M2 NPAC – Particle physics (2022-2023)

Exercise sheet № 4 - Introduction to strong interaction: isospin, SU(3)

Brief correction

Exercise 1

$\Lambda \to K^0\ X$

Λ is a baryon $\Rightarrow$ X is a baryon to ensure the conservation of the baryonic number (K^0 is a meson).

The lightest baryon with the required electric charge is the neutron n, but $m(\Lambda) < m(K^0) + m(n)$. Hence this decay is forbidden.

Although when writing Λ without the mass it designates the lightest state called Λ, if we consider the $\Lambda(1520)$ instead of the lightest Λ, the mass is high enough and the decay can happen. The PDG shows that $\Lambda(1520) \to K^0\ n$ and $\Gamma[\Lambda(1520)] \sim 16$ MeV $\Rightarrow$ strong interaction. Note that the K^0 in the final state guarantees the strangeness conservation.

Exercise 2

a) J=1 $\Rightarrow$ s=0, ℓ=1 or s=1, ℓ=0 (in principle, there is also s=1, ℓ=1 or 2, but these would have been excited states and we know that only the fundamental state exists).

We know that $P=P_pP_n(-1)^\ell=(-1)^\ell = +1 \Rightarrow$ ℓ even $\Rightarrow$ $\boxed{s=1,\ \ell=0}$

b) p-p and n-n are systems with two identical spin-½ fermions.

$\Rightarrow \quad \varepsilon = (-1)^{s+\ell+1} = -1 \ \Rightarrow$ s+ℓ even

$\Rightarrow$ s even ℓ even or s odd ℓ odd

First states: s=0, ℓ=0, $J^P=0^+$

s=1, ℓ=1, $J^P=0^-, 1^-, 2^-$

c) Given that strong interaction between nucleons does not depend on the electric charge, and as the deuteron has only one bound energy level, the only possible state is s=0, ℓ=0 (ℓ=1 and beyond cannot be bound states for p-p and n-n, because they are not bound states for the deuteron).

If the interaction between nucleons depends on their spin states, we can postulate that the state s=0, in the case of two nucleons, is energetically unflavored with respect to s=1. Then it is possible that this state does not exist as a bound state, and the puzzle is solved.

d) Isospin of a system of two nucleons:

$$I = \frac{1}{2} \otimes \frac{1}{2} \Rightarrow I_{tot} = 0 \text{ or } 1$$

$$\text{Triplet} \begin{cases} pp & \left|I_{tot}=1, I_{3tot}=1\right\rangle = \left|I=\frac{1}{2}, I_3=\frac{1}{2}\right\rangle \\ pn & \left|I_{tot}=1, I_{3tot}=0\right\rangle = \dfrac{\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{-1}{2}\right\rangle + \left|\frac{1}{2},\frac{-1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle}{\sqrt{2}} \\ nn & \left|I_{tot}=1, I_{3tot}=-1\right\rangle = \left|\frac{1}{2},\frac{-1}{2}\right\rangle\left|\frac{1}{2},\frac{-1}{2}\right\rangle \end{cases}$$

$$\text{Singlet } np \quad \left|I_{tot}=0, I_{3tot}=0\right\rangle = \frac{\left|\frac{1}{2},\frac{1}{2}\right\rangle\left|\frac{1}{2},\frac{-1}{2}\right\rangle - \left|\frac{1}{2},\frac{-1}{2}\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle}{\sqrt{2}}$$

Similarly to the spin, the function of the isospin triplet (singlet) is symmetric (anti-symmetric) under exchange.

As strong interaction does not depend on the electric charge, if a bound state $I=1$, $I_3=0$ existed, then the states $I_3 = +1,-1$ would also exist. Knowing that we only observe one bound state of n-p, it must be the isospin singlet ($I=0$).
Another way to answer: with the isospin formalism, p and n are two quantum states of a "single particle", the nucleon (N). The deuteron becomes a state of two identical spin-½ fermions. The total function has now an isospin component:
$\varepsilon = (-1)^{s+\ell+1} \cdot \varepsilon_{isospin} = -1$ (anti-symmetric).
As we know that $s=1$, $\ell=0 \Rightarrow \varepsilon_{isospin}=-1 \Rightarrow$ isospin singlet ($I=0$)

