QFT TD 2

1) ①

$$\mathcal{L}_\psi = \underset{3/2}{\bar\psi}(\underset{1}{i\not\partial} - m)\underset{3/2}{\psi} \Rightarrow [\mathcal{L}] = 4 \quad \text{if } [\psi] = 3/2$$

$$\mathcal{L}_\phi = \frac{1}{2}\underset{1}{\partial_\mu}\underset{1}{\phi}\underset{1}{\partial^\mu}\underset{1}{\phi} \qquad [\phi] = 1$$

a)

- $\underset{1}{g\phi}\underbrace{\bar\psi\psi}_{3} \Rightarrow [g] = 0$

- $G_F \underbrace{\bar\psi\psi}_{3}\underbrace{\bar\psi\psi}_{3} \Rightarrow [G_F] = -2$

b) g is marginal, G_F irrelevant

For G_F : $g_F = G_F E^2$

$g_F \gtrsim 1$ for $E > \frac{1}{\sqrt{G_F}}$ = some mass scale

c)

	ϕ^3	ϕ^4	$\phi\bar\psi\psi$	$(\partial_\mu\phi)\bar\psi\gamma^\mu\psi$	$\phi^2\bar\psi\psi$
D=	3	4	4	5	5

$\phi\partial_\mu\phi\partial^\mu\phi$	$(\phi^2\Box\phi)$ is the same	ϕ^5	$\bar\psi\Box\psi$
5	5	5	5

$\phi^2 \partial_\mu \phi \partial^\mu \phi$ $\quad (\Box\phi)^2 \quad \phi^2 \Box \phi^2 \quad \phi^3 \bar{\psi}\psi$

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$\phi \partial^\mu \phi \, \bar{\psi} \gamma_\mu \psi \quad (\bar{\psi}\psi)^2 \quad (\bar{\psi}\gamma^\mu \psi)(\bar{\psi}\gamma_\mu \psi)$

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$\bar{\psi} \Box \gamma^\mu \partial_\mu \psi \quad \phi^6$

6 $\quad$ 6

2) $G_N = 6.7 \times 10^{-11} \, Kg^{-1} \, m^3/s^2$

a) $[G_N] = M^{-1} L^3 S^{-2}$

in natuel units $L = S = M^{-1}$

$\rightarrow [G_N] = -1 + 2 - 3 = -2$

dimension $(\text{Lenght})^2 = \dfrac{1}{(\text{Mass})^2}$

$\rightarrow$ the interaction is <u>irrelevant</u>.

b) the effective coupling is $g_{eff} = G_N E^2$

strong coupling scale:

$E = \dfrac{1}{\sqrt{G_N}}$ $\quad$ how much is this ?

Call $G_N = \frac{1}{M_P^2}$ for some value M_P.

$G_N \simeq 7 \times 10^{-11}\ kg^{-1}\ m^3/s^2$

use the fact: $\hbar c \simeq (200\ MeV)(1\ fm)$
(cfr. Nucleon physics)

$(200\ MeV)(1\ fm) = 200 \times 10^6 \times 1{,}6 \times 10^{-19}\ J \times 10^{-15}\ m$

$\simeq 3 \times 10^{-26}\ kg\ m^3/s^2 = \hbar c = 1$

$\rightarrow$ in natural units: $1\ m^3/s^2 = \frac{1}{3} \times 10^{26}\ kg^{-1}$

$\rightarrow G_N \simeq 7 \times 10^{-11}\ kg^{-1} (0{,}3 \times 10^{26}\ kg^{-1})$

$\simeq 2 \times 10^{15}\ kg^{-2}$

now we want this in MeV (or GeV)

$1\ MeV = 1{,}6 \ 10^6 \times 10^{-19}\ kg\ m^2/s^2$

$\rightarrow 1\ kg \simeq 10^{13} \frac{MeV}{m^2/s^2} = 10^{13} \frac{MeV}{\underset{\underset{1}{\parallel}}{c^2}} \left(\frac{\overset{(3 \times 10^8)^2}{c^2}}{m^2/s^2}\right)$

$\simeq 10^{13} \times 9 \times 10^{16}\ MeV \simeq 10^{30}\ MeV$
$= 10^{27}\ GeV$

