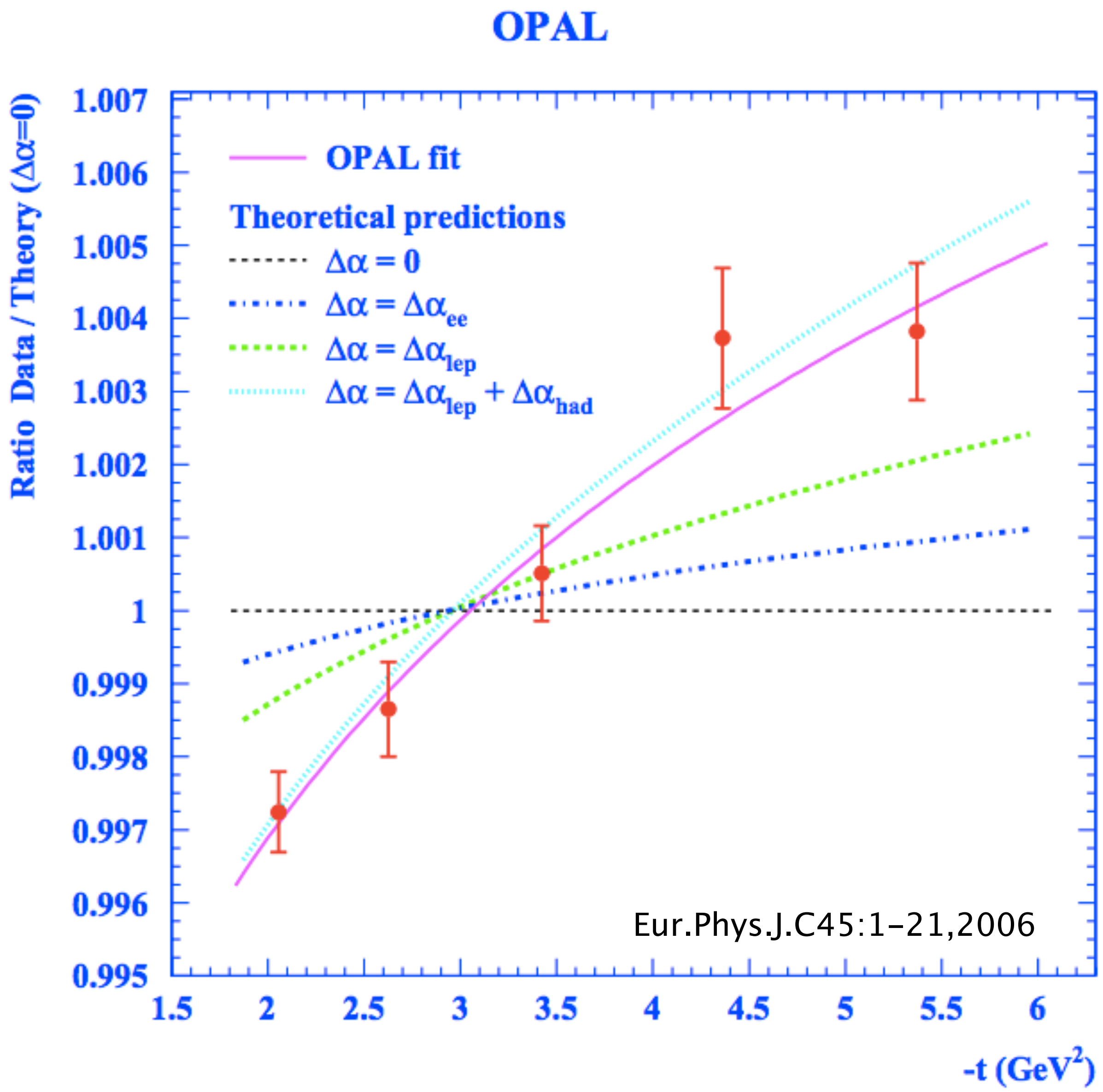
Screening effect



$$\alpha_{eff}(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\sum_f Q_f^2 \log\left(\frac{-q^2}{m_f^2}\right)}$$

$\alpha(q^2 = 0) = 1/137$

$\alpha(q^2 = M_Z^2) = 1/128$



Eur.Phys.J.C45:1-21,2006

Obtained from Bhabha scattering

TOPAZ @ Tristan (1997)