Exercise 3
The two processes are due to strong interaction $\Rightarrow$ isospin conservation

- $p + p \rightarrow d + \pi^+$. We have the isospin states:

$|p> = |1/2,1/2>$
$|d> = |0,0>$
$|\pi^+> = |1,1>$

$|pp> = |1/2,1/2>|1/2,1/2>=|1,1>$
$|d\pi^+> = |0,0>|1,1> = |1,1>$

$<pp\ |H_{int}|\ d\pi^+> = T_1$

- $n + p \rightarrow d + \pi^0$. We have the isospin states:

$|n> = |1/2,-1/2>$
$|\pi^0> = |1,0>$

$|np> = |1/2,-1/2>|1/2,1/2>=(|1,0>-|0,0>)/\sqrt{2}$
$|d\pi^0> = |1,0>$

$<np\ |H_{int}|\ d\pi^0> = <1,0\ |H_{int}(|1,0>-|0,0>)/\sqrt{2}) = <1,0\ |H_{int}|1,0>/\sqrt{2} = T_1/\sqrt{2}$

Given that isospin is conserved by the strong interaction, the Hamiltonian can only connect initial and final states with the same isospin (same I and same I_3).

$$\Rightarrow \frac{\sigma\left(pp \rightarrow d\pi^+\right)}{\sigma\left(np \rightarrow d\pi^0\right)} = \frac{|T_1|^2}{\left|\frac{1}{\sqrt{2}} T_1\right|^2} = 2$$ (in agreement with measurements).

Strong interaction depends on I but not on I_3 (inside an isospin multiplet there is a symmetry, and the strong interaction is the same for all the members of the multiplet). This behavior (charge-independence of strong interaction) is reflected in the amplitude.

Exercise 4

a) Due to baryonic-number and electric-charge conservation: $\begin{cases} \pi^+ p \rightarrow p\pi^+ \\ \pi^- p \rightarrow n\pi^0 \\ \pi^- p \rightarrow p\pi^- \end{cases}$

b) Nothing forbids the strong interaction in these reactions (only hadrons, no flavour violation). Moreover, the width of the Δ resonance is 120 MeV (PDG booklet and figure) $\Rightarrow$ $\tau(\Delta) = 0.5 \cdot 10^{-23}$ s, and $\Delta \rightarrow N\pi$ are dominant decay modes of the Δ.
We can conclude that the resonance is produced and decays via strong interaction.

c)

$$\left|\pi^+ p\right\rangle = \left|1,1\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle$$

$$\left|\pi^- p\right\rangle = \left|1,-1\right\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left|\frac{3}{2},\frac{-1}{2}\right\rangle - \frac{\sqrt{2}}{\sqrt{3}}\left|\frac{1}{2},\frac{-1}{2}\right\rangle$$

$$\left|\pi^0 n\right\rangle = \left|1,0\right\rangle\left|\frac{1}{2},\frac{-1}{2}\right\rangle = \frac{\sqrt{2}}{\sqrt{3}}\left|\frac{3}{2},\frac{-1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\frac{1}{2},\frac{-1}{2}\right\rangle$$

d)

$<\pi^+ p\ |H_{int}|\ \pi^+ p> = T(3/2)$

$<\pi^0 n\ |H_{int}|\ \pi^- p> = \sqrt{2}/3\ T(3/2) - \sqrt{2}/3\ T(1/2)$

$<\pi^- p\ |H_{int}|\ \pi^- p> = 1/3\ T(3/2) + 2/3\ T(1/2)$

The cross sections are proportional to $|\ |^2$

e) The resonance $\Delta(1232)$ has 4 charge states ($\Delta^{++},\Delta^+,\Delta^0,\Delta^-$). It has an isospin of 3/2 (see PDG booklet). If the Δ is an intermediate resonance in the process, the amplitudes $T_{1/2}$ are forbidden due to isospin conservation. Hence:

$$\Rightarrow \frac{\sigma\left(p\pi^+ \to \Delta^{++} \to p\pi^+\right)}{\sigma\left(\pi^- p \to \Delta^0 \to n\pi^0, \pi^- p\right)} = \frac{\left|T_{3/2}\right|^2}{\frac{2}{9}\left|T_{3/2}\right|^2 + \frac{1}{9}\left|T_{3/2}\right|^2} = 3$$ (in agreement with the figure)

f) In this region, we see resonances of π^- p but not of p $\pi^+ \Rightarrow$ for these resonances, necessarily I=1/2.

