QFT - PS3 Solutions

1| $(\Box + m^2)\phi = J$

1. $J(t<t_0) = 0 \quad , \quad \phi(t<t_0) = 0$

$$\Rightarrow \quad \phi(\vec{x},t) = \int d^3x' \int_{-\infty}^{+\infty} dt' \, G_R(\vec{x}-\vec{x}'; t-t') J(\vec{x}', t')$$

$$G_R(t-t') = 0 \text{ if } t<t'$$

Indeed: if $t<t_0$

$$\phi(t) = \int_{-\infty}^{+\infty} dt' \, G_R(t-t') J(t') = \int_{-\infty}^{t_0} dt' \, G_R(t-t') \underbrace{J(t')}_{=0} + \int_{t_0}^{+\infty} dt' \, \underbrace{G_R(t-t')}_{=0} J(t')$$

$$t - t' < t_0 - t' < 0$$

$$= 0$$

2. $J(\vec{x},t) = J_0 \theta(t) e^{-\mu t}$

$$\phi(\vec{x},t) = \int d^3x' \int_{-\infty}^{+\infty} dt' \, G_R(x-x', t-t') \, \theta(t') e^{-\mu t'}$$

$$G_R = \int \frac{d^3p}{(2\pi)^3} \int_{C_R} \frac{d\omega}{2\pi} \, \frac{e^{+i\vec{p}\cdot\vec{x} - i\omega_p t}}{p^2 - m^2}$$

$$\phi(\vec{x},t) = \int d^3x' \int dt' \int \frac{d^3p}{(2\pi)^3} \int_{C_R} \frac{d\omega}{2\pi} \frac{1}{p^2-m^2} \, e^{i\vec{p}\cdot(\vec{x}-\vec{x}')} e^{-i\omega(t-t')} J_0 \, \theta(t') e^{-\mu t'}$$

Let's take care of $\vec{p}$ and $\vec{x}$ first:

$$\int \frac{d\omega}{2\pi} \cancel{\int \frac{d^3p'}{(2\pi)^3} \delta^3(\vec{p}') \frac{(2\pi)^3}{p^2-m^2} e^{-i\vec{p}\cdot\vec{x}}} \int dt' \, e^{-i\omega(t-t')-\mu t'} J_0 \, \theta(t')$$

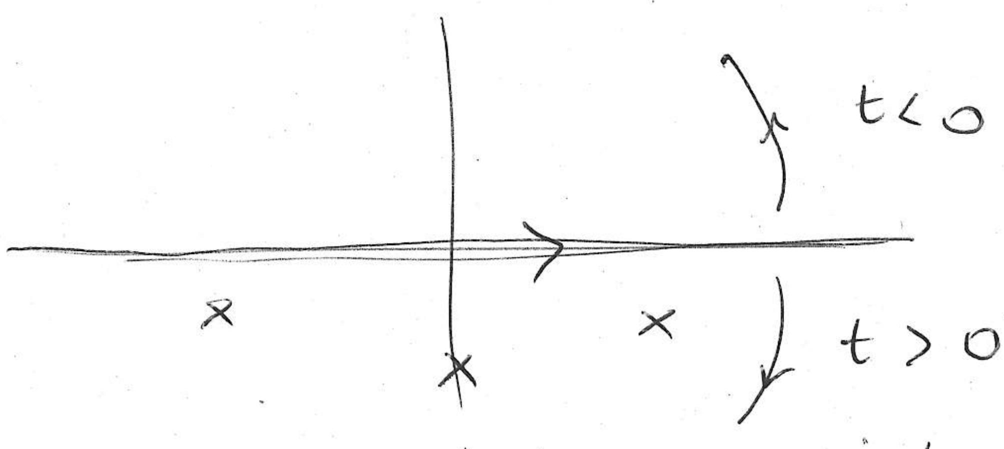
$$= \int_{C_R} \frac{d\omega}{2\pi} \frac{1}{\omega^2-m^2} \int_0^\infty dt' \, e^{-i\omega(t-t')-\mu t'}$$

