

$$e^+ e^- \longrightarrow \mu^+ \mu^-$$

• QED : A_μ, ψ_e, ψ_μ

$$L = \underbrace{\bar{\psi}_e (i \not{D} - m_e) \psi_e}_{e^+, e^-} + \underbrace{\bar{\psi}_\mu (i \not{D} - m_\mu) \psi_\mu}_{\mu^+, \mu^-} - \underbrace{\frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\text{photons}}$$

$$\not{D} = \gamma^\mu D_\mu = \gamma^\mu (\partial_\mu + ie A_\mu) \quad (q = -e)$$

$m_e \equiv$ mass of electron

$m_\mu \equiv$ mass of muon

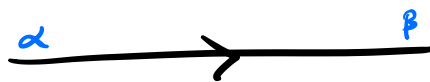
$e =$ charge of electron/muon

Interaction:

$$e \bar{\psi}_e \gamma^\mu \psi_e A_\mu : \text{diagram with } -ie\gamma^\mu_{\alpha\beta}$$

$$e \bar{\psi}_\mu \gamma^\mu \psi_\mu A_\mu : \text{diagram with } -ie\gamma^\mu$$

Propagators



$$i \frac{(\not{p} + m_e)_{\alpha\beta}}{p^2 - m_e^2 + i\epsilon}$$

e



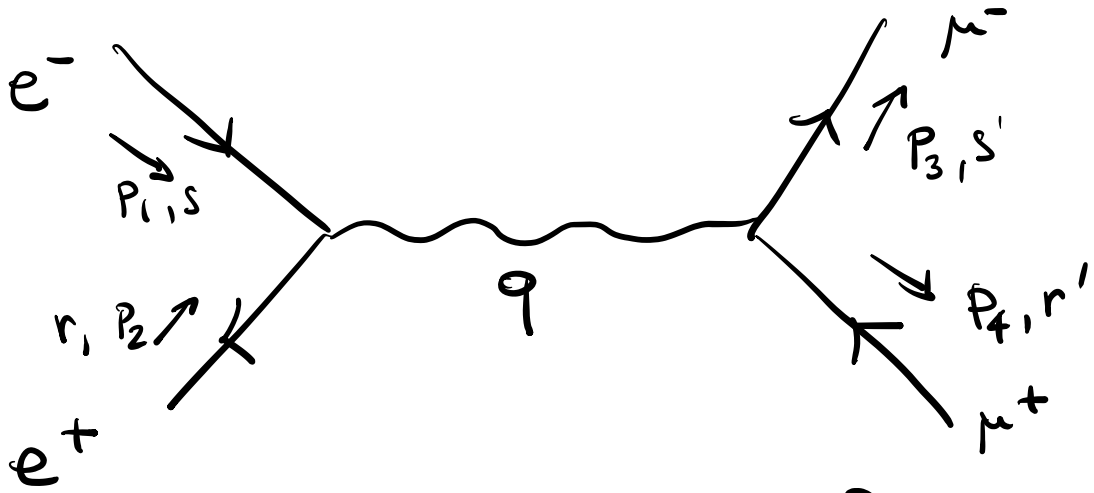
$$i \frac{(\not{p} + m_\mu)_{\alpha\beta}}{p^2 - m_\mu^2 + i\epsilon}$$

μ

A_μ



$$-i \frac{\gamma_{\mu\nu}}{p^2}$$

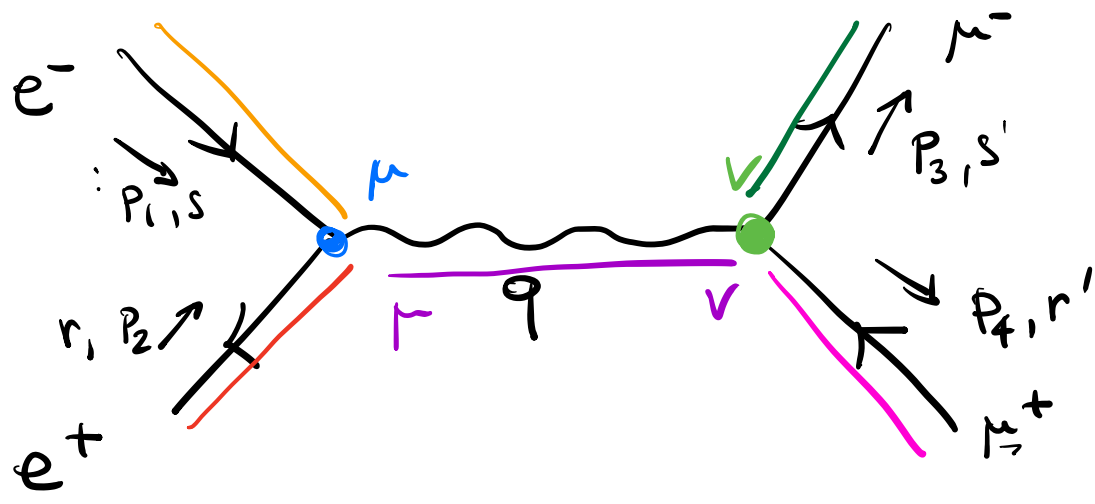


$$Q = P_1 + P_2 = P_3 + P_4$$

(momentum conservation at each vertex)

