1 - Massive Spin-1

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial A_\mu$$

1) Under $A_\mu \to A_\mu + \partial_\mu \alpha$:

. $F_{\mu\nu} \longrightarrow F_{\mu\nu}$ so the first term is invariant

. $A_\mu A^\mu \longrightarrow (A_\mu + \partial_\mu \alpha)(A^\mu + \partial^\mu \alpha) = A_\mu A^\mu + (\partial_\mu \alpha)(\partial^\mu \alpha)$
$+ 2A^\mu \partial_\mu \alpha$
$\neq A_\mu A^\mu$

$m^2 A_\mu A^\mu$ ~~this~~ is not invariant

2) Write $\mathcal{L} = -\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$
$+ \frac{m^2}{2} g^{\mu\nu} A_\mu A_\nu$

. $$\partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha A_\beta} = -\frac{1}{4} \partial_\alpha \Big[g^{\mu\rho} g^{\nu\sigma} (\delta^\alpha_\mu \delta^\beta_\nu - \delta^\alpha_\nu \delta^\beta_\mu)(\partial_\rho A_\sigma - \partial_\sigma A_\rho)$$
$$+ g^{\mu\rho} g^{\nu\sigma} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\delta^\alpha_\rho \delta^\beta_\sigma - \delta^\alpha_\sigma \delta^\beta_\rho)$$

$$= -\frac{1}{4} \partial_\alpha \Big[\partial^\alpha A^\beta - \partial^\beta A^\alpha - \partial^\beta A^\alpha + \partial^\alpha A^\beta$$
$$+ \partial^\alpha A^\beta - \partial^\beta A^\alpha - \partial^\beta A^\alpha + \partial^\alpha A^\beta \Big]$$

$$= - \partial_\alpha F^{\alpha\beta}$$

. $$\frac{\partial \mathcal{L}}{\partial A_\beta} = m^2 A^\beta$$

$\Rightarrow$ EL equations: $-\partial_\alpha \frac{\partial \mathcal{L}}{\partial \partial_\alpha A_\beta} + \frac{\partial \mathcal{L}}{\partial A_\beta} = 0$

are:

$$\Rightarrow \boxed{\partial_\alpha F^{\alpha\beta} + m^2 A^\beta = 0}$$

3) Act with ∂_β on EL equations and use

$\partial_\alpha \partial_\beta F^{\alpha\beta} = 0$ identically

$$\Rightarrow \boxed{m^2 \partial_\beta A^\beta = 0}$$

this looks like Lorentz's gauge in the $m^2=0$ case, but this follows from the field equations and it is not a choice. On the other hand, the Lagrangian is not gauge-invariant.

4) the field equations are $\partial^\mu F_{\mu\nu} + m^2 A_\nu = 0$

. for $\nu = 0$:

$$\partial^\mu F_{\mu 0} + m^2 A_0 = 0 \qquad \partial^\mu F_{\mu 0} = \partial^i F_{i0}$$

$$= \partial^i (\partial_i A_0 - \partial_0 A_i)$$

$$\Rightarrow -\vec{\nabla}^2 A_0 + \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A} + m^2 A_0 = 0$$

or $(-\vec{\nabla}^2 + m^2) A_0 = -\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A}$

this equation has no time-derivative acting on A_0

we can solve this by writing

$$A_0(\vec{x},t) = -\int d^3y \; G(\vec{x}-\vec{y}) \frac{\partial}{\partial t} \vec{\nabla}_y \cdot \vec{A}(\vec{y},t)$$

where $G(\vec{x}-\vec{y})$ is the Green's function for the time-independent operator, $-\nabla^2+m^2$:

$$(-\nabla_x^2 + m^2)\, G(\vec{x}-\vec{y}) = \delta^{(3)}(\vec{x}-\vec{y})$$

Therefore $A_0(\vec{x},t)$ is non-dynamical but it can be eliminated in terms of $\vec{A}(\vec{x},t)$ at each instant.

5) take $\nu = i$ in the field equation:

$$0 = \partial^\mu F_{\mu i} + m^2 A_i = \partial^0 F_{0i} + \partial^J F_{Ji} + m^2 A_i$$

$$= \frac{\partial}{\partial t}\left(\frac{\partial A_i}{\partial t} - \partial_i A_0\right) + \partial^J\left(\partial_J A_i - \partial_i A_J\right) + m^2 A_i$$

$$= \frac{\partial^2}{\partial t^2} A_i - \nabla^2 A_i - \partial_i \underbrace{\left[\frac{\partial A_0}{\partial t} + \partial^J A_J\right]}_{=0 \text{ by the constraint}} + m^2 A_i$$

$$\Rightarrow \quad \boxed{(\Box + m^2) A_i = 0}$$

6) $F_{\mu\nu} F^{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$$= 2(\partial_\mu A_\nu)(\partial^\mu A^\nu) - 2(\partial_\mu A_\nu)(\partial^\nu A^\mu) =$$

$$= -2 A_\nu \partial^\mu \partial_\mu A^\nu + 2 \partial_\mu (A_\nu \partial^\mu A^\nu)$$

$$+ 2 A_\nu \partial^\nu \partial_\mu A^\mu - 2 \partial_\mu (A_\nu \partial^\nu A^\mu)$$

$$= -2 \eta_{\mu\nu} A^\mu \Box A^\nu + 2 A^\nu \partial_\nu \partial_\mu A^\mu$$

