Exercise sheet № 8 - Around the Z boson

WARNING: This exercise contains matrix element computation with traces. In an actual exam, the result of the traces would be given.

1 QED $e^-e^+ \to f\bar{f}$

1. Draw the diagram of the process $e^-e^+ \to f\bar{f}$ considering only the QED interaction. Use for the quadri momenta the notations: k_1, k_2 4-vector of e^- and e^+ respectively and p_1, p_2 4-vectors of the f and $\bar{f}$ respectively. We are investigating data from a symmetrical $e^- - e^+$ collider and define the z axis as the beam axis oriented along the e^- beam therefore $k_1 \equiv (E, 0, 0, E)$, $k_2 \equiv (E, 0, 0, -E)$. We note $\sqrt{s} = 2E$. Note q the quadri-momentum circulating in the γ^*.

2. Write the matrix element as a function of Q_e and Q_f the electric charges of e^- and f.
We remind that the cross-section of a two-body decay can be written as:
$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2}{64\,\pi^2\,s}\,\frac{2\,p_f^*}{\sqrt{s}}, \tag{1}$$
with p_f^* the absolute momentum of f in the center of mass. Assuming the f-fermion and the electron masses are negligible, find the total QED cross section using the trace technology. Note that the total cross section is obtained averaging over the initial spins and summing over the final spins. Find that:
$$\frac{d\sigma(e^-e^+ \to f\bar{f})}{d\cos\theta} = Q_e^2\,Q_f^2\,\frac{\alpha^2 2\pi}{4\,s}(1+\cos^2\theta) \tag{2}$$

3. We now look at the initial and final polarisation separately. Noting X, X' the polarisation of e^- and e^+ respectively, and Y and Y' the polarisation of the f and $\bar{f}$ respectively, X, X', Y, Y' can be either L or R. There are a priori 16 different cross sections , $\sigma(e^-_X e^+_{X'} \to f_Y \bar{f}_{Y'})$. How many are not nul, conclude that we do not need the X' and Y' indices. Using the Wigner rotation matrices, find out the angular dependence for the different XY processes:
   - $d\sigma_{LL} = d\sigma(e^-_L e^+_R \to f_L \bar{f}_R)$,
   - $d\sigma_{LR} = d\sigma(e^-_L e^+_R \to f_R \bar{f}_L)$,
   - $d\sigma_{RL} = d\sigma(e^-_R e^+_L \to f_L \bar{f}_R)$,
   - $d\sigma_{RR} = d\sigma(e^-_R e^+_L \to f_R \bar{f}_L)$.

   From symmetry arguments find out that there are only two different polarized cross sections. Compute the total cross section as a function of the $d\sigma_{XY}$s and comparing to 2, find out the values of $d\sigma_{XY}$ normalisation factors.

4. Write the different matrix elements $\mathcal{M}_{XY}$ using the left and right chirality projectors P_X, P_Y (so either $P_L = \frac{1-\gamma_5}{2}$ or $P_R = \frac{1+\gamma_5}{2}$). You should find the same expression as in the unpolarised case with in addition one P_X and one P_Y in the expression. Define the charged independent matrix elements $\mathcal{M}^{EM}_{XY}$ in such a way that $\mathcal{M}_{XY} = Q_e\,Q_f\;\mathcal{M}^{EM}_{XY}$.

2 Z boson partial widths $\Gamma(Z \to \; f\bar{f})$

In this section, we use $s_w = \sin\theta_W$ and $c_w = \cos\theta_W$ where θ_W is the Weinberg angle. Reminders for Feynman diagrams and trace rules:

- incoming vector boson line, the polarisation ϵ_μ
- Polarisation summation rule for massless vector bosons:

$$\sum_{polarisation} \epsilon_\mu \epsilon_\nu^* \to -\eta_{\mu\nu}$$

- Polarisation summation rule for massive vector bosons:

$$\sum_{polarisation} \epsilon_\mu \epsilon_\nu^* \to -\eta_{\mu\nu} + \frac{k^\mu \, k^\nu}{m_V^2} \; ,$$

  with k^μ the 4-momentum of the vector boson

- $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = (\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\rho}\eta^{\nu\sigma})$
- $Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_5) = -4i\epsilon^{\mu\nu\rho\sigma}$ with $\epsilon^{\mu\nu\rho\sigma}$ the fully anti-symmetric tensor.