\Rightarrow overall momentum conservation
 $(P_1 + P_2 - P_3 - P_4 = 0)$

s-channel



- ingoing particle: $\xrightarrow{s, \vec{P}} \text{⊙} \leftarrow U^s(P)$
- ingoing antiparticle: $\xleftarrow{s, \vec{P}} \text{⊙} \leftarrow \bar{V}^s(\vec{P})$
- outgoing particle: $\text{⊙} \xrightarrow{s, \vec{P}} \leftarrow \bar{U}^s(\vec{P})$
- outgoing antiparticle: $\text{⊙} \xleftarrow{s, \vec{P}} \rightarrow V^s(\vec{P})$

$$A(p_1, p_2, p_3, p_4; s, s', r, r') =$$

$$\bar{v}_e^r(p_2) (-ie\gamma^\mu) u_e^s(p_1) \frac{(-i\gamma_{\mu\nu})}{q^2 + i\epsilon} \bar{u}_\mu^{s'}(p_3) (-ie\gamma^\nu) v_\mu^{r'}(p_4)$$

$|A|^2$ enters the cross section

$$= A^\dagger A \quad (\bar{\psi} \gamma^\mu \psi)^\dagger = \bar{\psi} \gamma^\mu \psi$$

$$A^\dagger = i \frac{e^2}{q^2} [\bar{v}^{r'}(p_4) \gamma_\mu u^{s'}(p_3)] [\bar{u}^s(p_1) \gamma^\mu v^r(p_2)]$$

$$A = -i \frac{e^2}{q^2} [\bar{u}^{s'}(p_3) \gamma_\nu v^{r'}(p_4)] [\bar{v}^r(p_2) \gamma^\nu u^s(p_1)]$$

$$|A|^2(p's, spins) = A^\dagger A = \frac{e^4}{(s)^2}$$

$$[\bar{v}^{r'}(p_4) \gamma_\mu u^{s'}(p_3)] [\bar{u}^s(p_1) \gamma^\mu v^r(p_2)] [\bar{u}^{s'}(p_3) \gamma_\nu v^{r'}(p_4)] [\bar{v}^r(p_2) \gamma^\nu u^s(p_1)]$$

$$(p_1 + p_2)^2 = s$$

Mandelstam variable s
 \equiv center of mass energy

- Suppose you cannot measure the spin of the final states

$$\Rightarrow \text{Total } |A|^2 = |A|_{\substack{s'=\uparrow \\ r'=\uparrow}}^2 + |A|_{\substack{s'=\uparrow \\ r'=\downarrow}}^2 +$$

$$|A|_{\substack{s'=\downarrow \\ r'=\downarrow}}^2 + |A|_{\substack{s'=\downarrow \\ r'=\uparrow}}^2$$

$$= \sum_{s', r'} |A_{s', r'}|^2$$

- Suppose that you cannot measure the initial spins either

\Rightarrow Average over initial spins

$$|A|^2 = \frac{1}{2} \frac{1}{2} \sum_{s, r} |A_{s, r}|^2$$

Average over initial spins

$$P_{\text{total}} = P_{\text{inclusive on final states averaged over initial states}} = P_{\text{unpolarized}}$$

$$= \frac{1}{4} \sum_{\substack{s, r \\ s', r'}} |A_{s, r, s', r'}|^2 \quad (P_1, P_2, P_3, P_4)$$

Recall:

$$\sum_{s=1}^2 \underline{v}_\alpha^s(P) \overline{v}_\beta^s(P) = (\not{P} - m \mathbb{1})_{\alpha\beta}$$

$$\sum_{s=1}^2 \underline{u}_\alpha^s(P) \overline{u}_\beta^s(P) = (\not{P} + m \mathbb{1})_{\alpha\beta}$$

$$\sum_{\substack{s, s' \\ r, r'}} [\underline{v}^{r'}(P_4) \delta_\mu \underline{u}^{s'}(P_3)] [\overline{u}^s(P_1) \delta^\nu \underline{v}^{r'}(P_2)] [\overline{u}^{s'}(P_3) \delta_\nu \underline{v}^{r'}(P_4)] [\overline{v}^{r'}(P_2) \delta^\mu \underline{u}^s(P_1)]$$

$$= \sum_{s', r'} (\underline{v}^{r'}(P_4) \delta_\mu \underline{u}^{s'}(P_3)) (\overline{u}^{s'}(P_3) \delta_\nu \underline{v}^{r'}(P_4)) \times$$

$$\times \sum_{r, r'} (\overline{u}^s(P_1) \delta^\nu \underline{v}^{r'}(P_2)) (\overline{v}^{r'}(P_2) \delta^\mu \underline{u}^s(P_1))$$