Exercise 5

a) $\Sigma \to \Lambda\,\gamma$ allowed, electromagnetic interaction (indicated by the photon).

$I(\Sigma)=1\ I_3 = \pm 1, 0$

$I(\Lambda)=0\ I_3 = 0$

Only the initial state with $I_3 = 0$ (Σ^0) is allowed due to charge conservation.

b) $\Lambda \to X + \gamma$

First order answer:

There is a photon $\Rightarrow$ electromagnetic interaction $\Rightarrow$ strangeness conservation

$\Rightarrow$ X is a strange baryon (baryonic quantum number conservation)

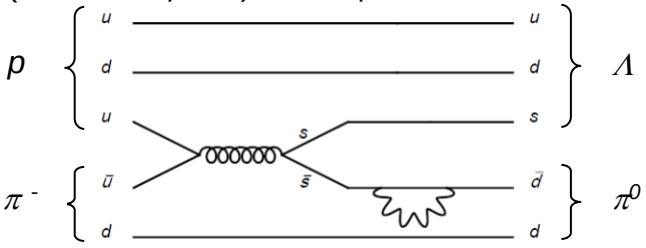
But the Λ is the lightest strange baryon and therefore this is impossible.

Second order answer:

Weak interaction is possible besides electromagnetic interaction. This is the case here, and the process is possible with a small branching ratio (it is useful to draw the diagram to be convinced).

c) $\pi\, p \to \Lambda\, \pi^0$ Strangeness is not conserved $\Rightarrow$ weak interaction. This reaction has a very low probability but is not forbidden. Experimentally, it is very difficult to observe among all the other final states of the π p reaction.

d) Possible diagrams (not the only one) of the question c:



Exercise 6

$\phi \rightarrow K^+K^-$

$\phi \rightarrow \rho\, \pi$

We have to use the formula of the width for the case a $\rightarrow$ 1+2 in the PDG. Under the hypothesis that the matrix elements corresponding to the different states of spin are the same:

$$R=\frac{\Gamma(\phi \rightarrow \rho\pi)}{\Gamma(\phi \rightarrow KK)} = \frac{\frac{1}{3}\cdot 3 \cdot p_\rho^{\,*} \cdot 3}{\frac{1}{3}\cdot 3 \cdot p_K^{\,*} \cdot 1} = 3\frac{p_\rho^{\,*}}{p_K^{\,*}}.$$

The numerical factors 1/3x3 that appear both in the numerator and the denominator are due to the average of the matrix elements for different spin states of the initial state. The factors 3 and 1 in the numerator and denominator, respectively, account for the sum of matrix elements for different spin states in each process. Note that the mean value on the initial state has no effect (3*1/3) because according to our hypothesis the matrix elements of the different spin states are the same (it has been kept here just as a reminder that in general it is necessary to do this average).

We also need to calculate the momenta in the centre of mass frame of the ϕ:

$$E_K^{\,*} = \frac{m_\phi}{2} \qquad p_K^{\,*} = \sqrt{\frac{m_\phi^2}{4} - m_K^2} \sim 0.127 \text{ GeV}$$

$$E_\pi^{\,*} = \frac{m_\phi^2 + m_\pi^2 - m_\rho^2}{2m_\phi} = 0.229 \text{ GeV} \Rightarrow p_\pi^{\,*} = p_\rho^{\,*} = 0.181 \text{ GeV}$$

Given that $m_K \sim \frac{m_\phi}{2}$, there is no much phase space available and this reduces the cross section of the decay into 2 kaons. We obtain: $\boxed{R=4.28}$

Experimentally we measure R = 0.35 $\pm$0.2. The matrix elements of these two decays must be different. This can be understood by the fact that the ϕ meson is a ss state, which yields two different diagrams of the two decays.

Exercise 7

p p $\rightarrow$ d π^+ (1)

d $\pi^+ \rightarrow$ p p (2)

a) The connection between (1) and (2) is a time reversal operation. Then, we conclude that the matrix elements connecting particular spin states must be identical (strong interaction conserves C and P, all interactions conserve CPT, hence T must be conserved).