$\rightarrow G_N = \cancel{\frac{2 \times 10^{15}}{10^{60}}}$

$$\to \quad G_N = \frac{2 \times 10^{15}}{\left(10^{27}\,\mathrm{GeV}\right)^2} = \frac{2\; 10^{15}}{10^{54}\,\mathrm{GeV}^2}$$

$$\simeq \frac{1}{10^{39}\,\mathrm{GeV}^2} \simeq \frac{1}{\left(10^{19}\,\mathrm{GeV}\right)^2}$$

$$\Rightarrow \quad \boxed{M_P = 10^{19}\,\mathrm{GeV}}$$

Quantum Gravity strong coupling scale

$\boxed{2}$ $$|\psi(t)\rangle_I = e^{iH_0 t}\,|\psi(t)\rangle_S$$

1) $$i\frac{d}{dt}|\psi\rangle_I = i\,(iH_0)\,e^{iH_0 t}\,|\psi\rangle_S + e^{iH_0 t}\,i\frac{d}{dt}|\psi\rangle_S$$

$$= \left(-H_0\, e^{iH_0 t}\,|\psi\rangle_S + e^{iH_0 t}\,H\,|\psi\rangle_S\right)$$

$$= e^{iH_0 t}\,(\underbrace{-H_0 + H}_{H_{int}})\,|\psi\rangle_S$$

$$= \underbrace{e^{iH_0 t}\,H_{int}\,e^{-iH_0 t}}_{H_I(t)}\;\underbrace{e^{iH_0 t}\,|\psi\rangle_S}_{|\psi\rangle_I}$$

2) $${}_I\langle f(t)|i(t)\rangle_I = {}_S\langle f(t)| \cancel{e^{-i\hat{H}_0 t}} \, \cancel{e^{i\hat{H}_0 t}} |i(t)\rangle_S$$

$$= {}_S\langle f(t)|i(t)\rangle_S$$

3) From previous result:

$${}_S\langle f(+\infty)|i(+\infty)\rangle_S = {}_I\langle f(+\infty)|i(+\infty)\rangle_I$$

By definition, $|i(+\infty)\rangle_I$ is obtained from $|i(-\infty)\rangle_I$ by the operator $U_I(+\infty,-\infty)$:

$$|i(+\infty)\rangle_I = \hat{U}_I(+\infty,-\infty)|i(-\infty)\rangle_I$$

$\Rightarrow$

$${}_S\langle f(+\infty)|i(+\infty)\rangle_S = {}_I\langle f(+\infty)|\hat{U}_I(+\infty,-\infty)|i(-\infty)\rangle_I$$

Identifying the initial and final states with free states, we have the result for the S-Matrix:

$${}_S\langle f(+\infty)|\hat{S}|i(-\infty)\rangle_S \equiv {}_S\langle f(+\infty)|i(+\infty)\rangle_S = \langle f|\hat{S}_I|i\rangle$$

where $\hat{S}_I = U_I(+\infty,-\infty)$

4) the operator $\tilde{U}_\pm(t, t_0)$ is defined by :

$$|\psi(t)\rangle_I = \hat{U}_\pm(t, t_0)\,|\psi(t_0)\rangle_\pm$$

now take a time-derivative on both sides:

$$i\frac{d}{dt}\,|\psi(t)\rangle_\pm = \left(i\frac{d}{dt}\,\hat{U}_\pm(t,t_0)\right)|\psi(t_0)\rangle_\pm$$

Use eq (1) on l.h.s. :

$$H_\pm(t)\,\underbrace{|\psi(t)\rangle}_{U_I(t,t_0)|\psi(t_0)\rangle_I}{}_\pm = \left(i\frac{d}{dt}\,\hat{U}_\pm(t,t_0)\right)|\psi(t_0)\rangle_\pm$$

$\Rightarrow$

$$\left(i\frac{d}{dt}\,\hat{U}_\pm(t,t_0)\right)|\psi(t_0)\rangle_I = H_\pm(t)\,U_\pm^-(t,t_0)\,|\psi(t_0)\rangle_I$$

this holds for any state $|\psi(t_0)\rangle_I$

$\Rightarrow$ the operators acting on each side must be equal

$$\boxed{i\frac{d}{dt}\,\hat{U}_\pm(t,t_0) = H_\pm(t)\,U_\pm^-(t,t_0)} \qquad (*)$$