$$A_{\mu\nu} = \sum_{s', r'} \sum_{\alpha, \beta, \delta, \epsilon} (\underline{v}_\alpha^{r'}(P_4) \delta_\mu \underline{u}_\beta^{s'}(P_3)) (\overline{u}_\delta^{s'}(P_3) \delta_\nu \underline{v}_\epsilon^{r'}(P_4)) =$$

$$= \sum_{\alpha, \beta, \delta, \epsilon} \left[\sum_{s'} \underline{u}_\beta^{s'}(P_3) \overline{u}_\delta^{s'}(P_3) \right] \left[\sum_{r'} \underline{v}_\epsilon^{r'}(P_4) \overline{v}_\alpha^{r'}(P_4) \right] (\delta_\mu)_\beta \delta_\nu \delta_\epsilon$$

$$= \sum_{\alpha, \beta, \delta, \epsilon} (\not{P}_3 + m \mathbb{1})_{\beta\delta} (\not{P}_4 - m \mathbb{1})_{\alpha\epsilon} (\delta_\mu)_\beta \delta_\nu \delta_\epsilon$$

$$\begin{aligned}
&= \sum_{\alpha\beta\delta\epsilon} \underbrace{(\not{p}_4 - m_f \mathbb{1})_{\epsilon\alpha} (\gamma_\mu)_{\alpha\beta} (\not{p}_3 + m_f \mathbb{1})_{\beta\delta} (\gamma_\nu)_{\delta\epsilon}} \\
&= \sum_{\epsilon} \{ (\not{p}_4 - m_f \mathbb{1}) \gamma_\mu (\not{p}_3 + m_f \mathbb{1}) \gamma_\nu \}_{\epsilon\epsilon} \\
&= \text{Tr} [(\not{p}_4 - m_f \mathbb{1}) \gamma_\mu (\not{p}_3 + m_f \mathbb{1}) \gamma_\nu] \\
&= A_{\mu\nu}
\end{aligned}$$

$$\tilde{A}^{\mu\nu}(P_1, P_2) = \text{Tr} [(\not{p}_2 - m_e \mathbb{1}) \gamma^\mu (\not{p}_1 + m_e \mathbb{1}) \gamma^\nu]$$

$$\begin{aligned}
&\text{Tr} [(\not{p}_2 - m_e \mathbb{1}) \gamma^\mu (\not{p}_1 + m_e \mathbb{1}) \gamma^\nu] = \text{expand} \\
&\text{contains } \text{Tr}_{\alpha\beta} \gamma^\mu \gamma^\nu = \text{Tr} \frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} = \text{Tr}_{\alpha\beta} \mathbb{1} \eta^{\mu\nu} \\
&\text{Tr} (\gamma^\mu \underbrace{P_\rho}_{\cancel{P}} \gamma^\rho \gamma^\nu) = P_\rho \text{Tr} (\gamma^\mu \gamma^\rho \gamma^\nu) \\
&= 0 \\
&\text{Tr} (P_{2\sigma} \gamma^\sigma \gamma^\mu \underbrace{P_\rho}_{\cancel{P}} \gamma^\rho \gamma^\nu) = P_{2\sigma} P_{1\rho} \text{Tr} (\gamma^\sigma \gamma^\mu \gamma^\rho \gamma^\nu)
\end{aligned}$$

$$\text{Tr } \gamma^\sigma \gamma^\mu \gamma^\rho \gamma^\nu = 4(\eta^{\sigma\mu} \eta^{\rho\nu} - \eta^{\sigma\rho} \eta^{\mu\nu} + \eta^{\sigma\nu} \eta^{\rho\mu})$$

$$\text{Tr } \gamma^\mu \gamma^\nu = 4 \eta^{\mu\nu}$$

$$\begin{aligned} \text{Tr} [(\not{P}_2 - m_e \mathbb{1}) \gamma^\mu (\not{P}_1 + m_e \mathbb{1}) \gamma^\nu] &= \\ &= -m_e^2 \text{Tr} [\gamma^\mu \gamma^\nu] + P_{2\sigma} P_{1\rho} \text{Tr} [\gamma^\sigma \gamma^\rho \gamma^\mu \gamma^\nu] \\ &= -4m_e^2 \eta^{\mu\nu} + P_{2\sigma} P_{1\rho} (\eta^{\sigma\mu} \eta^{\rho\nu} - \eta^{\sigma\rho} \eta^{\mu\nu} + \eta^{\sigma\nu} \eta^{\rho\mu}) \\ &= -4\eta^{\mu\nu} m_e^2 + (P_2^\mu P_1^\nu - (P_2 \cdot P_1) \eta^{\mu\nu} + P_2^\nu P_1^\mu) \\ &= 4[P_2^\mu P_1^\nu + P_2^\nu P_1^\mu - (P_2 \cdot P_1) \eta^{\mu\nu}] - \eta^{\mu\nu} m_e^2 \\ &\equiv \tilde{A}^{\mu\nu} \end{aligned}$$

$$A_{\mu\nu} = 4[P_{3\mu} P_{4\nu} + P_{3\nu} P_{4\mu} - (P_3 \cdot P_4) \eta_{\mu\nu} - \eta_{\mu\nu} m_\mu^2]$$