↑ Do this

$$= \int_{C_R} \frac{d\omega}{2\pi} \frac{1}{\omega^2-m^2} \frac{e^{-i\omega t}}{(+i\omega-\mu)}$$

now the last one: $\hookrightarrow \dfrac{1}{i(\omega+i\mu)(\omega-m)(\omega+m)}$

pole at $\omega = -i\mu$



no poles

3 poles

$$= -\frac{2\pi i}{2\pi} \left[\frac{e^{imt}}{(-2m)(-m+i\mu)} + \frac{e^{-imt}}{(2m)(m+i\mu)} + \frac{e^{-\mu t}}{i(-i\mu-m)(-i\mu+m)} \right]$$

$$(i\mu+m)(i\mu-m) = -(m+i\mu)(m-i\mu) = -(m^2+\mu^2)$$

$$= -i\left[\frac{e^{imt}}{i2m(m-i\mu)} + \frac{e^{-imt}}{i2m(m+i\mu)} - \frac{e^{-\mu t}}{i(\mu^2+m^2)}\right]$$

$$= \frac{e^{-\mu t}}{\mu^2+m^2} - \left(\frac{e^{imt}}{2m(m-i\mu)} + \frac{e^{-imt}}{2m(m+i\mu)}\right)$$

3. As $t \to +\infty$ the fist term goes away, and what's left is:

$$\frac{e^{imt}}{2m(m-i\mu)} + \frac{e^{-imt}}{2m(m+i\mu)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2m}} \left[\frac{(2\pi)^3\delta^3(p)}{\sqrt{2m}(m+i\mu)} e^{-imt}\right. + \text{c.c.}$$

since $\vec{p} = 0$ anyway:

$$= \int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{1}{\sqrt{2m}}}_{\sqrt{2\omega_p}} \left[\frac{(2\pi)^3\delta^3(\vec{p})}{\underbrace{\sqrt{2m}(m+i\mu)}_{\sqrt{2\omega_p}(\omega_p+i\mu)}} e^{-i\omega t + i\vec{p}\cdot\vec{x}}\right. + \text{c.c.}$$

$$\boxed{Q(\vec{p}) = \frac{(2\pi)^3\delta^3(\vec{p})}{\sqrt{2\omega_p}(\omega_p+i\mu)}}$$

2]

$$(-\nabla^2+m^2)\,G(\vec{x}) = \delta^{(3)}(\vec{x}) \qquad (6)$$

1) Write G as a Fourier trasform:

$$G(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\, \tilde{G}(\vec{k})$$

Apply $(-\nabla^2+m^2)$ and use (6)

$$(-\nabla^2+m^2)G(\vec{x}) = \int \frac{d^3k}{(2\pi)^3}\,(-\nabla^2+m^2)\, e^{i\vec{k}\cdot\vec{x}}\,\tilde{G}(\vec{k})$$

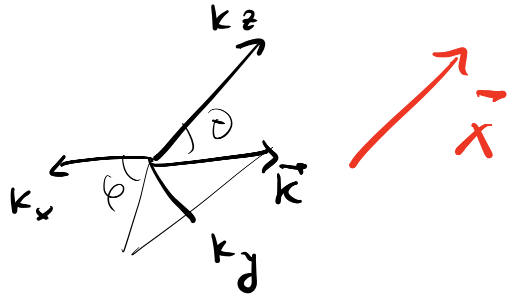
$$= \int \frac{d^3k}{(2\pi)^3}\,(|\vec{k}|^2+m^2)\, e^{i\vec{k}\cdot\vec{x}}\,\tilde{G}(\vec{k}) = \underbrace{\int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}}_{\delta^{(3)}(\vec{x})}$$

Inverting the Fourier-trasform:

$$(|\vec{k}|^2+m^2)\,\tilde{G}(\vec{k}) = 1 \quad \Rightarrow \quad \boxed{\tilde{G}(\vec{k}) = \frac{1}{|\vec{k}|^2+m^2}}$$

2) $\displaystyle\int d^3k \, \frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2+m^2}$ integral depends on a fixed vector $\vec{x}$

