M2 NPAC – Particle physics (2022-2023)

Exercise sheet № 7 - Weak interaction - basics

Brief correction

Exercise 1

a) [Γ] = [GeV]. $G^2 = [GeV^{-2}]^2$ so G^2E^5 = [GeV].

b) We remind the relation between the radioactive mean lifetime and the half-life:

$$N = N_0 e^{-t/\tau}$$

$$\frac{N_0}{2} = N_0 e^{-T_{1/2}/\tau} \Rightarrow \tau = \frac{T_{1/2}}{\ln 2}.$$

We have:

$$\Gamma_n = \frac{G_n^2 E^5}{30\pi^3} = \frac{\hbar}{\tau} \Rightarrow G_n \approx 1.76\ 10^{-5}\ \text{GeV}^{-2}$$

In this calculation, it is practical to use:

$$\hbar c = 197.3 MeV\ fm \quad ; \quad c = 3 \cdot 10^{23}\ fm/s.$$

c) $\frac{\hbar}{\tau} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} \Rightarrow G_\mu \approx 1.16\ 10^{-5}\ \text{GeV}^{-2}$

d) $G_\mu \sim G_n$: universality of weak interaction. The naïve Fermi approximation is not perfect and has to be completed with the Cabibbo angle and the CKM formalism (to come).

Exercise 2

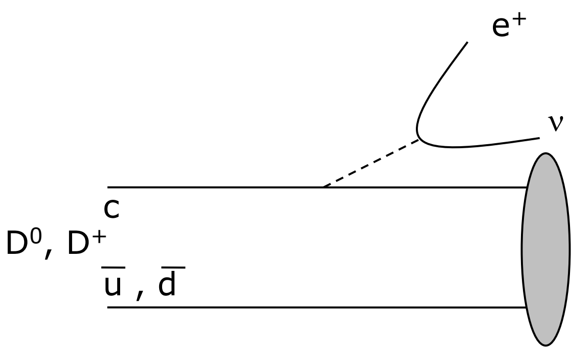
For these two processes, the PDG booklet gives the values of the ratios Γ_i/Γ, which are, by definition, the branching ratios. Note that the initial particles are different and their total widths must be taken into account (for each of them, $\Gamma = \hbar/\tau$). This gives:

$$\frac{\Gamma(D^+ \to e^+ X)}{\Gamma(D^0 \to e^+ X)} = \frac{\tau(D^0) BR(D^+ \to e^+ X)}{\tau(D^+) BR(D^0 \to e^+ X)}$$

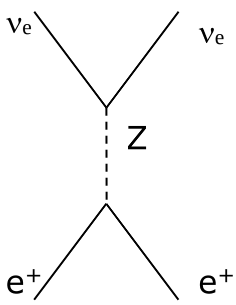
With the values of mean lifetimes and branching ratios given in the PDG booklet we find:

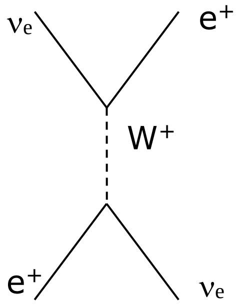
$$\frac{\Gamma(D^+ \to e^+ X)}{\Gamma(D^0 \to e^+ X)} \approx 1.$$

This result is due to the fact that the phase space factor is approximately the same in these two processes and that the matrix element is also the same. The only difference is the flavor of the spectator quark.

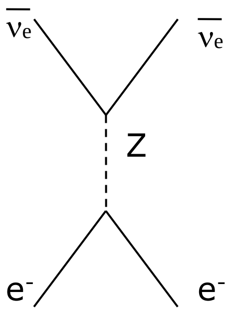


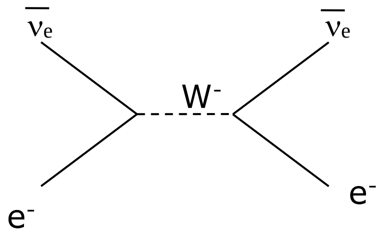
Exercise 3

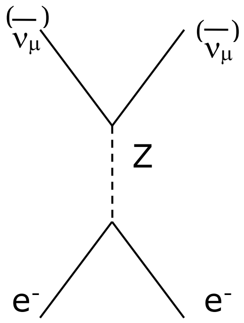




For each case there are 2 diagrams (with W and Z)