$$\frac{1}{4} \sum_{\text{polarizations}} |A|^2 = \frac{e^4}{4S^2} \tilde{A}^{\mu\nu} A_{\mu\nu} =$$

$$= \frac{e^4}{4S^2} \left[P_2^\mu P_1^\nu + P_2^\nu P_1^\mu - (P_2 \cdot P_1) \eta^{\mu\nu} \right] - \eta^{\mu\nu} m_e^2$$

$$= \frac{e^4}{4S^2} \left[P_{3\mu} P_{4\nu} + P_{3\nu} P_{4\mu} - (P_3 \cdot P_4) \eta_{\mu\nu} - \eta_{\mu\nu} m_\mu^2 \right] =$$

$$= \frac{8e^4}{S^2} \left[P_{14} P_{23} + P_{13} P_{24} + m_\mu^2 P_{12} + m_e^2 P_{34} + 2m_\mu^2 m_e^2 \right]$$

Where $P_{ij} = P_i \cdot P_j$ (e.g. $P_{14} = P_1 \cdot P_4$)
 $\equiv P_i^\mu P_{j\mu}$

Introduce Mandelstam variables:

$$s = (P_1 + P_2)^2 = (P_1 + P_2)_\mu (P_1 + P_2)^\mu = P_1^2 + P_2^2 + 2P_1 \cdot P_2 = 2m_e^2 + 2P_{12}$$

$$\Rightarrow P_{12} = \frac{1}{2} s - m_e^2$$

Also

$$P_1 + P_2 = P_3 + P_4 \Rightarrow s = (P_3 + P_4)^2 = 2m_\mu^2 + 2P_{34}$$

$$P_{34} = \frac{1}{2} s - m_\mu^2$$

$$t = (P_1 - P_3)^2 = P_1^2 + P_3^2 - 2P_{13} = m_e^2 + m_\mu^2 - 2P_{13}$$

$$u = (P_1 - P_4)^2 = P_1^2 + P_4^2 - 2P_{14} = m_e^2 + m_\mu^2 - 2P_{14}$$

⇒ Express P_{ij} in terms of S, t, u, m_e^2, m_μ^2

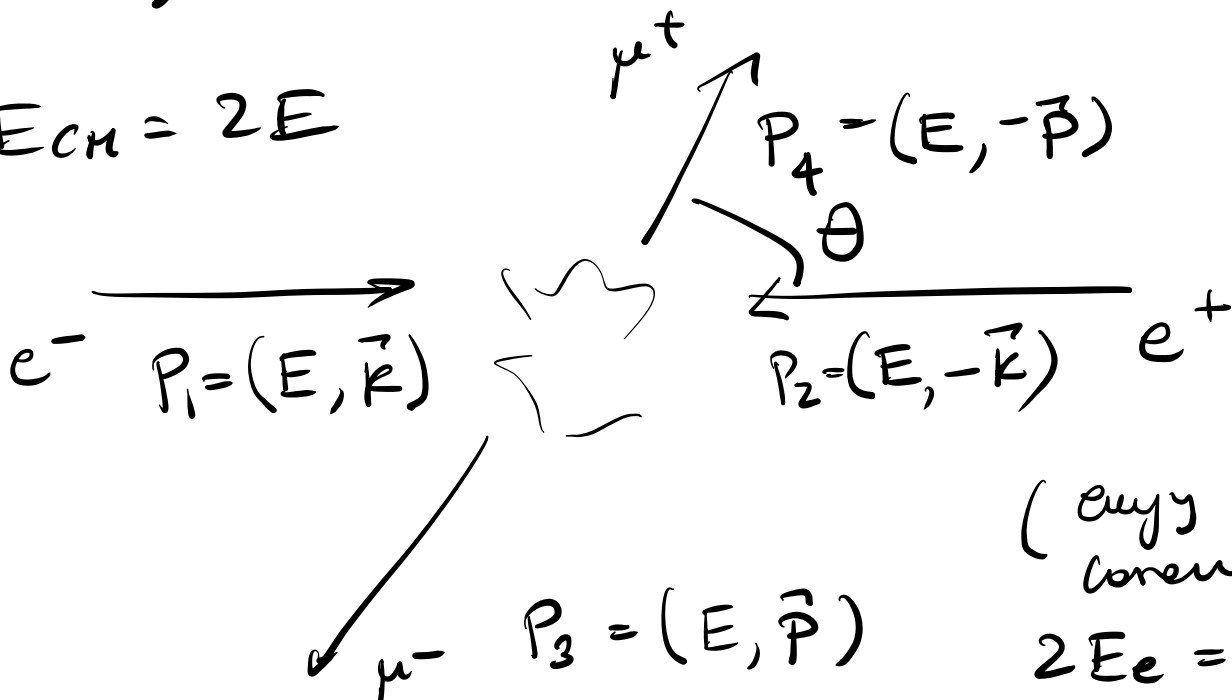
$$P_{23} = ? \quad u = (P_1 - P_4)^2 = (P_2 - P_3)^2 = m_e^2 + m_\mu^2 - 2P_{23}$$

$$= 0 \quad \left[\frac{1}{4} \sum_{\text{spins}} |A|^2 = \frac{2e^4}{s^2} \left[u^2 + t^2 + 4s(m_e^2 + m_\mu^2) - 2(m_e^2 + m_\mu^2)^2 \right] \right]$$