Measurement of the Electromagnetic Coupling at Large Momentum Transfer

PRL78, 424 (1997)

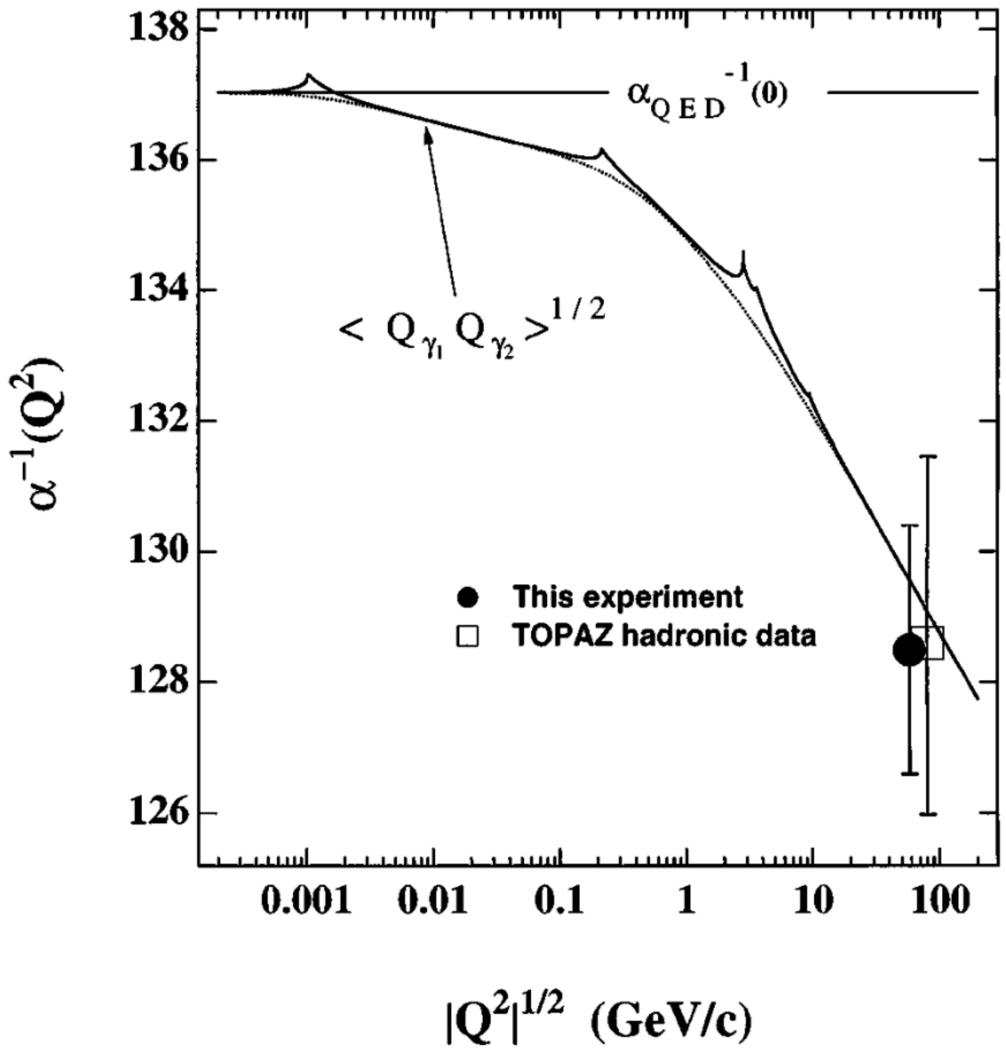
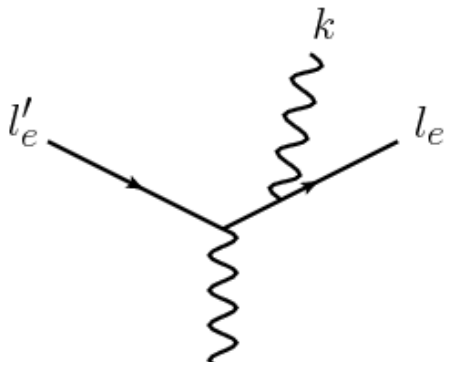