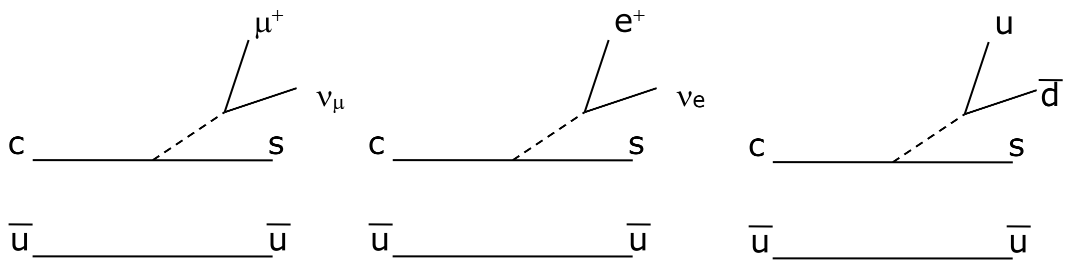




Only one diagram with Z; no similar diagram with a W

Exercise 4

a)



There are no contributions with $W \to \tau\, \nu$ and $W \to c\, \bar{s}$ because of the phase space. The transition $c \to d$ has a lower probability ($V_{CKM} \sim \lambda \Rightarrow$ observed rate suppressed by λ^2).

b) $BR(D^0 \to \mu^+ \nu_\mu X) = \dfrac{\Gamma(\mu\nu X)}{\Gamma_{tot}}$; $\Gamma_{tot} = \Gamma_e + \Gamma_\mu + 3\Gamma_{ud} \approx 5\ \Gamma_{\mu\nu}$

The factor 3 for Γ_{ud} is due to the color degree of freedom. Note that, unlike in the case of color suppression with a single meson created from the W boson, this factor 3 is applied to the widths and not the amplitudes. This is due to the fact that the hadronic state emerging from the W boson is not a single meson, and therefore, at first order of α_s, there is no gluon exchange between these two quarks (this, of course, does not work in case of a single meson). Thus, in the present case, the diagrams with different colors are supposed to have different quantum numbers and do not interfere.

The hypothesis of a roughly similar phase space is justified because the final state particles have low masses compared to the D^0. By taking $|V_{ud}|$~1 we get: $BR(D^0 \to \mu^+ \nu_\mu X) \sim 1/5 = 20\%$.

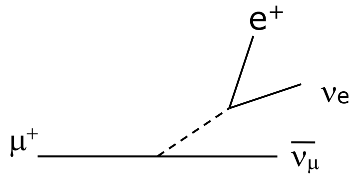
The PDG booklet gives BR = 6.5%. Our estimation has a discrepancy of a factor ~3. Probably this factor is due to other major contributions to the Γ_{tot}. In general, another source of discrepancy comes from QCD corrections (the quarks are not free, but bound inside hadrons).

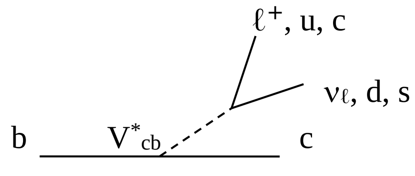
c) In the case of B^0, we obtain the result:

$$BR\left(B^0 \to \mu^+\nu_\mu X\right) = \frac{\Gamma_{\mu\nu}}{\Gamma_{e\nu} + \Gamma_{\mu\nu} + \Gamma_{\tau\nu} + 3\Gamma_{ud} + 3\Gamma_{cs}} = \frac{1}{9} \approx 11\%$$

PDG booklet: 10.33%. The estimation is much better.

d)





Only 1 diagram contributes to the μ decay, whereas there are 9 dominant ones for the b-quark. The nature of the interaction is the same in the two cases. With the Fermi approximation and by taking $|V_{ud}| \sim |V_{cs}| \sim 1$ we find:

$$\frac{1}{\tau_\mu} = Km_\mu^5 \quad ; \quad \frac{1}{\tau_b} = 9|V_{cb}|^2 Km_b^5$$

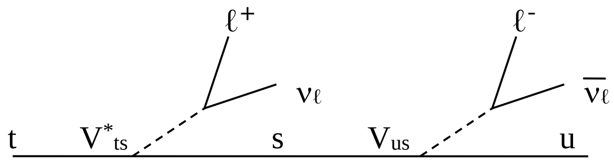
$$\tau_B \sim \tau_b = \frac{1}{9|V_{cb}|^2}\left(\frac{m_\mu}{m_b}\right)^5 \tau_\mu \sim 1.1\ 10^{-12}s \ \ (\text{PDG}: 1.5\ 10^{-12}s\ )$$

The difference, probably, comes mainly from QCD corrections.

e) In the B^0 case, there is a V_{cb} factor (~λ^2), whereas for the D^0 it is V_{cs} (~1). Without this effect, the B^0 lifetime would be shorter than the D^0 one ($m_B >> m_D$, 9 possible dominant modes instead of 5).