↑
exercise

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} = \frac{1}{64\pi^2} \frac{1}{E_{\text{cm}}^2} \frac{|\vec{P}_4|}{|\vec{P}_1|} \left(\frac{1}{4} \sum |A|^2 \right)$$

$$E_{\text{cm}} = 2E$$



(energy is conserved so $2E_e = 2E_\mu$)

$$S = (P_1 + P_2)^2 = \underbrace{\left((2E, \vec{0}) \right)^2}_{P_1 + P_2} = 4E^2 = (E_{cm})^2$$

$$t = (P_1 - P_3)^2 = \left((0, \vec{k} - \vec{p}) \right)^2 = m_e^2 + m_\mu^2 - 2P_1 \cdot P_3$$

$$P_1 \cdot P_3 = (E, \vec{k}) \cdot (E, +\vec{p}) = E^2 - \vec{k} \cdot \vec{p}$$

$$\Rightarrow t = m_e^2 + m_\mu^2 - 2(E^2 - \vec{k} \cdot \vec{p}) \quad (= (\vec{k} - \vec{p})^2)$$

$$u = m_e^2 + m_\mu^2 - 2(E^2 + \vec{k} \cdot \vec{p})$$

insert s, t, u into cross section
 $\alpha = e^2/4\pi$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega} \right)_{cm} = \frac{\alpha^2}{16E^6} \frac{|\vec{p}|}{|\vec{k}|} \left[E^4 + (\vec{k} \cdot \vec{p})^2 + E^2(m_e^2 + m_\mu^2) \right]$$

depends on E, θ

Relative Angle between e, μ

$$(\vec{k} \cdot \vec{p})^2 = \frac{(E^2 - m_e^2)}{|\vec{k}|^2} \frac{(E^2 - m_\mu^2)}{|\vec{p}|^2} \cos^2 \theta$$

suppose that $E \gg m_e, m_\mu$

$$(\vec{k} \cdot \vec{p})^2 \approx E^4 \cos^2 \theta, \quad |\vec{p}|/|\vec{k}| \approx 1$$

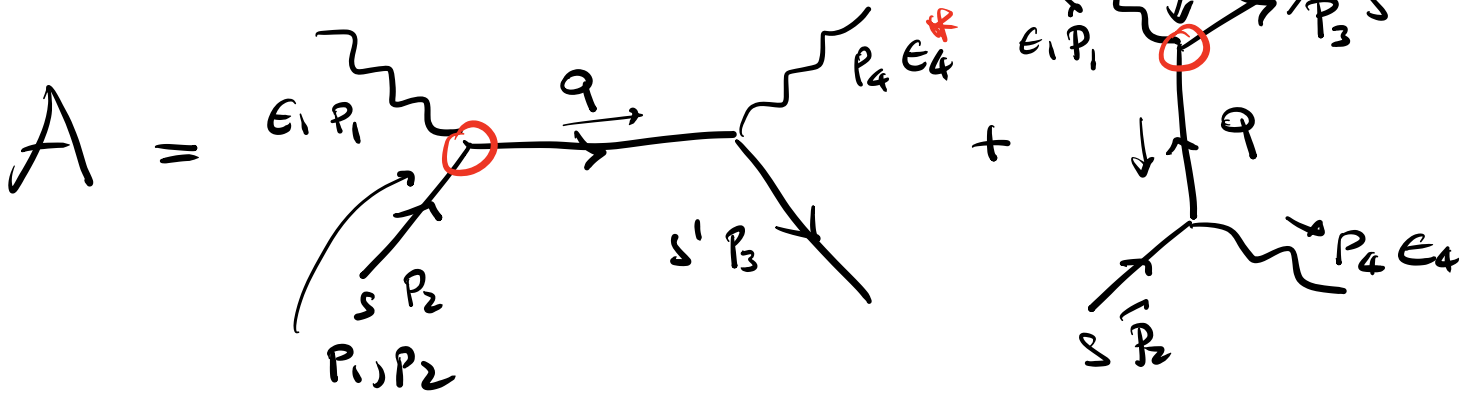
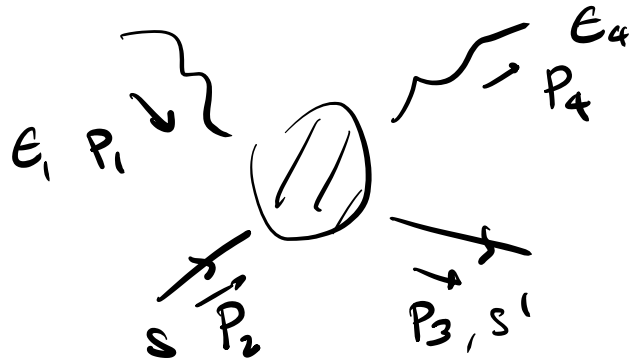
$$\left(\frac{d\sigma}{d\Omega} \right)_{CM} \approx \frac{\alpha^2}{16 E^6} (E^4 + E^4 \cos^2 \theta)$$

$$= \frac{\alpha^2}{16 E^2} (1 + \cos^2 \theta)$$

(unpolarised) ultra-relativistic $e^+e^- \rightarrow \mu^+\mu^-$
cross-section in CM frame