Go to spherical coordinates, choosing the $\hat{k}_z$-axis aligned with $\vec{x}$



$$\vec{k}\cdot\vec{x} = |\vec{k}||\vec{x}|\cos\theta$$

$$\int d^3k = \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\varphi \, k^2 \sin\theta \qquad k \equiv |\vec{k}|$$

$$\int d^3k \, \frac{e^{i\vec{k}\cdot\vec{x}}}{|\vec{k}|^2+m^2} = 2\pi \int_0^\infty \frac{dk\, k^2}{k^2+m^2} \underbrace{\int_0^\pi d\theta \sin\theta \, e^{ik|\vec{x}|\cos\theta}}_{\int_{-1}^{1} d\cos\theta \, e^{ik|\vec{x}|\cos\theta}} =$$

$$\int_{-1}^{1} d\cos\theta \, e^{ik|\vec{x}|\cos\theta} = \frac{e^{ik|\vec{x}|} - e^{-ik|\vec{x}|}}{ik|\vec{x}|}$$

$$= \frac{2\pi}{i|\vec{x}|} \int_0^\infty dk \, \frac{k}{k^2+m^2} \left(e^{ik|\vec{x}|} - e^{-ik|\vec{x}|} \right) = (*)$$

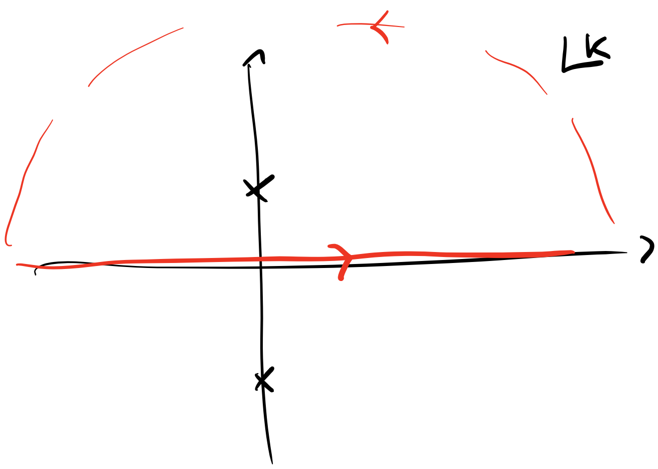
we can write the second term as follows:

$$-\int_0^{\infty} dk\, e^{-ik|\vec{x}|}\,\frac{k}{k^2+m^2} \underset{k\to -k}{=} +\int_0^{-\infty} dk\, e^{ik|\vec{x}|}\,\frac{(-k)}{k^2+m^2} = +\int_{-\infty}^{0} \frac{k}{k^2+m^2}\, e^{ik|\vec{x}|}$$

$\Rightarrow$ same integrand as the first term, but integrated between $-\infty$ and 0

$$(*) = \frac{2\pi}{i|\vec{x}|}\int_{-\infty}^{+\infty} dk\,\frac{k}{k^2+m^2}\, e^{ik|\vec{x}|} =$$

We can do this by Cauchy's theorem:



- poles at $k = \pm im$
- close contour in the upper half-plane

$e^{ik|\vec{x}|} \sim e^{-\mathrm{Im}\,k\,|\vec{x}|} \to 0$ for $\mathrm{Im}\,k > 0$ and large

$$= \frac{2\pi}{i|\vec{x}|}\times(2\pi i)\,\frac{k\, e^{ik|x|}}{(k+im)\cancel{(k-im)}}\Bigg|_{k=im} = \frac{(2\pi)^2}{|\vec{x}|}\,\frac{e^{-m|\vec{x}|}}{2}$$

Put back the $1/(2\pi)^3$

$$\Rightarrow \boxed{G(\vec{x}) = \frac{e^{-m|\vec{x}|}}{4\pi|\vec{x}|}}$$

3]

1) $\langle 0| \hat{\phi}(x) \hat{\phi}(x') |0\rangle =$

$$= \langle 0| \left(\left(\int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} \hat{a}_p e^{-ip\cdot x} + \hat{a}^\dagger_p e^{ip\cdot x} \right) \times \right.$$

$$\left. \left(\int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_q}} \hat{a}_q e^{-iq\cdot x'} + \hat{a}^\dagger_q e^{iq\cdot x'} \right) |0\rangle = \right.$$

$\hat{a}$ acting on $|0\rangle = 0$

$\hat{a}^\dagger$ " " $\langle 0| = 0$

$$= \langle 0| \int \frac{d^3p}{(2\pi)^3} \frac{a_p e^{-ip\cdot x}}{\sqrt{2\omega_p}} \int \frac{d^3q}{(2\pi)^3} \frac{a^\dagger_q e^{iq\cdot x'}}{\sqrt{2\omega_q}} |0\rangle =$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^2} \frac{e^{-ip\cdot x + iq\cdot x'}}{\sqrt{2\omega_p 2\omega_q}} \langle 0| \hat{a}_p \hat{a}^\dagger_q |0\rangle =$$