Exercise 5

We look for semileptonic decay processes of the type:



Below are 4 examples (the last one corresponds to the diagram above). We do not precise the decays of the W boson. The factor in the decay amplitude that comes from the CKM matrix elements is denoted A_{CKM}.

No.	Process	A_{CKM}
1	t→ W s ; s→ W u	$V_{ts}V_{us} \sim \lambda^2 \cdot \lambda = \lambda^3$
2	t→ W b ; b→ W u	$V_{tb}V_{ub} \sim 1 \cdot \lambda^3 = \lambda^3$
3	t→ W d ; d→ W u	$V_{td}V_{ud} \sim \lambda^3 \cdot 1 = \lambda^3$
4	t→ W s ; s→ W c ; c→ W d ;d→ W u	$V_{ts}V_{cs}\ V_{cd}V_{ud} \sim \lambda^2 \cdot 1 \cdot \lambda \cdot 1 = \lambda^3$

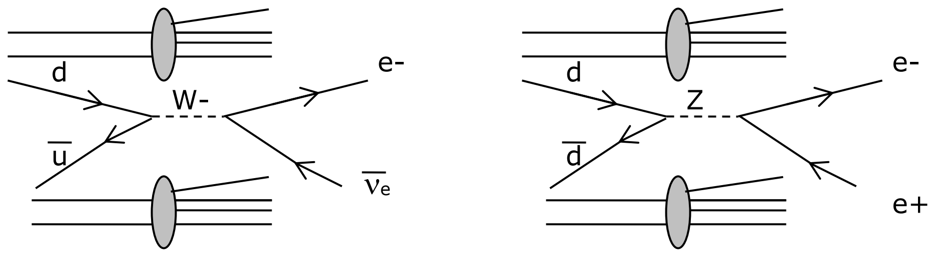
The fourth process, even though it is equivalent to the others with respect to the CKM-matrix-elements contribution, is highly suppressed. It has twice as much weak vertices as the first three processes (i.e. it has an additional factor of G^2 in the decay amplitude). The dominant processes are therefore 1, 2 and 3, which give, as far as we can tell from these simple arguments, equivalent contributions to the decay width.

Comment: we are looking for a rough estimation here, and therefore we do not take into account the parameters A, ρ and η of the Wolfenstein parameterization.

Exercise 6

a) If the W was a 0-spin particle, the problem would be identical to the pion decay from the angular-momentum point of view. The decay $\pi\rightarrow\mu\nu$ is favored with respect to the decay to $\pi\rightarrow e\nu$, because the electron, unlike the muon, is ultrarelativistic. For a 0-spin W, in principle we would have the same phenomenon, but it would become more than secondary due to the very-high mass of the W.

b)



c) Due to the weak-interaction coupling $\gamma^{\mu}\frac{(1-\gamma^5)}{2}$, and in the limit $m_q=0$, the quarks (antiquarks) are left-handed (right-handed). The anti-neutrino is always right-handed, and therefore the electron must be left-handed. We deduce that only one helicity configuration exists (single non-zero helicity amplitude).

d) The spin of the W is 1. The angular distribution is therefore given by:

$$d^1_{-1,-1}=d^1_{1,1}\propto 1+\cos\theta^* \Rightarrow \frac{d\sigma}{d\Omega^*}\propto\left|d^1_{1,1}\right|^2\propto\left(1+\cos\theta^*\right)^2$$

e) If the W was a spin-2 we would have:

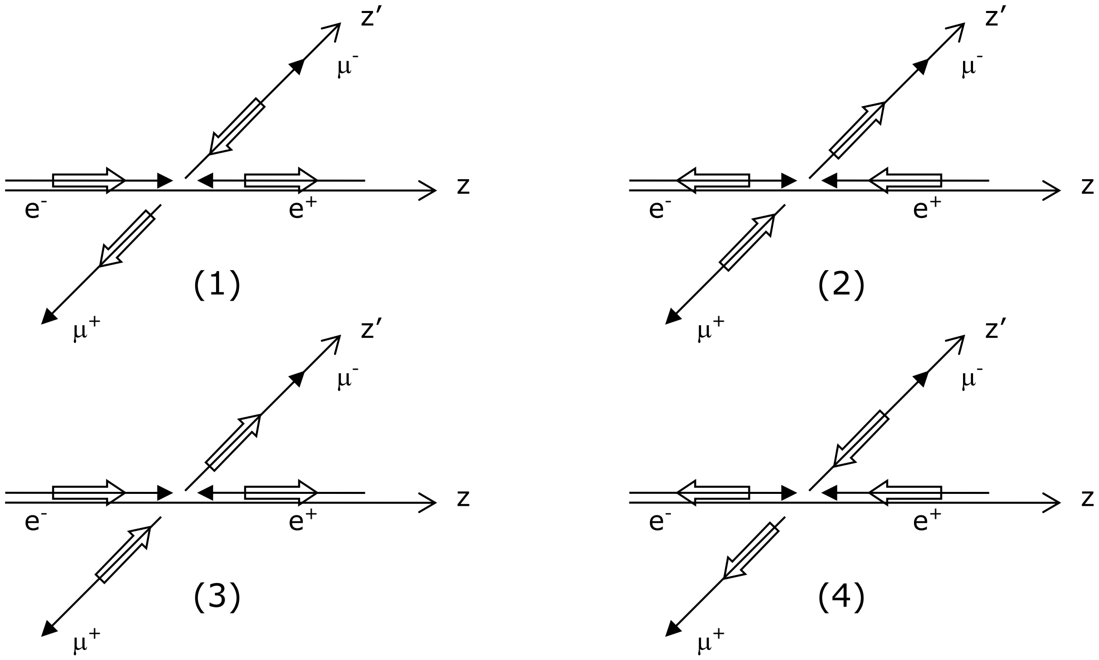
$$\frac{d\sigma}{d\Omega^*}\propto\left|d^2_{-1,-1}\right|^2=\left|d^2_{1,1}\right|^2\propto\left(\left(1+\cos\theta^*\right)\left(2\cos\theta^*-1\right)\right)^2$$

One can easily see which the right hypothesis is simply by looking at the angular distribution of the decay products.

Exercise 7

a) Because of the nature of electromagnetic interaction (vector coupling), the outgoing μ^+ and μ^-, as well as the incoming e^+ and e^- are of opposite chiralities. In high-energy regime (E>>m), helicity=chirality. From this, it is easy to see that the initial state of the reaction can be either S_z = +1 or -1, and the final state either $S_{z'}$ = +1 ou -1.

b)



The z- and z'-axes are oriented in the directions of the incoming e^- and the outgoing μ^-, respectively.

c) The schemes above are drown for a given θ. The θ–dependence of each one of them is given by the corresponding $d^J_{m,m'}$

(1) $T_1 = A_{+1,-1} d^{J=1}_{+1,-1} \propto \frac{1}{2}(1-\cos\theta)$

(2) $T_2 = A_{-1,+1} d^{J=1}_{-1,+1} = A_{-1,+1} d^{J=1}_{+1,-1} \propto \frac{1}{2}(1-\cos\theta)$

(3) $T_3 = A_{+1,+1} d^{J=1}_{+1,+1} \propto \frac{1}{2}(1+\cos\theta)$

(4) $T_4 = A_{-1,-1} d^{J=1}_{-1,-1} = A_{-1,-1} d^{J=1}_{+1,+1} \propto \frac{1}{2}(1+\cos\theta)$

Where $A_{m,m'}$ are the (angle-independent) helicity amplitudes. Notice that for θ=0, S_z is not conserved in configurations (1) and (2). Indeed, the matrices $d^J_{m,m'}$ ensure a 0-amplitude in this case.

d) To compare to the figure, we need to compute the total differential cross-section, i.e. the sum of squares of the 4 amplitudes above. Knowing that the four helicity amplitudes are the same (the total cross sections corresponding to the 4 helicity configurations are the same):