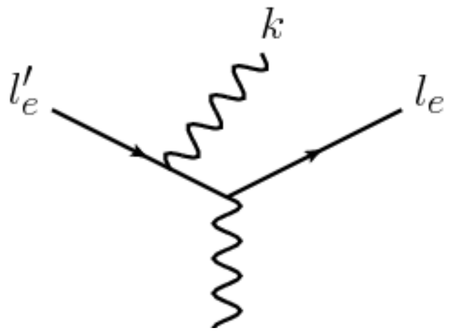
FIG. 2. The measured and theoretical electromagnetic coupling as a function of momentum transfer Q. The solid and dotted lines correspond to positive and negative Q^2 predictions, respectively. As we probe closer to the bare charge, its effective strength increases. $\langle Q_{\gamma_1} Q_{\gamma_2} \rangle^{1/2}$ denotes the square root of the median value for the product of the photon momentum transfers in the antitagged $e^+e^- \rightarrow e^+e^-\mu^+\mu^-$ sample. The hadronic data point has been shifted for display.

Complements

Infrared divergences (1)

Elastic scattering of
an electron





Well known problem: soft photon emission diverges at low k
Regularise with soft photon mass μ

$$\boxed{d\sigma(p\to p'+\gamma) \;=\; d\sigma_0(p\to p') \text{ x } \alpha/\pi\, f_{IR}(q^2)\log(-q^2/\mu^2)}$$

NB: $f_{IR}(q^2\to -\infty) = \log(-q^2/m_e^2)$
double logarithm!

Compensates precisely the IR divergence from the vertex!

$$\delta F_1^{IR}(q^2) = -\alpha/(2\pi)\, f_{IR}(q^2)\log(-q^2/\mu^2)$$

Can not measure elastic scattering independently from soft bremsstrahlung emission

$$\boxed{d\sigma_{meas}(p\to p') \equiv d\sigma(p\to p') + d\sigma(\,p\to p'+\gamma(k<E_{min})\,)}$$

Infrared divergences (2)

$$d\sigma_{meas}(p \to p') \equiv d\sigma_0(p \to p')[\, 1 - \alpha/\pi \; f_{IR}(q^2) \; \log(\, -q^2/\mathbf{E_{min}}^{\mathbf{2}}) \,]$$

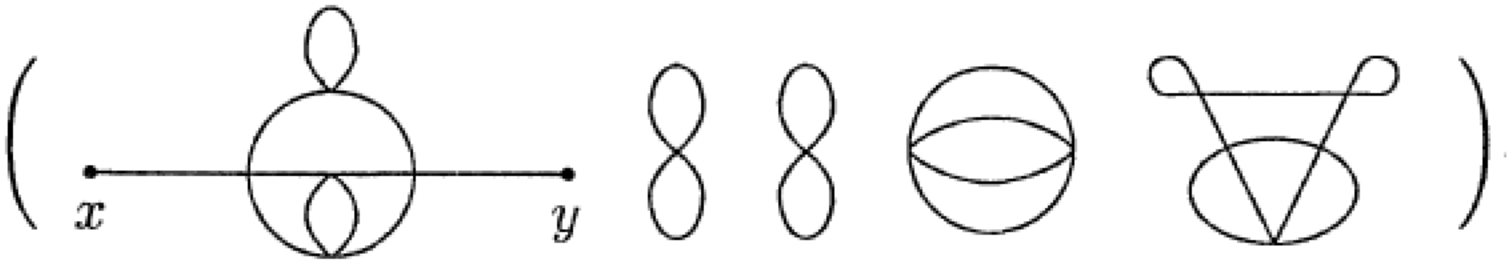
where E_{min} is the minimal energy one can measure for a photon
μ dependence has disappeared…

Emission of n soft photons adds terms $[\alpha/\pi\ (\,-q^2/m_e^2)\ \ \log(\,-q^2/\boldsymbol{\mu}^{\mathbf{2}})]^n$
Summing the logarithm to get to the Sudakov form factor

$$d\sigma_{meas}(p \to p') \equiv d\sigma_0(p \to p') \text{ x } \\ |\exp[\, -\alpha/\pi \; f_{IR}(q^2) \log(\, -q^2/E_{min}^2) \,]\, |^2$$

QED Feynman rules (3)

- For a given process the amplitudes of all diagrams with identical input and output particles should be summed
- Not connected diagram should be removed



- Only amputated diagrams (*i.e.* no loops on an external leg)