+ total divergence

$$\Rightarrow \mathcal{L} = \frac{1}{2} A^\mu (\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu) A^\nu + \frac{m^2}{2} A_\mu A^\mu$$

+ total divergence (where $\Box = \partial^\mu \partial_\mu$)

7) $$\frac{\partial \mathcal{L}_{source}}{\partial A_\nu} = - J^\nu$$

$$\Rightarrow \partial^\mu F_{\mu\nu} + m^2 A_\nu = J_\nu$$

or, using the form (5) of the Lagrangian:

$$(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu + m^2 \eta_{\mu\nu}) A^\mu = J_\nu$$

Eq. de Green:

$$(\Box_{(x)} \eta_{\mu\nu} - \partial_\mu^{(x)} \partial_\nu^{(x)} + m^2 \eta_{\mu\nu}) G^{\nu\rho}(x,y) = \delta_\mu^\rho \delta^{(4)}(x-y)$$

8) Green's equation in momentum space:

$\Box \longrightarrow -p^2$; $\partial_\mu \partial_\nu \longrightarrow -p_\mu p_\nu$; $\delta^4(x-y) \longrightarrow 1$

$$\left[(-p^2+m^2)\eta_{\mu\nu} + p_\mu p_\nu\right] G^{\nu\rho}(p) = \delta_\mu{}^\rho$$

$$\left(\text{where} \quad G^{\nu\rho}(p) = \int d^4x \, e^{-ipx} G^{\nu\rho}(x) \quad \right)$$

write $G^{\nu\rho}(p) = A\eta^{\nu\rho} + B p^\nu p^\rho$

$$\Rightarrow \left[(-p^2+m^2)\eta_{\mu\nu} + p_\mu p_\nu\right]\left[A\eta^{\nu\rho} + B p^\nu p^\rho\right] =$$

$$= (-p^2+m^2) A \delta_\mu^\rho + \left[(-p^2+m^2)B + A + p^2 B\right] p_\mu p^\rho = \delta_\mu^\rho$$

$$\Rightarrow \quad A = -\frac{1}{p^2-m^2} ; \quad B = \frac{1}{m^2}\frac{1}{p^2-m^2}$$

$$\Rightarrow \quad G_{\mu\nu}(p) = \frac{-\left(\eta_{\mu\nu} - \frac{p_\mu p_\nu}{m^2}\right)}{p^2 - m^2 + i\epsilon}$$

Add to give Feynman's prescription

9) $\partial^\mu G_{\mu\nu} = 0$ translates in momentum space to:

$$p^\mu G_{\mu\nu} \doteq \eta_{\mu\nu} p^\mu - \frac{p^\mu p_\mu}{m^2} p_\nu \underset{p^2=m^2}{=} 0$$

SU(2) Breaking by a doublet scalar

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad \phi \to U\phi \quad \text{or} \quad \delta\phi = i\sum_{a=1}^{3} \alpha^a \frac{\sigma^a}{2}\phi$$

Suppose ϕ gets a vacuum expectation value (no matter why): $\phi^\dagger\phi = |\varphi_1|^2 + |\varphi_2|^2 = v^2$

1) Choose the vacuum $\Phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$

the infinitesimal transformation acts on this as:

$$\delta\Phi_0 = i\begin{pmatrix} \alpha^3 & \alpha^1 - i\alpha^2 \\ \alpha^1 + i\alpha^2 & -\alpha^3 \end{pmatrix}\begin{pmatrix} 0 \\ v \end{pmatrix} = v\begin{pmatrix} \alpha^1 - i\alpha^2 \\ \alpha^3 \end{pmatrix}$$

the only solution to $\delta\Phi_0 = 0$ is $\alpha^1 = \alpha^2 = \alpha^3 = 0$ (remember that the $\alpha^a \in \mathbb{R}$), i.e. no transformation leaves the vacuum invariant

$\Rightarrow$ SU(2) is completely broken

"$SU(2) \longrightarrow \mathbb{1}$"

2)
$$D_\mu \phi = \partial_\mu \phi - igA^a_\mu \tau^a \phi \qquad (\text{“}\textstyle\sum_a\text{” understood})$$
$$= \left(\mathbb{1}_2 \partial_\mu - \frac{i}{2} g \begin{bmatrix} A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu \end{bmatrix}\right) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