5) Let us show that the expression:

$$\hat{U}_I(t,t_0) = T\left[\exp\left(-i\int_{t_0}^{t} H_I(t')\,dt'\right)\right]$$

solves equation (*), with initial condition:

$$\hat{U}(t_0,t_0) = \mathbb{1}$$

Indeed:

$$i\frac{d}{dt}\hat{U}_I(t,t_0) = i\frac{d}{dt}\left\{T\exp\left(-i\int_{t_0}^{t}\hat{H}_I(t')\,dt'\right)\right\}$$

T is a linear operation on operators (prove it!), so it commutes with the derivative

$$= \cancel{i}\,T\left\{\cancel{-i}\,\hat{H}_I(t)\exp\left(-i\int_{t_0}^{t}\hat{H}_I(t')\,dt'\right)\right\}$$

$t \geq$ any time inside the time-ordered product, inside the integrals $\Rightarrow$ it is always pushed to the left

$$= \hat{H}_I(t)\left\{T\exp\left(-i\int_{t_0}^{t}\hat{H}_I(t')\,dt'\right)\right\} = \hat{H}_I(t)\hat{U}_I(t,t_0)$$

$\Rightarrow$ the diff. eq. is solved.

$\hat{U}_I(t_0,t_0) = \hat{\mathbb{1}} \Rightarrow$ we found THE solution to the 1st order eq. + initial condition.

3.]

1. Consider the integrand in eq (4):

- the numerator (by eq (12)) is:

$$|\langle \vec{p}_3, \vec{p}_4 | S - \mathbb{1} | \vec{p}_1, p_2 \rangle|^2 = |A_{i\to f}|^2 \, \delta^{(4)}\Big(\sum p_i - \sum p_f\Big)(2\pi)^4 \underbrace{(2\pi)^4 \delta^{(4)}(0)}_{= VT \text{ by } (8)} =$$

$$= |A_{if}|^2 VT \, \delta^{(4)}\Big(\sum p_i - \sum p_f\Big)(2\pi)^4$$

- ~~the denominator is~~ the denominator is:

$$\langle p_1 p_2 | p_1 p_2 \rangle \langle p_3 p_4 | p_3 p_4 \rangle =$$

$$= \langle \vec{p}_1 | \vec{p}_1 \rangle \langle \vec{p}_2 | \vec{p}_2 \rangle \langle \vec{p}_3 | \vec{p}_3 \rangle \langle \vec{p}_4 | \vec{p}_4 \rangle =$$

$$= (2\omega_1 V)(2\omega_2 V)(2\omega_3 V)(2\omega_4 V)$$

$$\Rightarrow \quad dP = \int_{\vec{p}_3 \to d\Omega} \frac{|A_{if}|^2 VT (2\pi)^4 \delta^{(4)}\left(\sum p_i - \sum p_f\right) d\Pi_3 \, d\Pi_4}{(2\omega_1 V_1)(2\omega_2 V_2)(2\omega_3 V)(2\omega_4 V)}$$

the integration measures are:

$$d\Pi_3 = n_{\vec{p}_3} d^3 p_3 \qquad d\Pi_4 = n_{\vec{p}_4} d^3 p_4$$

to find $n_{\vec{p}}$ use (6) and the definition

(9) for single-particle states:

$$\langle \vec{p} | \vec{q} \rangle = \langle \vec{p} | \mathbb{1} | \vec{q} \rangle = \frac{1}{N} \langle \vec{p} | N \mathbb{1} | \vec{q} \rangle$$

by (9), (10)

$$= \frac{1}{\langle \vec{p} | \vec{p} \rangle} \int d^3p' \, n_{\vec{p}'} \langle \vec{p} | \vec{p}' \rangle \langle \vec{p}' | \vec{q} \rangle$$

$$= \frac{1}{2\omega_p V} \int d^3p' \, n_{\vec{p}'} (2\pi)^3 2\omega_p \delta^{(3)}(\vec{p}-\vec{p}') \times (2\pi)^3 2\omega_q \delta^{(3)}(\vec{p}'-\vec{q})$$

