M2 NPAC 2022/2023

QFT - part 2 **Problem set n°1**

Spinor algebra

The Dirac matrices are defined in terms of the basic property :

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4 \tag{1}$$

where $g_{\mu\nu}$ is the Minkowski metric $diag(1, -1 - 1 - 1)$ and $\mathbf{1}_4$ is the identity matrix.

A basis for positive and negative frequency solutions of the Dirac equation is given by :

$$u^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\xi^s \\ \sqrt{p_\mu \bar{\sigma}^\mu}\xi^s \end{pmatrix}, \qquad v^s(p) = \begin{pmatrix} \sqrt{p_\mu \sigma^\mu}\xi^s \\ -\sqrt{p_\mu \bar{\sigma}^\mu}\xi^s \end{pmatrix}, \qquad s = 1, 2 \tag{2}$$

where ξ^s are the two-component spinors

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

In the expressions (2)

$$\sigma^\mu = (\mathbf{1}_2, \sigma^i), \qquad \bar{\sigma}^\mu = (\mathbf{1}_2, -\sigma^i)$$

where σ^i are the Pauli matrices.

1 Traces of γ matrices

1. Without using the explicit representation of the γ-matrices, but only equation (1), show that

$$\begin{aligned} &\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \quad , \\ &\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \quad . \end{aligned}$$

   where the trace is over the spinor indices.
2. Deduce expressions for $\mathrm{Tr}\,(\not{a}\,\not{b})$ and $\mathrm{Tr}\,(\not{a}\,\not{b}\,\not{c}\,\not{d})$, where a_μ, b_μ etc. are 4-vectors.
3. Show that $\mathrm{Tr}\,(\gamma^\mu) = 0$

2 Spin sums

In what follows, α, β... are spinor indices, and run from 1 to 4, s, s'... run over the polarisation (1,2).

1. Show that

$$\bar{u}^s(p)u^{s'}(p) = 2m\delta^{ss'}, \quad \bar{v}^s(p)v^{s'}(p) = -2m\delta^{ss'}, \quad \bar{v}^s(p)u^{s'}(p) = \bar{u}^s(p)v^{s'}(p) = 0$$

2. Show that

$$\bar{u}^s(p)\gamma^\mu u^{s'}(p) = 2\delta^{ss'}p^\mu$$

3. Show that

$$\sum_{s=1}^{2} u^s_\alpha(p)\bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v^s_\alpha(p)\bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

Notice that in excerices 1 and 2 spinor indices are contracted, while in excercise 3 they are not.