Compton Scattering

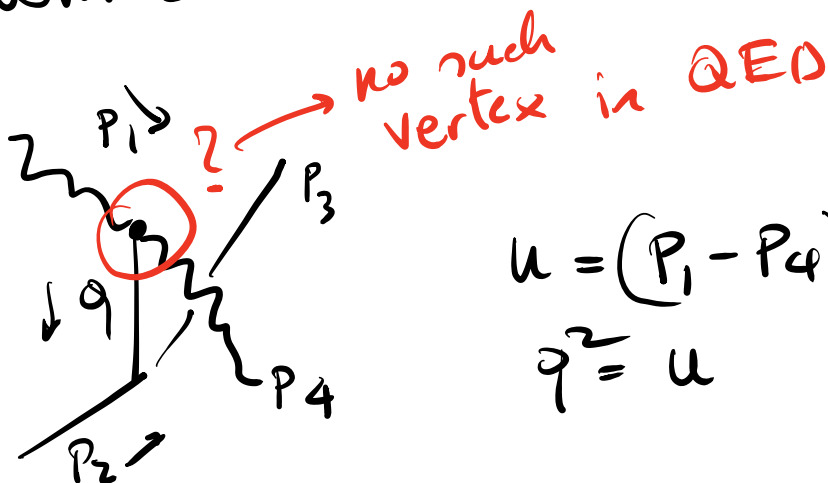
$$e^- \gamma \longrightarrow e^- \gamma$$



$$q = p_1 + p_2 \quad q^2 = s \quad \text{s-channel}$$

$$q = p_1 - p_3 \quad q^2 = t \quad \text{t-channel}$$

no
u-channel
here



$$u = (p_1 - p_4)^2$$

$$q^2 = u$$

ingoing particle: $\rightarrow_{s, \vec{p}} \otimes \rightarrow U^s(p)$

ingoing antiparticle: $\leftarrow_{s, \vec{p}} \otimes \rightarrow \bar{V}^s(\vec{p})$

outgoing particle: $\otimes \rightarrow_{s, \vec{p}} \rightarrow \bar{U}^s(\vec{p})$

outgoing antiparticle: $\otimes \leftarrow_{s, \vec{p}} \rightarrow V^s(\vec{p})$

incoming photon: $\text{wavy } \otimes \rightarrow \epsilon_\mu(p)$

outgoing photon: $\otimes \text{wavy } \rightarrow \epsilon_\mu^*(p)$

$$\epsilon_1^\mu p_{1\mu} = 0$$

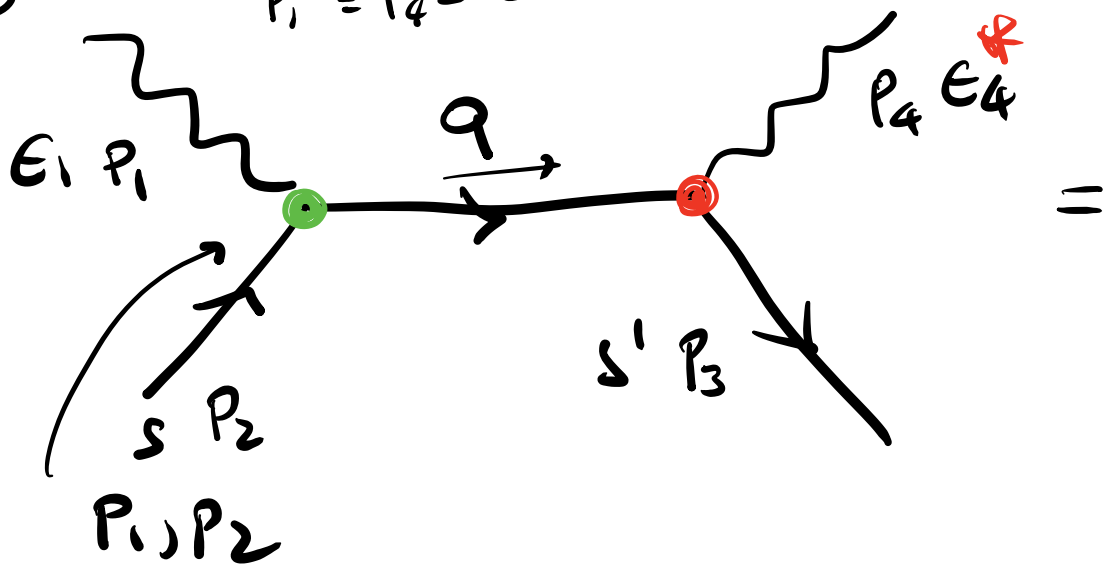
$$\vec{\epsilon}_1 \cdot \vec{p}_1 = 0$$