1. Draw the Feynman diagram of the $Z \to f\bar{f}$ decay, we note p_1 and p_2 the 4-momenta of f and $\bar{f}$ respectively.

2. Quoting $\Gamma_L = \Gamma(Z \to f_L\bar{f}_R)$ and $\Gamma_R = \Gamma(Z \to f_R\bar{f}_L)$, write the matrix element for these two decays. We will also note:

$$\mathcal{A}_X = \bar{u}(p_1)\gamma^\mu \, P_X \, v(p_2)\epsilon_\mu$$

   with P_X the projector on chirality X, *i.e.* $P_L = \frac{1-\gamma_5}{2}$ and $P_R = \frac{1+\gamma_5}{2}$.

3. Compute $|\mathcal{A}_L|^2$ and $|\mathcal{A}_R|^2$, you should find:

$$|\mathcal{A}_L|^2 = |\mathcal{A}_R|^2 = 2\, m_Z^2 \tag{3}$$

4. For a two-body decay of a particle R, the decay rate is given by:

$$\frac{d\Gamma}{d\Omega} = \frac{p_f^*}{32\,\pi^2\, m_R^2}|\mathcal{M}|^2 \tag{4}$$

   Using this compute Γ_L and Γ_R

5. Compute the total partial width $\Gamma(Z \to f\bar{f})$, remember that we have 3 colors for quarks!

6. We define the left-right asymmetry as

$$A_{LR}^f = \frac{\Gamma_L - \Gamma_R}{\Gamma_L + \Gamma_R} \tag{5}$$

   Find the expression of A_{LR}^f as a function of s_w^2.

3 QED + Weak neutral currents $e^-e^+ \to f\bar{f}$

In this section, we will note $s_w = \sin\theta_W$ and $c_w = \cos\theta_W$.

1. Draw the diagram(s) of the process $e^-e^+ \to f\bar{f}$ considering both QED and the Weak interaction. We use the same notation as for QED.

2. Justify we have only 4 different cross sections:

   - $d\sigma_{LL} = d\sigma(e^-_L e^+_R \to f_L \bar{f}_R)$,
   - $d\sigma_{LR} = d\sigma(e^-_L e^+_R \to f_R \bar{f}_L)$,
   - $d\sigma_{RL} = d\sigma(e^-_R e^+_L \to f_L \bar{f}_R)$,
   - $d\sigma_{RR} = d\sigma(e^-_R e^+_L \to f_R \bar{f}_L)$.

   The Z boson propagator in the unitary gauge can be written:

$$D_Z^{\mu\nu}(q) = \frac{-i}{q^2 - m_Z^2}\left(\eta^{\mu\nu} - \frac{q^\mu q^\nu}{m_Z^2}\right) \tag{6}$$

   Write the matrix element $\mathcal{M}_{XY}$ for the different polarized cross sections using the chirality projectors P_X and P_Y.

3. In the $\mathcal{M}_{XY}$ expression, extract the terms proportional to $q_\mu q_\nu$ and show they are zero for massless fermions.

4. Neglecting, $q^\mu q^\nu / m_Z^2$ terms, show that this can be put under the form (use the fact that $e = g s_w$):

$$\mathcal{M}_{XY} = \mathcal{M}_{XY}^{EM} \times |F_{XY}(f)|^2 \tag{7}$$

   with

$$\begin{aligned}
F_{LR}(f) &= -Q_f + \frac{(\frac{1}{2} - s_w^2)\,Q_f}{c_w^2}\frac{s}{s - m_Z^2} \\
F_{RL}(f) &= -Q_f + \frac{T_3^f - Q_f s_w^2}{c_w^2}\frac{s}{s - m_Z^2} \\
F_{RR}(f) &= -Q_f - Q_f \frac{s_w^2}{c_w^2}\frac{s}{s - m_Z^2} \\
F_{LL}(f) &= -Q_f + \frac{(-\frac{1}{2} + s_w^2)(T_3^f - Q_f s_w^2)}{s_w^2 c_w^2}\frac{s}{s - m_Z^2}
\end{aligned} \tag{8}$$

   Since the Z is a resonance, the propagator diverges for $s = m_Z^2$, it is in fact regular and the propagator should be replace by a Breit-Wigner resonance: $\frac{s}{s - m_Z^2} \longrightarrow \frac{s}{s - m_Z^2 + i m_Z \Gamma_Z}$.