3 vector fields $A^1_\mu, A^2_\mu, A^3_\mu$

3) Setting $\Phi = \phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}$:

$$D_\mu \phi_0 = -\frac{i}{2} g \begin{pmatrix} A^3_\mu & A^1_\mu - iA^2_\mu \\ A^1_\mu + iA^2_\mu & -A^3_\mu \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} = -\frac{igv}{2} \begin{pmatrix} A^1_\mu - iA^2_\mu \\ -A^3_\mu \end{pmatrix}$$

the kinetic term, evaluated on $\phi = \phi_0$, is:

$$(D_\mu \phi_0)^\dagger (D^\mu \phi) = \frac{g^2 v^2}{4} \left(A^1_\mu + iA^2_\mu, -A^3_\mu\right) \begin{pmatrix} A^{1\mu} - iA^{2\mu} \\ -A^{3\mu} \end{pmatrix}$$

$$= \frac{g^2 v^2}{4} \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu} + A^3_\mu A^{3\mu} \right)$$

So all 3 gauge bosons get the same mass,

$$m_A^2 = \frac{1}{2} g^2 v^2$$

SU(2) breaking by a Triplet scalar

Remember that $SU(2) \simeq SO(3)$, and the 3-dim representation of $SU(2)$ is the vectorial representation of $SO(3)$:

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad \Phi \to R\,\Phi$$ where R is a 3-d Rotation

(ϕ_i Real)

$$\delta \Phi = i \alpha^a T^a \Phi$$

these are not the space coordinates but some "internal" labels

T^a are the generators of infinitesimal rotations around x, y, z (Taken Hermitian, i.e. purely imaginary)

Say Φ Takes a v.e.v., $\Phi^t \Phi = \phi_1^2 + \phi_2^2 + \phi_3^2 = v^2$

1) Choose vacuum $\Phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

Under a generic transformation:

$$\delta \Phi_0 = \alpha^1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} + \alpha^2 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} + \alpha^3 \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$$

$$= \alpha^1 \begin{pmatrix} 0 \\ -v \\ 0 \end{pmatrix} + \alpha^2 \begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} + \alpha^3 \cdot 0 = v \begin{pmatrix} \alpha^2 \\ -\alpha^1 \\ 0 \end{pmatrix}$$

So $\delta \phi_0 = 0$ requires $\alpha^1 = \alpha^2 = 0$ but α^3 can be $\neq 0$

$\Rightarrow$ Transformations generated by T^3 <u>do not</u> break the symmetry

$$\delta_3 \phi_0 \equiv \alpha^3 T^3 \phi_0 = 0$$

These transformations form an $SO(2)$ subgroup of $SO(3)$: the rotations around the 3 axis:

$$R_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_3(\theta)\phi_0 = \phi_0$$

$$\text{“} SO(3) \longrightarrow SO(2) \text{”}$$

or in terms of the universal covering groups:

$$\text{“} SU(2) \longrightarrow U(1) \text{”}$$

$SU(2)$ is broken down to $U(1)$

To see it in terms of $SU(2)$ and $U(1)$ arrange the triplet Φ into a 2×2 complex matrix:

$$\Phi = \begin{pmatrix} \phi^3 & \phi^1 - i\phi^2 \\ \phi^1 + i\phi^2 & -\phi^3 \end{pmatrix}$$

$SU(2)$ acts by adjoint action:

$$\underset{\text{(finite)}}{\phi \to U \phi U^{-1}} \qquad \underset{\text{(infinitesimal)}}{\delta\phi = i\alpha^a [\tau^a, \phi]}$$

if $\phi_0 = \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix}$ $\quad \delta\phi_0 = 0$ if $\alpha^1 = \alpha^2 = 0$ $([\sigma^3, \phi_0] = 0)$

The unbroken $U(1)$ acts as: $\phi \to e^{i\theta\frac{\sigma^3}{2}} \phi \, e^{-i\theta\frac{\sigma^3}{2}}$

$$\delta\phi = i\theta [\sigma^3/2, \phi]$$

2) $D_\mu \phi = \partial_\mu \phi - ig A^a_\mu T^a \phi =$

$$= \left[\mathbb{1}_3 \partial_\mu - g \begin{pmatrix} 0 & -A^3_\mu & A^2_\mu \\ A^3_\mu & 0 & -A^1_\mu \\ -A^2_\mu & A^1_\mu & 0 \end{pmatrix} \right] \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

3) set $\phi = \phi_0 = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$

$$D_\mu \phi = -g \begin{pmatrix} 0 & -A^3_\mu & A^2_\mu \\ A^3_\mu & 0 & -A^1_\mu \\ -A^2_\mu & A^1_\mu & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} = gv \begin{pmatrix} A^2_\mu \\ -A^1_\mu \\ 0 \end{pmatrix}$$

$$\Rightarrow \frac{1}{2}(D_\mu \phi_0)^\dagger (D^\mu \phi_0) = \frac{1}{2} g^2 v^2 \left(A^1_\mu A^{1\mu} + A^2_\mu A^{2\mu} \right)$$

- <u>only</u> A^1_μ and A^2_μ obtain a mass, $m_A = gv$
- A^3_μ stays massless

We learned that, for an SU(2) gauge group:

- a doublet breaks SU(2) completely
- a triplet leaves a U(1) subgroup unbroken
- The vector bosons mass is $\div\ gv$