$$\langle 0| \hat{a}_p \hat{a}^\dagger_q |0\rangle = \underbrace{\langle 0| \hat{a}^\dagger_q \hat{a}_p |0\rangle}_{=0} + \langle 0| \underbrace{[a_p, a^\dagger_q]}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \hat{\mathbb{1}}} |0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \int \frac{d^3q}{(2\pi)^2} \frac{e^{-ip\cdot x + iq\cdot x'}}{\sqrt{2\omega_p 2\omega_q}} (2\pi)^3 \delta^{(3)}(\hat{p} - \vec{q}) =$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{e^{-ip\cdot(x-x')}}{2\omega_{\vec{p}}} = D_f(x-x')$$

$$D_-(x-x') = D_+(x'-x) = \langle 0|\,\hat{\phi}(x')\,\hat{\phi}(x)\,|0\rangle$$

$$\Downarrow$$

$$-i\,G_F(x-x') = \theta(t-t')\,\langle 0|\,\phi(x)\,\phi(x')\,|0\rangle + \theta(t'-t)\,\langle 0|\,\phi(x')\,\phi(x)\,|0\rangle$$

$$= \langle 0|\Big(\theta(t-t')\,\phi(x)\,\phi(x') + \theta(t'-t)\,\phi(x')\,\phi(x)\Big)|0\rangle$$

$$= \langle 0|\,T\big(\phi(x)\,\phi(x')\big)\,|0\rangle$$

2) set $t' = 0$ for simplicity

$$G_R = G_F - iD_- = i\theta(t)\,\langle 0|\,\phi(x)\phi(0)\,|0\rangle + i\theta(-t)\,\langle 0|\,\phi(0)\phi(x)\,|0\rangle$$

$$-i\,\langle 0|\,\phi(0)\,\phi(x)\,|0\rangle =$$

$$= i\theta(t)\,\langle 0|\,\phi(x)\,\phi(0)\,|0\rangle + i\underbrace{\big(\theta(-t) - 1\big)}_{-\theta(t)}\langle 0|\,\phi(0)\,\phi(x)\,|0\rangle$$

$$= i\theta(t)\,\langle 0|\Big(\phi(x)\,\phi(0) - \phi(0)\,\phi(x)\Big)|0\rangle =$$

$$=0 \;\; \boxed{G_R(x) = i\,\theta(t)\,\langle 0|\,[\hat{\phi}(x), \hat{\phi}(0)]\,|0\rangle}$$

similarly, from $G_A = G_F - iD_+$ one gets:

$$G_A(x) = -i\theta(-t)\,\langle 0|\,[\hat{\phi}(x), \hat{\phi}(0)]\,|0\rangle$$

3) $$\Delta_F(p_1,p_2) = \int d^4x_1\, d^4x_2\, e^{ip_1\cdot x_1} e^{ip_2\cdot x_2} \Delta_F(x_1-x_2)$$

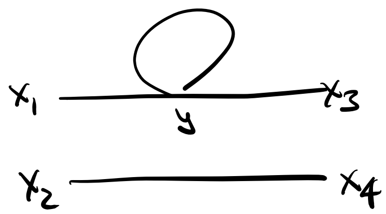
$$= \int d^4x_1 d^4x_2\, e^{ip_1\cdot x_1} e^{ip_2\cdot x_2} \int \frac{d^4q}{(2\pi)^4} \frac{ie^{-iq\cdot(x_1-x_2)}}{q^2-m^2+i\epsilon} =$$

$$= i\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2-m^2+i\epsilon} \underbrace{\int d^4x_1\, e^{i(p_1-q)x_1}}_{(2\pi)^4\delta^{(4)}(p_1-q)} \underbrace{\int d^4x_2\, e^{i(p_2+q)x_2}}_{(2\pi)^4\delta^{(4)}(p_2+q)}$$

$$= i\int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2-m^2+i\epsilon} (2\pi)^4 \delta^{(4)}(q-p_1)(2\pi)^4\delta^{(4)}(q+p_2)$$

$$= (2\pi)^4 \delta^{(4)}(p_1+p_2) \frac{i}{p_1^2-m^2+i\epsilon}$$

4)