5. Using the QED results, find ou the 4 different cross sections:

$$\begin{aligned}
\frac{d\sigma_{LR}}{d\cos\theta} &= \frac{\pi\alpha^2}{2\,s}(1 - \cos\theta)^2\ |F_{LR}(f)|^2 \\
\frac{d\sigma_{RL}}{d\cos\theta} &= \frac{\pi\alpha^2}{2\,s}(1 - \cos\theta)^2\ |F_{RL}(f)|^2 \\
\frac{d\sigma_{RR}}{d\cos\theta} &= \frac{\pi\alpha^2}{2\,s}(1 + \cos\theta)^2\ |F_{RR}(f)|^2 \\
\frac{d\sigma_{LL}}{d\cos\theta} &= \frac{\pi\alpha^2}{2\,s}(1 + \cos\theta)^2\ |F_{LL}(f)|^2
\end{aligned} \tag{9}$$

6. Compute the total cross section as a function of $|F_{XY}|^2$

7. Put F_{XY} under the form $F_{XY} = -Q_f + A_{XY}\frac{s}{(s-m_Z^2)+im_Z\Gamma_Z}$ and compute $|F_{XY}|^2$. Deduce that the interference term is zero for $\sqrt{s} = m_Z$. What is the order of magntitude of the σ_{EM}/σ_Z at $\sqrt{s} = m_Z$. Compute σ_Z/σ_{EM} at the Z pole for $f \equiv \mu$. Find out the EM contribution is negligible at the Z pole.

8. We define the left-right asymmetry as

$$A_{LR}^f = \frac{\Gamma(Z \to f_R\bar{f}_L) - \Gamma(Z \to f_L\bar{f}_R)}{\Gamma(Z \to f_R\bar{f}_L) + \Gamma(Z \to f_L\bar{f}_R)} \tag{10}$$

Demonstrate that

$$A_{LR}^f = \frac{(T_3^f - Q_s\, s_w^2)^2 - Q_f^2\ s_w^4}{(T_3^f - Q_s\, s_w^2)^2 + Q_f^2\ s_w^4} \tag{11}$$

9. We define the forward (backward) cross section σ_F (σ_B) as:

$$\begin{aligned} \sigma_F &= \int_0^1 d\cos\theta \frac{d\sigma(e^-e^+ \to f\bar{f})}{d\cos\theta} \\ \sigma_B &= \int_{-1}^0 d\cos\theta \frac{d\sigma(e^-e^+ \to f\bar{f})}{d\cos\theta} \end{aligned} \tag{12}$$

Compute σ_F and σ_B as a function of $|F_{XY}|^2$.

10. Eventually compute the forward backward asymmetry as a function of $|F_{XY}|^2$ and find out that

$$A_{FB}^f \equiv \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4}\frac{|F_{LL}|^2 + |F_{RR}|^2 - |F_{RL}|^2 - |F_{LR}|^2}{|F_{LL}|^2 + |F_{RR}|^2 + |F_{RL}|^2 + |F_{LR}|^2} \tag{13}$$

11. At $\sqrt{s} = m_Z$ (we add the upper-script 0 for the asymmetry at the Z pole), find out that:

$$A_{FB}^{0f} = \frac{3}{4} A_{LR}^e A_{LR}^f \tag{14}$$

12. Argue that this gives an access to the value of $\sin^2\theta_W$

13. Compute the value of A_{LR}^f for leptons ℓ, up-type quarks and down-type quarks, using the value $\sin^2\theta_W \approx 0.23$. Deduce the values of A_{FB}^f for ℓ, up-type quarks and down-type quarks (like bottom quarks). The values measured at LEP were:

$$\begin{aligned} A_{FB}^\ell &= 0.0171 \pm 0.0010 \\ A_{FB}^b &= 0.1000 \pm 0.0017 \\ A_{FB}^c &= 0.0699 \pm 0.0036 \end{aligned} \tag{15}$$

conclude...