Exercise 1

1.8 Traces of gamma matrices

Now compute the traces using the fundamental relation $\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}$, and the cyclic property of the trace, $tr(A \ldots BC) = tr(CA \ldots B)$. First, we can show that the trace of the product of any odd number of γ-matrices vanishes by using $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma^a\} = 0$,

$$\begin{aligned}
tr\left(\underbrace{\gamma^\alpha \ldots \gamma^\beta}_{2n+1}\right) &= tr\left(1\gamma^\alpha \ldots \gamma^\beta\right) \\
&= tr\left(\gamma_5\gamma_5\gamma^\alpha \ldots \gamma^\beta\right) \\
&= -tr\left(\gamma_5\gamma^\alpha\gamma_5 \ldots \gamma^\beta\right) \\
&= (-1)^{2n+1} tr\left(\gamma_5\gamma^\alpha \ldots \gamma^\beta\gamma_5\right) \\
&= (-1)^{2n+1} tr\left(\gamma_5\gamma_5\gamma^\alpha \ldots \gamma^\beta\right) \\
&= -tr\left(\gamma^\alpha \ldots \gamma^\beta\right) \\
&= 0
\end{aligned}$$

For even products, we will need traces of products of $2, 4, 6$ and 8 gamma matrices.

$$\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\right) &= tr\left(-\gamma^\beta\gamma^\alpha + 2\eta^{\alpha\beta}1\right) \\
&= -tr\left(\gamma^\beta\gamma^\alpha\right) + 2\eta^{\alpha\beta} tr(1) \\
&= -tr\left(\gamma^\alpha\gamma^\beta\right) + 8\eta^{\alpha\beta} \\
tr\left(\gamma^\alpha\gamma^\beta\right) &= 4\eta^{\alpha\beta}
\end{aligned}$$

and

$$\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= tr\left(\left(-\gamma^\beta\gamma^\alpha + 2\eta^{\alpha\beta}1\right)\gamma^\mu\gamma^\nu\right) \\
&= -tr\left(\gamma^\beta\gamma^\alpha\gamma^\mu\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
&= -tr\left(\gamma^\beta\left(-\gamma^\mu\gamma^\alpha + 2\eta^{\mu\alpha}\right)\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
&= tr\left(\gamma^\beta\gamma^\mu\gamma^\alpha\gamma^\nu\right) - 2\eta^{\mu\alpha} tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
&= tr\left(\gamma^\beta\gamma^\mu\left(-\gamma^\nu\gamma^\alpha\right) + 2\eta^{\nu\alpha}\right) - 2\eta^{\mu\alpha} tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
&= -tr\left(\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha\right) + 2\eta^{\nu\alpha} tr\left(\gamma^\beta\gamma^\mu\right) - 2\eta^{\mu\alpha} tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
2tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= 2\eta^{\nu\alpha} tr\left(\gamma^\beta\gamma^\mu\right) - 2\eta^{\mu\alpha} tr\left(\gamma^\beta\gamma^\nu\right) + 2\eta^{\alpha\beta} tr(\gamma^\mu\gamma^\nu) \\
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= 4\eta^{\nu\alpha}\eta^{\beta\mu} - 4\eta^{\mu\alpha}\eta^{\beta\nu} + 4\eta^{\alpha\beta}\eta^{\mu\nu}
\end{aligned}$$

For six, we use the simple pattern to more quickly find

$$\begin{aligned}
tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= tr\left(\left(-\gamma^\beta\gamma^\alpha + 2\eta^{\alpha\beta}1\right)\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= tr\left(-\gamma^\beta\left(2\eta^{\alpha\mu} - \gamma^\mu\gamma^\alpha\right)\gamma^\nu\gamma^\rho\gamma^\sigma + 2\eta^{\alpha\beta}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
&= tr\left(-\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\alpha + 2\eta^{\alpha\sigma}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\rho - 2\eta^{\alpha\rho}\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\sigma + 2\eta^{\alpha\nu}\gamma^\beta\gamma^\mu\gamma^\rho\gamma^\sigma - 2\eta^{\alpha\mu}\gamma^\beta\gamma^\nu\gamma^\rho\gamma^\sigma + 2\eta^{\alpha\beta}\gamma^\mu\gamma
\end{aligned}$$