As each of the particular-spin-state matrix elements are identical, we may write:

$$\sum_{\text{int}(i)}\sum_{\text{int}(f)} \left|\langle f|T|i\rangle\right|^2 = \sum_{m_d,m_\pi,m_p,m_p} \left|\left\langle p_{m_p} p_{m_p} \left| T_{m_d,m_\pi,m_p,m_p} \right| d_{m_d}\pi_{m_\pi}\right\rangle\right|^2 \equiv |T|^2$$

Due to the two indistinguishable particles (protons) in reaction (2), a factor ½ appears in the cross section (see additional notes). Indeed, the final state, where the direction of one of the protons is given by the coordinates (θ, φ), is the same as that with the coordinates (π-θ, φ+π) $\Rightarrow$ we need to integrate only on half of the total solid angle.

b) We must ensure the equality of s in these two reactions.

In p p $\rightarrow$ d π^+

$s = 2M_p^2 + 2E_{p1}E_{p2} - 2\vec{p}_{p1}\cdot\vec{p}_{p2}$. One of the protons is at rest

$\Rightarrow s = 2M_p^2 + 2\left(E_p^c + M_p\right)M_p = 4M_p^2 + 2E_p^c M_p$

$\Rightarrow \sqrt{s} = 2.04$ GeV.

In d $\pi^+ \rightarrow$pp

$s = \left(2M_p\right)^2 + M_\pi^2 + 2\left(E_\pi^c + M_\pi\right)2M_p = 4M_p^2 + M_\pi^2 + 4E_\pi^c M_p + 4M_\pi M_p$

S is equal in the two reactions and therefore

$\Rightarrow 2E_p^c M_p = M_\pi^2 + 4E_\pi^c M_p + 4M_\pi M_p$

$\Rightarrow E_\pi^c = \frac{1}{2}E_p^c - \frac{M_\pi^2}{4M_p} - M_\pi \sim 25$ MeV

It is necessary to have both experiments with the same energy in the centre of mass frame to ensure that the matrix elements are equal. In principle they could depend on the momenta.

c) Conservation of total angular momentum.

J = L + S

$J_{pp} = S_{pp} + L_{pp}$ S_{pp} =0 or 1 L_{pp} = 0, 1, 2… so J_{pp} is integer.

$J_{\pi d} = S_{\pi d} + L_{\pi d} = 1 + S_\pi + L_{\pi d}$ $L_{\pi d}$ = 0, 1, 2…

Conservation of J $\Rightarrow$ $J_{\pi d}$ is integer $\Rightarrow$ S_π is integer

d)

$$s_{pp\rightarrow pd}\left(\sqrt{s}\right) \propto \frac{p_f^*}{p_i^*}\cdot\frac{1}{s}\left|T\right|^2\cdot\left(2s_p+1\right)\left(2s_d+1\right)$$

$$s_{pd\rightarrow pp}\left(\sqrt{s}\right) \propto \frac{p_i^*}{p_f^*}\cdot\frac{1}{s}\left|T\right|^2\cdot\frac{1}{2}\cdot\left(2s_p+1\right)\left(2s_p+1\right)$$

In these two expressions, there are factors corresponding to the sum of matrix elements for all possible spins of the final state particles. These are just multiplicative factors as we neglect the spin dependence of the matrix elements. For this reason, it is useless to average for different spins in the initial state. The factor ½ in the second expression corresponds to the two indistinguishable protons. The centre-of-mass momentum of particles in the final (initial) state of the second reaction is the same as the one of particles in the initial (final) state of the first.

$$\Rightarrow \frac{\sigma_{pp\rightarrow\pi d}}{\sigma_{\pi d\rightarrow pp}} = \frac{3}{2}\left(2s_\pi+1\right)\left(\frac{p_f^*}{p_i^*}\right)^2$$

$$\left(2s_\pi+1\right) = \frac{2}{3}\left(\frac{p_i^*}{p_f^*}\right)^2\frac{\sigma_{pp\rightarrow\pi d}}{\sigma_{\pi d\rightarrow pp}}$$

p_i^* and p_f^* are calculated with the reaction pp$\rightarrow\pi$d.

$$p_f^* = \sqrt{E_\pi^2 - M_\pi^2} = \sqrt{\left(\frac{s + M_\pi^2 - M_d^2}{2\sqrt{s}}\right)^2 - M_\pi^2}$$