$$= \frac{n_{\vec{p}} (2\pi)^3 2\omega_p \delta^{(3)}(\vec{p}-\vec{q}) (2\pi)^3 2\omega_p}{\cancel{2\omega_p} V} =$$

by (6) this must also be equal to

$$(2\pi)^3 2\omega_{\vec{p}} \delta^{(3)}(\vec{p}-\vec{q})$$

$$\Rightarrow \quad n_{\vec{p}} = \frac{V}{(2\pi)^3}$$

So:

$$dP = \int_{\vec{p}_3 \to d\Omega} \frac{|A_{if}|^2 \, V T (2\pi) \delta^{(4)}(\Sigma p_i - \Sigma p_f) \cancel{V}}{(2\omega_1 V)(2\omega_2 V)(2\omega_3 \cancel{V})(2\omega_4 \cancel{V})} \frac{d^3p_3}{(2\pi)^3} \cancel{V} \frac{d^3p_4}{(2\pi)^3}$$

From (4) and $F = \frac{1}{V}|\vec{v}_2 - \vec{v}_1|$:

$$d\sigma = \frac{\cancel{V}}{\cancel{T}|\vec{v}_2 - \vec{v}_1|}\frac{\cancel{V}\,\cancel{T}}{(2\omega_1 \cancel{V})(2\omega_2 \cancel{V})}\int_{\substack{\vec{p}_3 \to d\Omega \\ \vec{p}_4 \to \text{anywhere}}} |A_{if}|^2 \left\{(2\pi)^4 \delta^4\left(\sum_i p_i - \sum_f p_f\right) \frac{d^3 p_3}{(2\pi)^3 2\omega_3}\frac{d^3 p_4}{(2\pi)^3 2\omega_4}\right\}$$

$\omega_1 = E_1 \qquad \omega_2 = E_2$

$$= \frac{1}{2E_1\, 2E_2\, |\vec{v}_2 - \vec{v}_1|}\int_{\substack{\vec{p}_3 \to d\Omega \\ \vec{p}_4 \to \text{anywhere}}} |A_{i\to f}|^2 \, d\Pi_{LIPS}$$

2. In the CM frame:

$$\vec{p}_1 + \vec{p}_2 = 0 \qquad E_1 = \sqrt{p_1^2 + m_1^2} \qquad E_1 + E_2 \equiv E_{CM}$$

$$\to |\vec{p}_1| = |\vec{p}_2| \qquad E_2 = \sqrt{p_{\cancel{1}2}^2 + m_2^2}$$

$$\delta^{(4)}\left(p_1 + p_2 - (p_3 + p_4)\right) = \delta^{(3)}\left(-(\vec{p}_1 + \vec{p}_2) + \vec{p}_3 + \vec{p}_4\right)\delta\left((E_3 + E_4) - (E_1 + E_2)\right)$$

$$= \delta^{(3)}\left(\vec{p}_3 + \vec{p}_4\right)\delta\left((E_3 + E_4) - E_{CM}\right)$$

$$\int_{\substack{\vec{P}_3 \to d\Omega \\ \vec{P}_4 \to \text{anywhere}}} |A_{i\to f}|^2 \, d\Pi_{LIPS} =$$

Go to spherical coordinates in momentum space for P_3: $d\vec{P}_3^3 = P_3^2 \, dP_3 \, d\Omega$

$P_3 \equiv |\vec{P}_3|$

$$= d\Omega \int \frac{P_3^2 \, dP_3}{(2\pi)^3 2E_3} \int \frac{d\vec{P}_4^3}{(2\pi)^3 2E_4} \left[|A_{i\to f}|^2 \times (2\pi)^4 \delta(E_3+E_4-E_{CM}) \delta^{(3)}(\vec{P}_3+\vec{P}_4) \right]$$

we do not integrate over direction of $\vec{P}_3$.

$$= \frac{d\Omega}{4\pi^2} \int_0^{+\infty} \frac{dP_3 \, P_3^2}{2E_3 \, 2E_4} |A_{i\to f}|^2_{(\vec{P}_3, -\vec{P}_3)} \delta(E_3+E_4-E_{CM}) =$$