$$p_1^2 = p_4^2 = 0$$

$$\vec{\epsilon}_4 \cdot \vec{p}_4 = 0$$

$$\epsilon_4^\mu p_{4\mu} = 0$$

$$A_s =$$



$$= \bar{u}_\alpha^{s'}(p_3) \underline{(-ie\gamma_{\alpha\beta}^\mu)} (\not{q} + m_e \mathbb{1})_{\beta\delta} \underline{(-ie\gamma_{\delta\epsilon}^\nu)} u_\epsilon^s(p) \underline{\epsilon_{1\nu}(p_1)} \underline{\epsilon_{4\mu}^*(p_4)}$$

$$s - m_e^2 + i\epsilon$$

$$A_t = -e^2 \epsilon_1^\mu \epsilon_4^{\nu*} \bar{u}(p_3) \gamma_\mu [(\not{p}_1 - \not{p}_3) + m] \gamma_\nu u(p_2)$$

$$t - m_e^2 + i\epsilon$$

Unpolarized probability

$$\frac{1}{4} \sum_{\text{spins}} \sum_{\text{polarizations}} |A_s + A_t|^2 =$$

$$= \frac{1}{4} \sum_{\text{spins}} \sum_{\text{polarizations}} \left(A_s^\dagger A_s^\dagger A_t^\dagger A_t + A_s^\dagger A_t + A_t^\dagger A_s \right)$$

physical

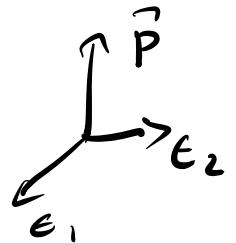
Polarization sums

$$\sum_n |n\rangle \langle n| = \mathbb{1}$$

~~Complete basis of vectors~~

$$\sum_{\mu} \epsilon_{\mu}^{\nu}(\vec{p}) \epsilon_{\mu}^{\lambda}(\vec{p}) = -\eta^{\nu\lambda}$$

$$p_{\mu}, \bar{p}_{\mu}, \epsilon_1, \epsilon_2$$



$$\left\{ \begin{array}{l} p_{\mu} = (|\mathbf{p}|, \vec{p}) \\ \bar{p}_{\mu} = (|\mathbf{p}|, -\vec{p}) \end{array} \right.$$

$$\boxed{\epsilon_1, \epsilon_2 = (0, \vec{\epsilon}_{1,2})}$$

physical

$$-\eta^{\mu\nu} = \sum_{\vec{p}} \epsilon_{\vec{p}}^{\mu} \epsilon_{\vec{p}}^{\nu} = \sum_{\substack{\perp \text{ polarizations} \\ i=1}}^2 \epsilon_i^{\mu} \epsilon_i^{\nu} + \sum_{\substack{\parallel \text{ polarizations}}}$$

Complete basis of vectors

$$\epsilon^{\mu \parallel} = \{ (|\vec{p}|, \vec{p}), (|\vec{p}|, -\vec{p}) \} \quad \text{longitudinal}$$

$$\epsilon_i^{\mu} = \{ 0, \vec{e}_1, \vec{e}_2 \}$$

$$\sum_{\text{longitudinal}} \epsilon_{\vec{p}}^{\mu} \epsilon_{\vec{p}}^{\nu} = \frac{p^{\mu} \bar{p}^{\nu} + \bar{p}^{\mu} p^{\nu}}{2E^2}$$

$$\Rightarrow \sum_{\perp i=1}^2 \epsilon_i^{\mu} \epsilon_i^{\nu} = -\eta^{\mu\nu} + \frac{p^{\mu} \bar{p}^{\nu} + \bar{p}^{\mu} p^{\nu}}{2E^2}$$

this never contributes!

$$A_s = \bar{u}_{\alpha}^{s'}(p_3) \underbrace{(-ie\gamma_{\alpha\beta}^{\mu})}_{\substack{\vec{p}_1 + \vec{p}_2 \\ S - m_e^2 + i\epsilon}} \underbrace{(-ie\gamma_{\delta\epsilon}^{\nu})}_{\text{green}} u_{\epsilon}^s(p_1) \underbrace{\epsilon_{\nu}}_{\text{green}}(p_1) \underbrace{\epsilon_{\mu}^*}_{\text{red}}(p_2)$$

$$= \epsilon_{\nu}(p_1) A_s \quad \checkmark$$

$$\text{Fact: } P_{1\mu} A_s^\mu = 0$$

$$\bar{P}_{1\mu} A_s^\mu = 0$$

Effectively:

$$\text{" } \sum_{\substack{\uparrow \\ \text{polarizations}}} \epsilon_i^\mu(p) \epsilon_i^\nu(q) = -\eta^{\mu\nu} \text{"}$$

(true when contracted into an amplitude)

$$\frac{1}{4} \sum_{\text{spins}} \sum_{\text{pol}} |\mathcal{A}|^2 = \dots \dots \dots$$