$$= \Delta_F(x_4 - x_2)\int d^4y\, \Delta_F(y-x_1)\Delta_F(x_3-y)\,\Delta_F(0)$$

$$\equiv \Delta_{NC}(x_1, x_2, x_3, x_4) \qquad NC \equiv \text{non-connected}$$

$$\tilde{\Delta}_{nc}(p_1,p_2,p_3,p_4) = \int d^4x_1 \dots d^4x_4\, e^{ip_1\cdot x_1 + ip_2\cdot x_2 + ip_3\cdot x_3 + ip_4\cdot x_4}\, \Delta_{nc}(x_1,x_2,x_3,x_4)$$

$$= \left(\int d^4x_2\, d^4x_4\, \Delta_F(x_4-x_2)\, e^{ip_2\cdot x_2 + ip_4\cdot x_4}\right) \times$$

$$\left(\int d^4x_1\, d^4x_3\, e^{ip_3\cdot x_3 + ip_4\cdot x_4}\int d^4y\, \Delta_F(y-x_1)\Delta_F(x_3-y)\right)\Delta_F(0)$$

$$= \left[(2\pi)^4\, \delta^{(4)}(p_2+p_4)\frac{i}{p_2^2 - m^2 + i\epsilon}\right] \times \quad \Delta_F(0) \times$$

see part 3

$$\times \int d^4x_1 d^4x_3 d^4y\, e^{ip_1\cdot x_1 + ip_3\cdot x_3}\int \frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\frac{i e^{-iq_1\cdot(y-x_1)}\, i e^{-iq_3(x_3-y)}}{(q_1^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)}$$

$$= \left[(2\pi)^4\, \delta^{(4)}(p_2+p_4)\frac{i}{p_2^2-m^2+i\epsilon}\right]\times \Delta_F(0)\times$$

$$\times\, i^2 \int\frac{d^4q_1\, d^4q_3}{(2\pi)^4(2\pi)^4}\int d^4x_1 d^4x_3 d^4y\, \frac{e^{i(p_1+q_1)\cdot x_1}\, e^{i(p_3-q_3)\cdot x_3}\, e^{i(q_3-q_1)\cdot y}}{(q_1^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)} =$$

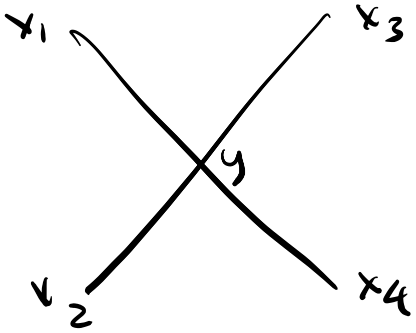
now do the x_1, x_3 and y integrals:

$$= (2\pi)^4 \delta^{(4)}(p_2+p_4) \frac{i}{p_2^2 - m^2 + i\epsilon} \times \Delta_F(0)$$

$$\times i^2 \int \frac{d^4 q_1}{(2\pi)^4} \int \frac{d^4 q_3}{(2\pi)^4} \frac{(2\pi)^4\delta^{(4)}(p_1+q_1)(2\pi)^4\delta^{(4)}(p_3-q_3)(2\pi)^4\delta^{(4)}(q_3-q_1)}{(q_1^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)}$$

$$= i^2 \frac{\left[(2\pi)^4\delta^{(4)}(p_2+p_4)\right]\times\left[(2\pi)^4\delta^4(p_3+p_1)\right]}{(p_1^2-m^2+i\epsilon)(p_3^2-m^2+i\epsilon)} \underbrace{\int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}}_{\Delta_F(0)}$$

in the disconnected diagram, the momentum in each line is separately conserved.



$$= \int d^4y\, \Delta_F(y-x_1)\,\Delta_F(y-x_2)\,\Delta_F(x_3-y)\,\Delta_F(x_4-y)$$

$$\equiv \Delta_c(x_1, x_2, x_3, x_4)$$

$c \equiv$ connected

$$\tilde{\Delta}_c(p_1, p_2, p_3, p_4) =$$

$$= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \, e^{ip_1\cdot x_1 + ip_2\cdot x_2 + ip_3\cdot x_3 + ip_4\cdot x_4} \quad \Delta_C(x_1, x_2, x_3, x_4) =$$

$$= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \, d^4y \; e^{ip_1\cdot x_1 + ip_2\cdot x_2 + ip_3\cdot x_3 + ip_4\cdot x_4} \quad \Delta_F(y-x_1)\Delta_F(y-x_2)\Delta_F(x_3-y)\Delta_F(x_4-y) =$$