[call $\vec{P}_3 = \vec{P}_f$, $\vec{P}_4 = -\vec{P}_f$

$$E_3 = \sqrt{P_f^2 + m_3^2} \qquad E_4 = \sqrt{P_f^2 + m_4^2}$$

(for now look at the general case where m_1, m_2, m_3 and m_4 may all be different.

change variable from P_f to $x = E_3 + E_4 = \sqrt{P_f^2+m_3^2} + \sqrt{P_f^2+m_4^2}$,

$$dP_f = \left(\frac{dx}{dP_f}\right)^{-1} dx \qquad \frac{dx}{dP_f} = \frac{P_f}{E_3} + \frac{P_f}{E_4}$$

$$= P_f \frac{E_3+E_4}{E_3 E_4} = \frac{P_f \, x}{E_3 E_4} \Big]$$

$$\Rightarrow \quad dp_f = \frac{E_3 E_4}{(E_3+E_4)p_f}\, dx \qquad x \equiv E_3 + E_4$$

$$\rightarrow \quad \cancel{d\sigma} = \frac{d\Omega}{16\pi^2} \int_{x_{min}}^{+\infty} \frac{dx}{x}\, p_f\, |A_{i\to f}|^2\, \delta(x - E_{CM})$$

$$x_{min} = x(p_f = 0) = m_3 + m_4 > 0$$

$$= \begin{cases} 0 & \text{if } E_{CM} < m_3 + m_4 \\ \dfrac{d\Omega}{16\pi^2} |A_{i\to f}|^2 \dfrac{p_f}{E_{CM}} & \text{if } E_{CM} \geq m_3+m_4 \end{cases}$$

$$= \frac{d\Omega}{16\pi^2} \frac{p_f}{E_{CM}} |A_{i\to f}|^2\, \Theta(E_{CM} - (m_3 + m_4))$$

~~La probabilité de diffusion est zero~~

The scattering probability is zero if the center-of-mass energy is less than the sum of the final state masses: there is not enough energy to produce the final states.

Finally:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2E_1 2E_2} \frac{1}{|\vec{v}_2 - \vec{v}_1|} \frac{|A_{i\to f}|^2 p_f}{16\pi^2 E_{CM}} \times \theta\left(E_{CM} - (m_3 + m_4)\right)}$$

one must obtain p_f as a function of E_{CM} and the masses. this can be obtained knowing that:

$$E_3 + E_4 = E_{CM} \qquad (*)$$

$$E_3^2 - E_4^2 = \left(p_f^2 + m_3^2\right) - \left(p_f^2 + m_4^2\right) = m_3^2 - m_4^2$$

$$\Rightarrow (E_3 - E_4)(E_3 + E_4) = m_3^2 - m_4^2$$

$$\Rightarrow E_3 - E_4 = \frac{m_3^2 - m_4^2}{E_{CM}} \qquad (**)$$

From $(*)$ and $(**)$ we get:

$$E_3 = \frac{1}{2}\left(E_{CM} + \frac{m_3^2 - m_4^2}{E_{CM}}\right), \quad E_4 = \frac{1}{2}\left(E_{CM} - \frac{m_3^2 - m_4^2}{E_{CM}}\right)$$

and $p_f = \sqrt{E_3^2 - m_3^2}$

We can further simplify : $|\vec{V}_2 - \vec{V}_1| = \frac{|\vec{P}_1|}{E_1} + \frac{|\vec{P}_2|}{E_2}$

$$|\vec{P}_1| = P_i \qquad \vec{P}_2 = -\vec{P}_1$$
$$|\vec{P}_2| = P_i$$

$$\frac{d\sigma}{d\Omega} = \frac{|A_{i\to f}|^2 \, P_f}{64\pi^2 \left(\frac{P_i}{E_1} + \frac{P_i}{E_2}\right)(E_1 E_2)} \frac{1}{E_{CM}} \, \theta\left(E_{CM} - (m_3 + m_4)\right)$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{|A_{i\to f}|^2}{64\pi^2 E_{CM}^2} \frac{P_f}{P_i} \, \theta\left(E_{CM} - (m_3 + m_4)\right)}$$

this is the general result for all masses arbitrary.