We had $\sqrt{s}$= 2.04 GeV $\Rightarrow$ p_f^*=0.080 GeV

$$p_i^* = \sqrt{E_p^2 - M_p^2} = \sqrt{\left(\frac{\sqrt{s}}{2}\right)^2 - M_p^2} = 0.40 \text{ GeV}$$

$\Rightarrow \left(2s_\pi+1\right) = 0.97$

$\Rightarrow s_\pi = -0.02$

With error propagation, one can get:

$$\frac{\sigma(2s_\pi+1)}{2s_\pi+1}=\sqrt{\left(\frac{\sigma(\sigma_{pp})}{\sigma_{pp}}\right)^2+\left(\frac{\sigma(\sigma_{\pi d})}{\sigma_{\pi d}}\right)^2}=0.35$$

$$\Rightarrow \sigma(s_\pi)=(2s_\pi+1)\times 0.35\times\frac{1}{2}=.17$$

$$\Rightarrow s_\pi=-0.02\pm 0.17$$

Exercise 8

a) $$R=\frac{\sigma(e^+e^-\to \text{hadrons})}{\sigma(e^+e^-\to\mu^+\mu^-)}=3\sum_q e_q^2$$

b)

1st level $R=3\left[\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^2\right]=2$ (u, d, s)

2nd level $R=2+3\left[\left(\frac{2}{3}\right)^2\right]=\frac{10}{3}$ (u, d, s, c)

3rd level $R=\frac{10}{3}+3\left[\left(\frac{1}{3}\right)^2\right]=\frac{11}{3}$ (u, d, s, c, b)

ρ ω : containing u ubar and d dbar
φ : s s sbar
J/ψ, ψ(2S) : c cbar
Υ : b bbar

This result is an experimental evidence for the number of colours $N_c=3$.

Exercise 9

(i) $\omega \to \eta\ \pi^0$
(ii) $\rho^0 \to \eta\ \pi^0$
(iii) $\rho^+ \to \eta\ \pi^+$
(iv) $\rho^+ \to\ \pi^0 \pi^+$

According to the PDG-booklet: $I^G\ (J^{PC})$

ω (uu+dd):	$0^-(1^{--})$	η (uu+dd) :	$0^+(0^{-+})$
π^0 (uu-dd):	$1^-(0^{-+})$	π^+ (ud):	$1^-(0^-)$
ρ^0 (uu-dd):	$1^+(1^{--})$	ρ^+ (ud):	$1^+(1^-)$

a) For (i) and (ii) there is no conservation of the charge conjugation C, and therefore the reactions are forbidden via strong and electromagnetic interactions. They occur by weak interaction.
b) In reactions (iii) and (iv) there is conservation of the electric charge, of the isospin I_3 and of the parity (with ℓ=1 between the final-state particles). The baryonic, leptonic and strangeness quantum numbers are equal to zero in the initial and final states of both reactions. Charge conjugation cannot be considered because the initial particles are charged. Hence, if we do not take into account the G-parity, these decays are possible via strong or electromagnetic interactions (strong interaction dominates).
c) The two reactions have the same initial state, and therefore the ratio of branching fractions is simply the ratio of partial widths. To estimate it without any hypothesis, one needs to take into account the matrix elements and the phase space factors.

$$\frac{\Gamma(\rho^+ \to \eta\pi^+)}{\Gamma(\rho^+ \to \pi^0\pi^+)} = \frac{p^*_{\eta\pi}}{p^*_{\pi\pi}} \frac{\left|T_{\eta\pi}\right|^2}{\left|T_{\pi\pi}\right|^2}.$$

Averaging the initial spin states has no effect. The final states are spin-0. This calculation is done under the suggested assumption that the matrix elements are identical, so finally only the phase-space factors are to consider. Note that in $\rho^+ \to \eta\ \pi^+$ the phase space is smaller compared to the one in $\rho^+ \to \pi^0 \pi^+$.

$$\frac{\Gamma(\rho^+ \to \eta\pi^+)}{\Gamma(\rho^+ \to \pi^0\pi^+)} \approx \frac{p^*_{\eta\pi}}{p^*_{\pi\pi}} = \frac{\sqrt{\left(m^2_{\rho^+} + m^2_{\pi^+} - m^2_{\eta}\right)^2 - 4m^2_{\rho^+} m^2_{\pi^+}}}{\sqrt{\left(m^2_{\rho^+} + m^2_{\pi^+} - m^2_{\pi^0}\right)^2 - 4m^2_{\rho^+} m^2_{\pi^+}}} \approx 0.4.$$