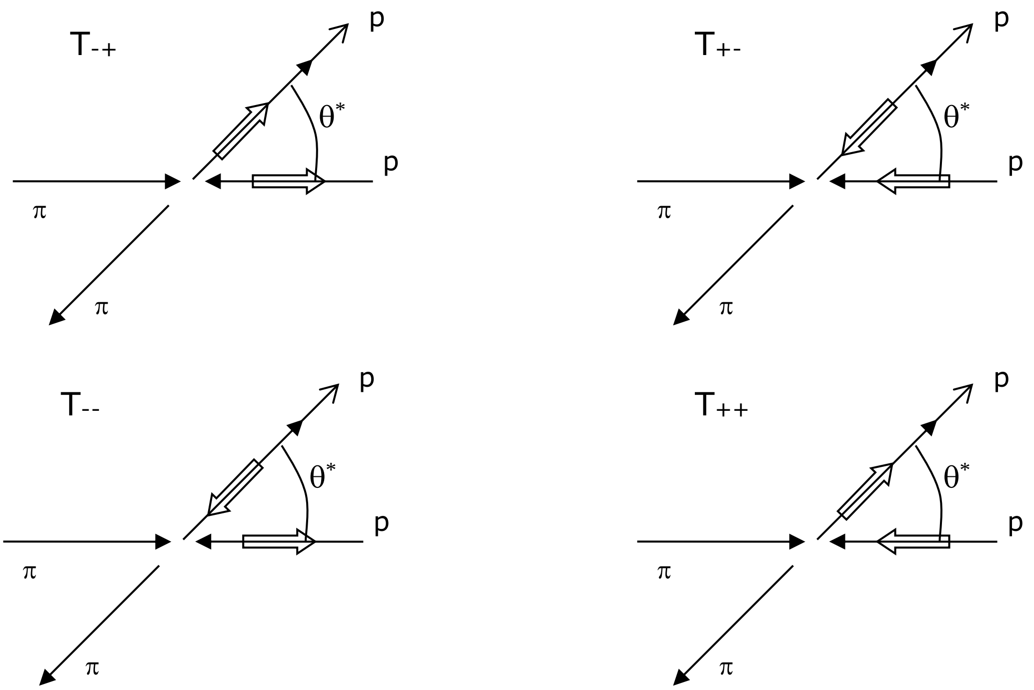
$$\frac{d\sigma}{d\Omega^*} \propto 2\left|d^{J=1}_{1,+1}\right|^2 + 2\left|d^{J=1}_{1,-1}\right|^2 = 2\left(\frac{1}{4}(1+\cos\theta)^2 + \frac{1}{4}(1-\cos\theta)^2\right) = 1+\cos^2\theta \cdot$$

This clearly describes the behavior shown in Figure 1.

Exercise 8

a) $\pi^+ p \to \pi^+ p$

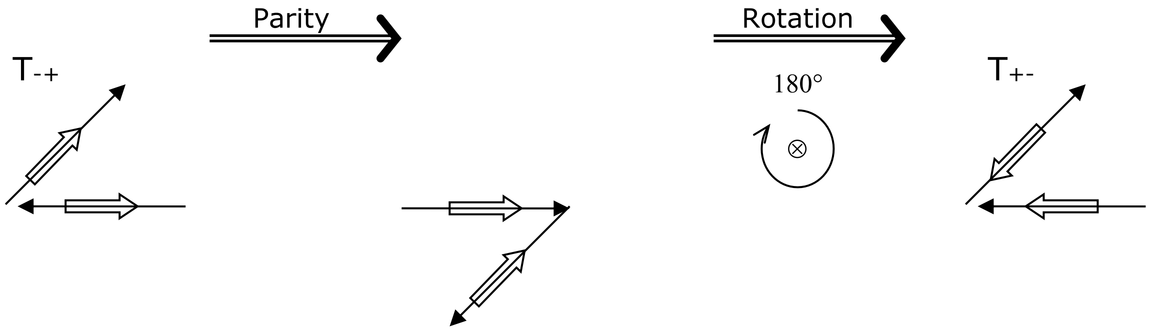
We note by (+) the states λ=+1/2 and by (-) the states λ=-1/2. The π is spin-0, and therefore λ_π=0. In the center-of-mass frame:



The configurations corresponding to T_{+-} and T_{++} can be obtained from the ones corresponding to T_{-+} et T_{--}, respectively, by two consecutive operations:

1. Parity (P), which flips the directions of momenta with no effect on the angular momenta (resulting in inversed helicities);
2. Rotation of 180° about an axis perpendicular to the plane of the sheet. This transformation does not affect the helicities that are invariant under rotation ([h,J]=0).