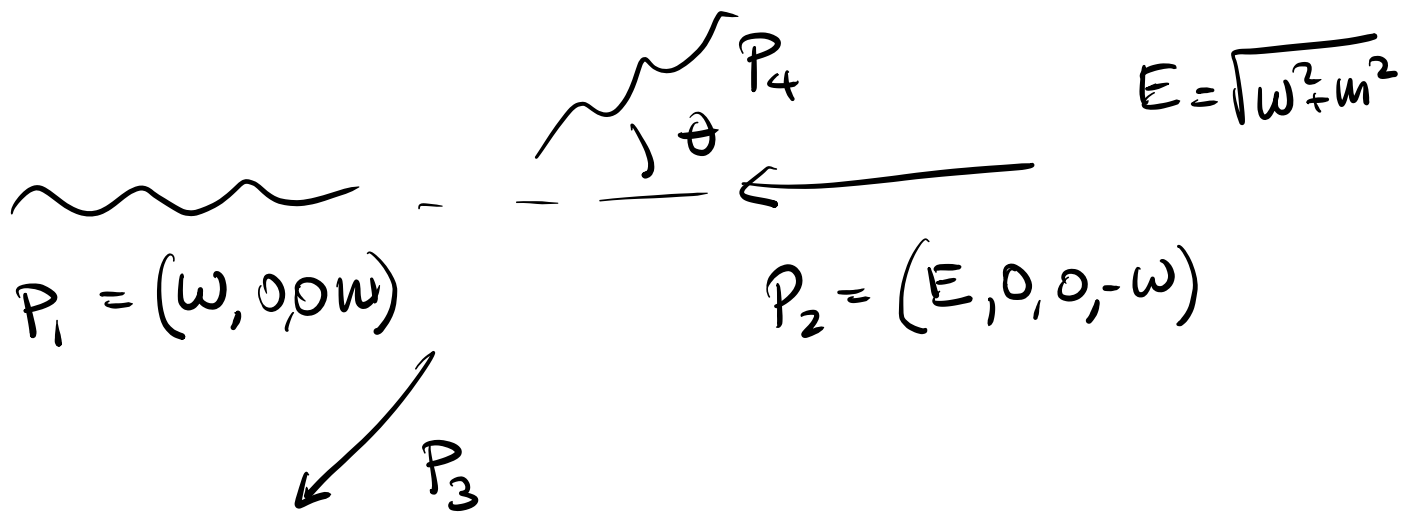
Spin sum rules +
polarization sum rule

$$= e^4 \text{Tr} \left[(\not{p}_3 + m) \left(\frac{\gamma^\nu (\not{p}_1 + \not{p}_2 + m) \gamma^\mu}{s - m^2} + \frac{\gamma^\mu (\not{p}_2 - \not{p}_4 + m) \gamma^\nu}{t - m^2} \right) \right. \\ \left. (\not{p}_2 + m) \left(\frac{\gamma_\mu (\not{p}_1 + \not{p}_2 + m) \gamma_\nu}{s - m^2} + \frac{\gamma_\nu (\not{p}_2 - \not{p}_4 + m) \gamma_\mu}{t - m^2} \right) \right]$$

use trace identities

$$= 2e^4 \left[\frac{P_{24}}{P_{12}} + \frac{P_{12}}{P_{24}} + 2m^2 \left(\frac{1}{P_{12}} - \frac{1}{P_{24}} \right) + m^4 \left(\frac{1}{P_{12}} - \frac{1}{P_{24}} \right)^2 \right]$$

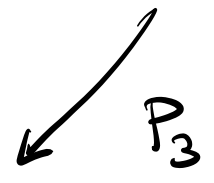
Kinematics in CM frame



$$P_1 = (\omega, 0, 0, \omega)$$

$$P_2 = (E, 0, 0, -\omega)$$

$$E = \sqrt{\omega^2 + m^2}$$



$$P_4 = (\omega, \omega \sin\theta, 0, \omega \cos\theta)$$

$$P_3 = (E, -\omega \sin\theta, 0, -\omega \cos\theta)$$

$$\Rightarrow P_{12} = \omega E + \omega^2$$

$$P_{24} = \omega (E + \omega \cos\theta)$$

$$\omega \gg m$$

(photon has very high energy)

$$\frac{1}{4} \sum |A|^2 \approx 4e^4 \left[\frac{1 + \cos\theta}{4} + \frac{1}{\frac{m^2}{4\omega^2} + 1 + \cos\theta} \right]$$

↑ small

$$\left. \frac{d\sigma}{d\Omega} \right|_{CM} = \frac{1}{64\pi^2 (2\omega)^2} \times \frac{1}{4} \sum |A|^2 =$$

$$= \frac{\pi \alpha^2}{2\omega^2} \left[\frac{1 + \cos\theta}{4} + \frac{1}{\frac{m^2}{2\omega^2} + 1 + \cos\theta} \right]$$

Compton cross section in the
CM frame in the high-energy limit