now introduce the Fourier-transforms of each Δ_F

$$= \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 \, d^4y \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4q_3}{(2\pi)^4} \int \frac{d^4q_4}{(2\pi)^4} \times$$

$$\times \; e^{ip_1\cdot x_1 + ip_2\cdot x_2 + ip_3\cdot x_3 + ip_4\cdot x_4} \frac{e^{-iq_1\cdot(y-x_1) - iq_2\cdot(y-x_2) - iq_3\cdot(x_3-y) - iq_4\cdot(x_4-y)}}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)(q_4^2-m^2+i\epsilon)}$$

do the $x_1 \ldots x_4$ integral $\rightarrow$ 4 δ-functions

$$= \int d^4y \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \int \frac{d^4q_3}{(2\pi)^4} \int \frac{d^4q_4}{(2\pi)^4} \; (2\pi)^4 \times (2\pi)^4 \times (2\pi)^4 \times (2\pi)^4$$

$$\times \; \delta^{(4)}(p_1+q_1)\,\delta^{(4)}(p_2+q_2)\,\delta^{(4)}(p_3-q_3)\,\delta^{(4)}(p_4-q_4)$$

$$\times \; \frac{e^{-i(q_1+q_2-q_3-q_4)\cdot y}}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)(q_4^2-m^2+i\epsilon)} =$$

now do the $d^4q_1 \dots d^4q_4$ integrals $\Rightarrow$ $p_1 = -q_1 \quad p_3 = q_3$
$p_2 = -q_2 \quad p_4 = q_4$

$$= \int d^4y \, \frac{e^{i(p_1+p_2+p_3+p_4)\cdot y}}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)(q_4^2-m^2+i\epsilon)}$$

do the y-integral $\Rightarrow$ gives another δ

$$= (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3+p_4) \frac{1}{(q_1^2-m^2+i\epsilon)(q_2^2-m^2+i\epsilon)(q_3^2-m^2+i\epsilon)(q_4^2-m^2+i\epsilon)}$$

connected graph: only one overall δ-function

3

$$T\,\phi(x_1)\ldots\phi(x_n) = :\phi(x_1)\cdots\phi(x_n): + \sum :\text{all possible contractions}:$$

- it is true for $n=2$, since we proved

$$T\,\phi(x)\phi(y) = :\phi(x)\phi(y): + \Delta_F(x-y)\,\mathbb{1}$$

(*) Assume true for $n-1$ fields. then

$$T(\phi(x_1)\underbrace{\phi(x_2)\ldots\phi(x_n)}_{n-1}) =$$

Assume $t_1 > t_2, t_3 \ldots t_n$ (we can always move the latter field to the front inside T)

$$= \phi(x_1)\,T\,\phi(x_2)\ldots\phi(x_n) \overset{\text{by assumption } (*)}{=}$$

$$= \phi(x_1)\left[:\phi(x_2)\cdots\phi(x_n): + :\text{contraction of } n-1 \text{ fields}:\right]$$

$=$ now move $\phi(x_1)$ inside the $:\ :$

$\Rightarrow$ ~~every time~~ $\phi(x_1) = \underset{\substack{\uparrow\\ a^{\dagger}\text{'s}}}{\phi_+(x_1)} + \underset{\substack{\uparrow\\ a\text{'s}}}{\phi_-(x_1)}$

$\phi_-(x_1)$ is already normal-ordered in front. We have to move ϕ_+ to the right every-time $\phi_+(x_1)$ commutes with one of the $\phi_-(x_i)$ we get a contraction

$$[\phi_+(x_1), \phi_-(x_i)] = iD_+(x_1 - x_i)$$

$$= \underbrace{i\,\Theta(t_1 - t_i)}_{\substack{\text{this is } 1 \\ \text{since we assumed } t_1 > t_i}} D_+(x_1 - x_i) + \underbrace{i\,\Theta(t_i - t_1)}_{\substack{\text{this is zero} \\ \text{anyway} \\ \text{since } t_i < t_1}} D_-$$

$$\rightarrow \phi(x_1) :\phi(x_2)\cdots\phi(x_n): =$$

$$= :\phi(x_1)\cdots\phi(x_n): +$$

$$+ \sum_i{}' D_F(x_1 - x_i) :\prod_{J \neq i,1} \phi(x_J):$$

and the same with the other terms with more contractions.