Below we present an illustration of the action of these two operations on the protons of T_{-+} (the transformations are trivial for the pions).



As the reactions occur by strong interaction, which is invariant both under rotation and parity, we obtain only 2 independent helicity amplitudes.

b)

$T_{--} = \langle J,- | T | J,- \rangle = f^J_{--} \cdot d^J_{1/2,-1/2}\left(\theta^*\right)$,

$T_{-+} = \langle J,+ | T | J,- \rangle = f^J_{-+} \cdot d^J_{1/2,1/2}\left(\theta^*\right)$,

where f^J_{--}, f^J_{-+} do no depend on neither θ* nor φ*

$$\frac{d\sigma}{d\Omega^*} = 2\frac{2J+1}{4\pi}\left|d^J_{1/2,-1/2}\left(\theta^*\right)\right|^2 \left|f^J_{--}\right|^2 + 2\frac{2J+1}{4\pi}\left|d^J_{1/2,1/2}\left(\theta^*\right)\right|^2 \left|f^J_{-+}\right|^2$$

Using the hypothesis $f^J_{++} = f^J_{+-} \equiv f^J_{1/2}$ we finally obtain:

$$\frac{d\sigma}{d\Omega^*} \propto \left\{\left|d^J_{1/2,1/2}\left(\theta^*\right)\right|^2 + \left|d^J_{1/2,-1/2}\left(\theta^*\right)\right|^2\right\}\left|f^J_{1/2}\right|^2 \propto \left|d^J_{1/2,1/2}\left(\theta^*\right)\right|^2 + \left|d^J_{1/2,-1/2}\left(\theta^*\right)\right|^2$$

c) If J = ½ :

$d^{1/2}_{1/2,1/2}\left(\theta^*\right) = \cos\frac{\theta^*}{2}$ et $d^{1/2}_{1/2,-1/2}\left(\theta^*\right) = -\sin\frac{\theta^*}{2}$ $\Rightarrow \frac{d\sigma}{d\Omega^*} \propto$ constent

If J =3/2 :

$d^{3/2}_{1/2,1/2}\left(\theta^*\right) = \frac{3\cos\theta^* - 1}{2}\cos\frac{\theta^*}{2}$ and $d^{3/2}_{1/2,-1/2}\left(\theta^*\right) = -\frac{3\cos\theta^* + 1}{2}\sin\frac{\theta^*}{2}$

$$\Rightarrow \frac{d\sigma}{d\Omega^*} \propto \left(\left(3\cos\theta^* - 1\right)^2\left(1 + \cos\theta^*\right) + \left(3\cos\theta^* + 1\right)^2\left(1 - \cos\theta^*\right)\right) = 1 + 3\cos^2\theta^*$$

Comparing the expressions to Figure 2 we conclude that the spin of the Δ is 3/2.

d)

$$\sigma\left(\sqrt{s}\right) = \frac{4\pi}{p^2}\cdot\frac{2J+1}{2}\cdot\frac{\Gamma^2/4}{\left(\sqrt{s} - m_\Delta\right)^2 + \Gamma^2/4}$$

$$\Rightarrow \sigma^{\text{Max}} = \sigma\left(m_\Delta\right) = \frac{4\pi}{p^2}\cdot\frac{2J+1}{2}$$

$\sqrt{s} = m_\Delta = 1.232$ GeV

$$E^*_\pi = \frac{m^2_\Delta + m^2_\pi - m^2_p}{2m_\Delta} = 0.267 \text{ GeV} \Rightarrow p^* = 0.227 \text{ GeV}$$

$\Rightarrow \sigma^{\text{Max}} = \frac{4\pi}{.227^2}\cdot\frac{2J+1}{2}\cdot\left(\hbar c\right)^2$ obtained by dimentional analysis, restauring $\left(\hbar c\right)^2 = .389$ mb GeV²

$\sigma^{\text{Max}} \simeq$ 95 mb for J=1/2

$\sigma^{\text{Max}} \simeq 190$ mb for J=3/2

$\sigma^{\text{Max}} \simeq 285$ mb for J=5/2

$\Rightarrow$ We confirm that the graph corresponding to Δ^{++} describes a resonance of spin 3/2.