and from here we can use the result for the trace of four,

$$\begin{aligned}
tr\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= tr\left(\eta^{\alpha\sigma}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}-\eta^{\alpha\rho}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}+\eta^{\alpha\nu}\gamma^{\beta}\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}-\eta^{\alpha\mu}\gamma^{\beta}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}+\eta^{\alpha\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) \\
&= 4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho}-\eta^{\beta\nu}\eta^{\mu\rho}+\eta^{\rho\beta}\eta^{\mu\nu}\right)-4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma}-\eta^{\beta\nu}\eta^{\mu\sigma}+\eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma}-\eta^{\beta\rho}\eta^{\mu\sigma}+\eta^{\beta\sigma}\eta^{\mu\rho}\right)-4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma}-\eta^{\beta\rho}\eta^{\nu\sigma}+\eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}\right)
\end{aligned}$$

or, perhaps more mnemonically,

$$\begin{aligned}
tr\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\right) &= 4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}\right)-4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma}-\eta^{\beta\rho}\eta^{\nu\sigma}+\eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma}-\eta^{\beta\rho}\eta^{\mu\sigma}+\eta^{\beta\sigma}\eta^{\mu\rho}\right)-4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma}-\eta^{\beta\nu}\eta^{\mu\sigma}+\eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho}-\eta^{\beta\nu}\eta^{\mu\rho}+\eta^{\rho\beta}\eta^{\mu\nu}\right)
\end{aligned}$$

From this, if we don't run out of Greek letters, we can immediately write the result for eight gammas:

$$\begin{aligned}
tr\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma}\gamma^{\lambda}\gamma^{\tau}\right) &= 4\eta^{\alpha\beta}\left(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}\right)-4\eta^{\alpha\mu}\left(\eta^{\beta\nu}\eta^{\rho\sigma}-\eta^{\beta\rho}\eta^{\nu\sigma}+\eta^{\beta\sigma}\eta^{\nu\rho}\right) \\
&\quad +4\eta^{\alpha\nu}\left(\eta^{\beta\mu}\eta^{\rho\sigma}-\eta^{\beta\rho}\eta^{\mu\sigma}+\eta^{\beta\sigma}\eta^{\mu\rho}\right)-4\eta^{\alpha\rho}\left(\eta^{\beta\mu}\eta^{\nu\sigma}-\eta^{\beta\nu}\eta^{\mu\sigma}+\eta^{\sigma\beta}\eta^{\mu\nu}\right) \\
&\quad +4\eta^{\alpha\sigma}\left(\eta^{\beta\mu}\eta^{\nu\rho}-\eta^{\beta\nu}\eta^{\mu\rho}+\eta^{\rho\beta}\eta^{\mu\nu}\right)
\end{aligned}$$

Problem set 1, exercise 2

$$u^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\, \xi^s \\ \sqrt{p_\mu \bar\sigma^\mu}\, \xi^s \end{pmatrix} \qquad \begin{matrix} \sigma^0 = \mathbb{1} \\ \bar\sigma^0 = \mathbb{1} \end{matrix} \qquad \xi^s = \left\{ \underset{s=1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{s=2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}$$

$$\bar u = u^\dagger \gamma^0 = \left(\xi^s \sqrt{p_\mu \sigma^\mu}^\dagger \;\; \xi^s \sqrt{p_\mu \bar\sigma^\mu}^\dagger \right) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} =$$

$$= \left(\xi^s \sqrt{p_\mu \bar\sigma^\mu}^\dagger \;\; \xi^s \sqrt{p_\mu \sigma^\mu}^\dagger \right)$$