A better estimation of this ratio can be obtained by taking isospin into account. This makes sense only because we suppose here that the two reactions are due to strong interaction[1]. The isospin states are:
$|\rho^+\rangle = |1,+1\rangle$; $|\eta\rangle = |0,0\rangle$; $|\pi^+\rangle = |1,+1\rangle$; $|\pi^0\rangle = |1,0\rangle$,
therefore: $|\eta\ \pi^+\rangle = |1,+1\rangle$ and $|\pi^0\ \pi^+\rangle = |1,0\rangle \otimes |1,+1\rangle = (|2,+1\rangle - |1,+1\rangle)/\sqrt{2}$.
Under isospin conservation, the only allowed final state is $|1,+1\rangle$.
Using $\langle\eta\ \pi^+|H_{int}|\rho^+\rangle = T_1$, we get:

$$\frac{\left|T_{\eta\pi}\right|^2}{\left|T_{\pi\pi}\right|^2} = \frac{T_1^2}{\frac{1}{2}T_1^2} = 2,$$

and finally:

$$\frac{\Gamma(\rho^+ \to \eta\pi^+)}{\Gamma(\rho^+ \to \pi^0\pi^+)} = \frac{p^*_{\eta\pi}}{p^*_{\pi\pi}} \frac{\left|T_{\eta\pi}\right|^2}{\left|T_{\pi\pi}\right|^2} \approx 1.$$

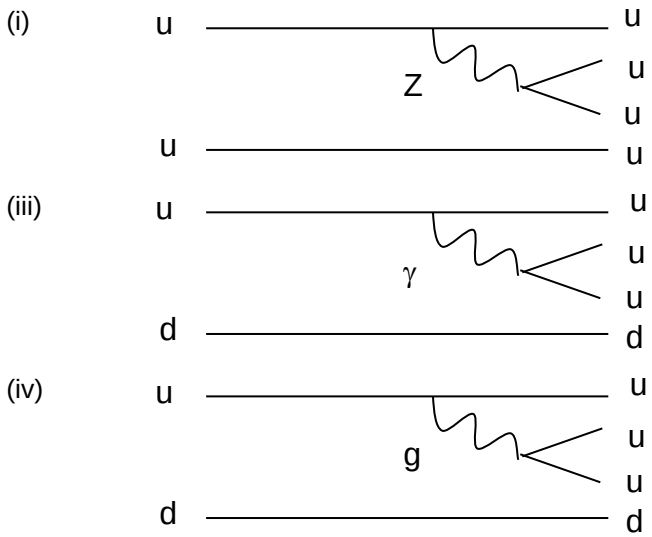
[1] Under this hypothesis, the identical matrix elements are isospin matrix elements.

The widths given in the PDG-booklet for these reactions are nonetheless different by a few orders of magnitude: $\Gamma(\rho^+ \to \eta\ \pi^+) < 6\ 10^{-3}$ and $\Gamma(\rho^+ \to \pi^0\ \pi^+) \approx 1$. This indicates without any doubt that the matrix elements differ for these two processes, which are therefore due to different interactions.

d) $\eta_G = \eta_C(-1)^I$, which gives $\eta_G(\rho^+)= +1$, $\eta_G(\eta)= +1$, $\eta_G(\pi^+)= -1$, $\eta_G(\pi^0)= -1$.
η_G is a multiplicative quantum number, hence $\eta_G(\eta\pi)=-1$, $\eta_G(\pi\pi)=1$.

e) It is clear from this calculation that the G-parity is conserved in reaction (iv) but not in reaction (iii). Together with the other quantum number, this leads to the conclusion that (iii) and (iv) are due to electromagnetic and strong interactions, respectively. This explains the large differences in the branching ratios. The source of the wrong conclusions in question c is, of course, the fact that we did not take into account the G-parity, whose necessity is illustrated by this example.

f)



g) The Strong-interaction lagrangian is invariant under isospin transformation. Consider a process that is forbidden by charge-conjugation violation. Other processes that may be obtained by isospin-space rotations of this forbidden process must be forbidden as well, at least by strong interaction. Note that this is true despite the fact that certain states and particles in the "rotated" processes are not necessarily eigenvalues of the charge conjugation operator (e.g. charged particles). The above description is, in fact, synonym of G-parity conservation. We stress that as isospin is not a symmetry with respect to the electromagnetic interaction, the latter does not conserve G-parity.