If $m_1 = m_2 = m_3 = m_4 \equiv m$

$$\Rightarrow P_1 = P_2 = P_3 = P_4 = \frac{1}{2} E_{CM}$$

also $E_{CM} \geq 2m$ by definition

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{|A_{i\to f}|^2}{64\pi^2 E_{CM}^2}}$$

All particles have the same mass

$2 \to 2$ scattering amplitude in φ^4 theory.

4.2) Consider $H_{int} = \int d^3x \, \frac{\lambda}{4!} \varphi^4(x)$

suppose we want to find the amplitude for $i \longrightarrow f$ with

$$i = |P_1, P_2\rangle \qquad f = |P_1', P_2'\rangle$$

We have to calculate:

$$\langle P_1', P_2' | S - \mathbb{1} | P_1, P_2 \rangle \overset{\text{to lowest order in } \lambda}{\simeq}$$

$$\langle P_1', P_2' | -i \int_{-\infty}^{+\infty} dt \, H_I(t) | P_1, P_2 \rangle$$

We have to time-order this

Remember: $T(\quad) = :\quad: +$ Contraction.

$$|P_1, P_2\rangle = \sqrt{2w_1}\sqrt{2\omega_2}\, a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} |0\rangle$$

$$|P_1', P_2'\rangle = \sqrt{2\omega_1'}\sqrt{2\omega_2'}\, a^\dagger_{\vec{p}_1'} a^\dagger_{p_2'} |0\rangle$$

so what we have is :

$$\langle 0| a_{\vec{p}_1'} a_{\vec{p}_2'} \int d^4x\, \phi^4(x)\, a^\dagger_{\vec{p}_1} a^\dagger_{\vec{p}_2} |0\rangle$$

$$\times \; -\frac{i\lambda}{4!} \sqrt{2\omega_1 2\omega_2 2\omega_1' 2\omega_2'}$$

now

$$\int d^4x\, \phi^4(x) = \int d^4x \int \frac{d^3q_1}{(2\pi)^3\sqrt{2\omega_{q_1}}} \left(a_{q_1} e^{-iq_1x_0} + a^\dagger_{q_1} e^{iq_1x_0}\right)$$

$$\times \int \frac{d^3q_2}{(2\pi)^3\sqrt{2\omega_2}} \left(a_{q_2} + a^\dagger_{q_2}\right) \int \frac{d^3q_3}{(2\pi)^3\sqrt{2\omega_3}} \left(a_{q_3} + a^\dagger_{q_3} \cdots\right)$$

$$\int \frac{d^3q_4}{(2\pi)^3\sqrt{2\omega_4}} \left(a_{q_4} + a^\dagger_{q_4}\right)$$

only terms with an equal # of a and $a^\dagger$ survive.

$\Rightarrow$ We have to pick a term from ϕ^4 with 2 $a^\dagger$'s and 2 a's.

Moreover, any term like:

$$\langle 0| a_{p_1'} a_{p_2'} \; a_{q_1} a^\dagger_{q_2} a_{q_3} a^\dagger_{q_4} \; a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle$$

gives the same as:

$$\langle 0| a_{p_1'} a_{p_2'} a^\dagger_{q_2} a^\dagger_{q_4} a_{q_1} a_{q_3} a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle$$

the difference has internal commutators like

$$[a_{q_1}, a^\dagger_{q_2}] = \delta^3(q_2 - q_1)$$

$$[a_{q_3}, a^\dagger_{q_4}] = \delta^3(q_4 - q_3)$$

and leaves

$$\doteq \langle 0| a_{p_1'} a_{p_2'} a^\dagger_{p_1} a^\dagger_{p_2} |0\rangle$$

$$\doteq \delta^{(4)}(p_1' - p_1)\,\delta^{(4)}(p_2' - p_2)$$

+ permutations

these are not interesting (and in fact zero ~~null~~ unless some of the initial p's are equal to the final p's

$\Rightarrow$ We can look at the normal ordered operator $:\phi^4(x):$ $\rightarrow$ all the a's are on the right

$$\rightarrow = \int d^4x \int \prod_{i=1}^{4} \frac{d^3q_i}{(2\pi)^3 \sqrt{2\omega_i}} \langle 0 | a_{p_1'} a_{p_2'} a^\dagger_{\vec{q}_1} a^\dagger_{\vec{q}_2} a_{\vec{q}_3} a_{\vec{q}_4} a^\dagger_{p_1} a^\dagger_{p_2} | 0 \rangle \, e^{ixq_1 + ixq_2 - ixq_3 - ixq_4}$$