2.1)

$$\bar u^s u^{s'} = \left(\xi^s \sqrt{p_\mu \bar\sigma^\mu}^\dagger ,\; \xi^s \sqrt{p_\mu \sigma^\mu}^\dagger \right) \begin{pmatrix} \sqrt{p_\nu \sigma^\nu}\, \xi^{s'} \\ \sqrt{p_\nu \bar\sigma^\nu}\, \xi^{s'} \end{pmatrix}$$

$$= \xi^s \sqrt{p_\mu \bar\sigma^\mu}^\dagger \sqrt{p_\nu \sigma^\nu}\, \xi^{s'} + \xi^s \sqrt{p_\mu \sigma^\mu}^\dagger \sqrt{p_\nu \bar\sigma^\nu}\, \xi^{s'}$$

now $\bar\sigma^{\mu\dagger} = \bar\sigma^\mu$; $\sigma^{\mu\dagger} = \sigma^\mu$

$$= \xi^s \sqrt{(p_\mu \bar\sigma^\mu)(p_\nu \sigma^\nu)}\, \xi^{s'} + \xi^s \sqrt{(p_\mu \sigma^\mu)(p_\nu \bar\sigma^\nu)}\, \xi^{s'}$$

$$= (*)$$

We have:

$$(p_\mu \bar{\sigma}^\mu)(p_\nu \sigma^\nu) = (p_\mu \sigma^\mu)(p_\nu \bar{\sigma}^\nu) = m^2 \mathbb{1}$$

$$p_\mu \sigma^\mu p_\nu \bar{\sigma}^\nu = p_\mu p_\nu \sigma^\mu \bar{\sigma}^\nu = (p_0 \mathbb{1} + \vec{p}\cdot\vec{\sigma})(p_0 \mathbb{1} - \vec{p}\cdot\vec{\sigma})$$

$$= p_0^2 \mathbb{1} - \underbrace{(\vec{p}\cdot\vec{\sigma})^2}_{p^2 \mathbb{1}} = m^2 \mathbb{1}$$

and similarly for the other one

$$\Rightarrow (*) = 2m\, \xi^s \xi^{s'} = 2m \delta^{ss'}$$

2.2)

$$\bar{u}^s \gamma^\mu u^{s'} =$$

$$\left(\xi^s \sqrt{p_\rho \bar{\sigma}^\rho}^{\dagger}, \xi^s \sqrt{p_\rho \sigma^\rho}^{\dagger} \right) \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p_\nu \sigma^\nu}\, \xi^{s'} \\ \sqrt{p_\nu \bar{\sigma}^\nu}\, \xi^{s'} \end{pmatrix}$$

$$= \left(\xi^s \sqrt{p_\rho \bar{\sigma}^\rho}, \xi^s \sqrt{p_\rho \sigma^\rho} \right) \begin{pmatrix} \sigma^\mu \sqrt{p_\nu \bar{\sigma}^\nu}\, \xi^{s'} \\ \\ \bar{\sigma}^\mu \sqrt{p_\nu \sigma^\nu}\, \xi^{s'} \end{pmatrix} =$$

$$= \xi^{s} \sqrt{p_\rho \bar{\sigma}^\rho}\; \sigma^\mu \sqrt{p_\nu \bar{\sigma}^\nu}\, \xi^{s'} + \xi^{s} \sqrt{p_\rho \sigma^\rho}\; \bar{\sigma}^\mu \sqrt{p_\nu \sigma^\nu}\, \xi^{s'} = (*)$$

$$(p_\mu \sigma^\mu)(p_\nu \bar{\sigma}^\nu) = (p_\mu \bar{\sigma}^\mu)(p_\nu \sigma^\nu) = m^2 \mathbb{1} \qquad (\text{sec } 2.1)$$

$$\Rightarrow \begin{cases} \sqrt{p\cdot\bar{\sigma}} = \dfrac{\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})}}{m}\sqrt{p\cdot\bar{\sigma}} = \dfrac{\sqrt{p\cdot\sigma}}{m}(p\cdot\bar{\sigma}) \\[2ex] \sqrt{p\cdot\sigma} = \sqrt{p\cdot\sigma}\,\dfrac{\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})}}{m} = (p\cdot\sigma)\dfrac{\sqrt{p\cdot\bar{\sigma}}}{m} \end{cases}$$