Exercice 10

a) $U \to U^* = e^{-i\vec{\theta}\cdot\vec{T}^*} = e^{i\vec{\theta}\cdot(-\vec{T}^*)}$
The $-\vec{T}^*$ are therefore the SU(3) generators for antiquarks.

b) Antiquarks states in (I_3, Y): corresponding eigenvalues of

$$I_3 = -T_3^* = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad Y = -\frac{2}{\sqrt{3}}T_8^* = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow$$

$$|I_3, Y\rangle(\bar{u}) = \left|-\frac{1}{2}, -\frac{1}{3}\right\rangle \quad ; \quad |I_3, Y\rangle(\bar{d}) = \left|\frac{1}{2}, -\frac{1}{3}\right\rangle \quad ; \quad |I_3, Y\rangle(\bar{s}) = \left|0, \frac{2}{3}\right\rangle \quad ,$$

which is what we denoted as 3 bar.

c) T_2 already satisfies $-T^*_2 = T_2$ and therefore must commute with R_I. As T_1 and T_3 are real, they verify $R_I(-T_i)R_I^\dagger = T_i$, which yields immediately the anti-commutation.

From these (anti-) commutation relations or by simple matrix products, it is easy to show that the given expression of R_I works.

d) The unitary matrix R_I performs a change of basis: a matrix M transforms to the new basis as $R_I M R_I^\dagger$ and a vector *v* as $R_I v$. Thus, working with the matrices $R_I(-T_i^*)R_I^\dagger = T_i$ jointly with the vectors $R_I|\bar{q}\rangle$ is equivalent to working with the matrices $-T_i^*$ and the vectors $|\bar{q}\rangle$. Physics results are basis independent.

Mathematically:

$$\langle \bar{q}'|(-T_i^*)|\bar{q}\rangle = \langle \bar{q}'|I(-T_i^*)I|\bar{q}\rangle = \langle \bar{q}'|R_I^\dagger R_I(-T_i^*)R_I^\dagger R_I|\bar{q}\rangle = \left(\langle \bar{q}'|R_I^\dagger\right) T_i \left(R_I|\bar{q}\rangle\right)$$

e) A simple matrix algebra:

$$|\bar{u}\rangle \to R_I|\bar{u}\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad ; \quad |\bar{d}\rangle \to R_I|\bar{d}\rangle = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

($|\bar{u}\rangle$ is the same as $|d\rangle$, $|\bar{d}\rangle$ is the same as $-|u\rangle$).

f) Observables do not depend on a particular basis, and here we illustrate this point for I_3. In the new basis, I_3 is computed as:

$$I_3(\bar{u}) = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} \quad ; \quad I_3(\bar{d}) = \begin{pmatrix} -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \frac{1}{2}$$

g) Quark-quark states can be simply obtained by copying the spin triplet and singlet. In terms of ($|I, I_3\rangle$):

$$|1,1\rangle = |u,u\rangle \quad , \quad |1,0\rangle = \frac{|u,d\rangle + |d,u\rangle}{\sqrt{2}} \quad , \quad |1,-1\rangle = |d,d\rangle \quad ; \quad |0,0\rangle = \frac{|u,d\rangle - |d,u\rangle}{\sqrt{2}}$$

h) From e) and f) above, to obtain the corresponding states made of a quark and an antiquark, we simply need to substitute, for the second particle of the composite system,

$|u\rangle \to -|\bar{d}\rangle \quad , \quad |d\rangle \to |\bar{u}\rangle.$

Thus, the analog isospin system writes:

$$|1,1\rangle = -|u,\bar{d}\rangle \quad , \quad |1,0\rangle = \frac{|u,\bar{u}\rangle - |d,\bar{d}\rangle}{\sqrt{2}} \quad , \quad |1,-1\rangle = |d,\bar{u}\rangle \quad ; \quad |0,0\rangle = \frac{|u,\bar{u}\rangle + |d,\bar{d}\rangle}{\sqrt{2}}.$$