$+$ permutations

$$\times \left(-i\frac{\lambda}{4!}\right) \sqrt{2\omega_1 2\omega_2 2\omega_1' 2\omega_2'}$$

Now use $[a_{\vec{q}_i}, a^\dagger_{\vec{p}_i}] = (2\pi)^3 \delta^3(\vec{q}_i - \vec{p}_i)$

$= \cancel{\int d^4x \int \prod \frac{dq_i}{(2\pi)^3 \sqrt{2\omega_i}}}$ the matrix element is :

$$\langle 0 | a_{p_1'} a^\dagger_{q_1} a_{p_2'} a^\dagger_{q_2} a_{q_3} a_{q_4} a^\dagger_{p_1} a^\dagger_{p_2} | 0 \rangle$$

$$+ \langle 0 | a_{p_1'} a^\dagger_{q_2} a_{q_3} a_{q_4} a^\dagger_{p_1} a^\dagger_{p_2} | 0 \rangle \times \delta^{(3)}(q_1 - p_2')(2\pi)^3$$

$$= (2\pi)^3 \delta^{(3)}(q_1 - p_1') \langle 0 | a_{p_2'} a^\dagger_{q_2} a_{q_3} a_{q_4} a^\dagger_{p_1} a^\dagger_{p_2} | 0 \rangle$$

$$(2\pi)^3 \delta^{(3)}(q_1 - p_2') \langle 0 | a_{p_1'} a^\dagger_{q_2} a_{q_3} a_{q_4} a^\dagger_{p_1} a^\dagger_{p_2} | 0 \rangle$$

$$= \left((2\pi)^3\right)^4 \delta^{(3)}(q_1 - p_1') \delta^3(q_2 - p_2') \delta^{(3)}(q_4 - p_1) \delta^3(q_3 - p_2)$$

+ All possible permutations –

Let us look at one term:

$$= -\frac{i\lambda}{4!}\sqrt{2\omega_1 2\omega_2 2\omega_1' 2\omega_2'} \times$$

$$\times \int d^4x \int \frac{d^3q_1 d^3q_2 d^3q_3 d^3q_4}{\cancel{(2\pi)^3(2\pi)^3(2\pi)^3(2\pi)^3}} \frac{\cancel{((2\pi)^3)^4}}{\sqrt{2\omega_1' 2\omega_2' 2\omega_3 2\omega_4}} \times$$

$$e^{ix(q_1+q_2-q_3-q_4)}$$

$$\delta^{(3)}(q_1-P_1')\,\delta^{(3)}(q_2-P_2')\,\delta^{(3)}(q_4-P_1)\,\delta^{(3)}(q_3-P_2)$$

~~3 and~~ 4 delta-functions + 4 integrations

$\longrightarrow$

$$-i\frac{\lambda}{4!}\frac{\cancel{\sqrt{2\omega_1 2\omega_2 2\omega_2' 2\omega_1'}}}{\cancel{\sqrt{2\omega_1 2\omega_2 2\omega_1' 2\omega_2}}} \times \underbrace{\overbrace{\int d^4x\, e^{i(P_1'+P_2'-P_1-P_2)}}^{(2\pi)^4\delta^{(4)}(P_f-P_i)}}_{\text{overall}}$$

overall δ-function.
We take it out...

So each term gives

$$-\frac{i\lambda}{4!}(2\pi)^4\delta^{(4)}(P_f-P_i)$$

How many terms?

we can contract $\vec{q}_1$ with 4 momenta (p_1, p_2, p_1', p_2');

then $\vec{q}_2$ with 3 momenta ; etc

$\Rightarrow$ 4! terms

$$\boxed{\langle f | S - \mathbb{1} | i \rangle = -i\lambda (2\pi)^4 \delta^{(4)}(p_f - p_i)}$$

To lowest order in λ

the next order would be:

$$\langle f | \frac{\lambda^2}{(4!)^2} T \int d^4x \, \varphi^4(x) \int d^4y \, \varphi^4(y) | i \rangle$$