$$(*) = \xi^{s} \frac{\sqrt{p\cdot\sigma}}{m}\, p\cdot\bar{\sigma}\; \sigma^\mu \sqrt{p\cdot\bar{\sigma}}\, \xi^{s'} + \xi^{s} \sqrt{p\cdot\sigma}\; \bar{\sigma}^\mu (p\cdot\sigma) \frac{\sqrt{p\cdot\bar{\sigma}}}{m}\, \xi^{s'}$$

$$= \frac{\xi^{s}}{m} \sqrt{p\cdot\sigma}\left(p\cdot\bar{\sigma}\,\sigma^\mu + \bar{\sigma}^\mu (p\cdot\sigma)\right)\sqrt{p\cdot\bar{\sigma}}\, \xi^{s'} =$$

$$p_\rho\left(\bar{\sigma}^\rho \sigma^\mu + \bar{\sigma}^\mu \sigma^\rho\right) = \begin{pmatrix} \{\mathbb{1},\mathbb{1}\} & 0 \\ 0 & -\{\sigma^i,\sigma^j\} \end{pmatrix} = 2 p_\rho \eta^{\mu\rho} \mathbb{1}$$

$$= \frac{1}{m}\, \xi^{s} \underbrace{\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})}}_{m\mathbb{1}}\, 2p^\mu \mathbb{1}\, \xi^{s'} = 2p^\mu \delta^{ss'}$$

2.3)

we want to show: $\sum_{s=1}^{2} u^s_\alpha \, \bar{u}^s_\beta = \cancel{p}_{\alpha\beta} + m\delta_{\alpha\beta}$

$2\times 1 \quad \times \quad 1\times 2 \quad = 2\times 2$

$$\sum_s \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\, \xi^s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\, \xi^s \end{pmatrix} \left(\xi^s \sqrt{p_\mu \bar{\sigma}^\mu}^\dagger \quad \xi^s \sqrt{p_\mu \sigma^\mu}^\dagger \right) =$$

$\xi^s = |s\rangle \qquad \sum_s \xi^s \xi^s = \sum_{s=1}^{2} |s\rangle\langle s|$

$\sum_s \xi^s_a \xi^s_b = \delta_{ab}$ $\quad a, b = 1,2$ $\quad \leftarrow \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right]$

$$= \begin{array}{cc} \left(\begin{array}{c|c} \sqrt{(p_\mu\sigma^\mu)(p_\nu\bar{\sigma}^\nu)} & \sqrt{(p_\mu\sigma^\mu)(p_\nu\sigma^\nu)} \\ \hline \sqrt{(p_\mu\bar{\sigma}^\mu)(p_\nu\bar{\sigma}^\nu)} & \sqrt{(p_\mu\bar{\sigma}^\mu)(p_\nu\sigma^\nu)} \end{array}\right) & \begin{array}{c} 2 \\ 2 \end{array} \\ \begin{array}{cc} 2 & \qquad\qquad\qquad 2 \end{array} & \end{array} = (*)$$

Now :

$$(p_\mu\sigma^\mu)(p_\nu\bar{\sigma}^\nu) = m^2 \mathbb{1}$$

$$(p_\mu\bar{\sigma}^\mu)(p_\nu\sigma^\nu) = m^2 \mathbb{1}$$

(sec 2.1)

Also:

$$p_\mu\sigma^\mu \, p_\nu\sigma^\nu = (p_\mu\sigma^\mu)^2$$

$$\Rightarrow (*) = \begin{pmatrix} m\mathbb{1}_2 & \sqrt{(p\cdot\sigma)^2} \\ \sqrt{(p\cdot\bar{\sigma})^2} & m\mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} m\mathbb{1}_2 & p\cdot\sigma \\ p\cdot\bar{\sigma} & m\mathbb{1}_2 \end{pmatrix}$$

$$= m\mathbb{1} + p\cdot \underbrace{\begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}}_{\gamma^\mu}$$

$$= m\mathbb{1} + \gamma^\mu p_\mu$$

Similarly for the other identity