M2 NPAC 2022/2023

**QFT - part 2** **problem set n°2**

# Interacting field theories and the S-matrix

## 1 Dimensional Analysis

1. Consider a theory with one scalar field $\phi$ and one Dirac spinor $\psi$.
   a) Find the dimension of the coupling constants of the following interactions :
   $$g\,\phi\,\overline{\psi}\psi\;, \qquad G_F\,\overline{\psi}\psi\,\overline{\psi}\psi$$
   b) Is each of the interactions above relevant, marginal, or irrelevant ? For the irrelevant ones, what is the energy scale at which perturbation theory breaks down ?
   c) Write all possible interactions up to dimension six compatible with Lorentz invariance and parity.
2. In Einstein's theory of gravitation, the coupling constant is the same as Newton's gravitational constant $G_N$ (in S.I. units : $G_N = 6.7\,10^{-11}\,m^3 Kg^{-1} s^{-2}$).
   a) Find the dimension of $G_N$ in natural units. Is this coupling relevant or irrelevant ?
   b) If the latter is true, find the value (in GeV) of corresponding strong coupling energy scale.

## 2 Dyson's equation

*This problem completes the derivations which were left out of the Jan 3 lecture.*

We denote $|\psi\rangle_S$ and $|\psi\rangle_I$ a quantum state in the Schrödinger and Interaction picture, respectively. They are related by
$$|\psi(t)\rangle_I = e^{iH_0 t}|\psi(t)\rangle_S$$
where the Schrödinger picture Hamiltonian is $H = H_0 + H_{int}$, and $H_0$ is the Hamiltonian of the free theory.

Similarlty, operators in the Schrödinger and Interaction pictures are related by :
$$O_I(t) = e^{iH_0 t} O_S e^{-iH_0 t}$$

Recall that Schrödinger picture states satisfy the full Schrödinger equation :
$$i\frac{d}{dt}|\psi(t)\rangle_S = (H_0 + H_{int})|\psi(t)\rangle_S$$
and that Schrödinger picture operators are time-independent.

We denote by $U_I(t, t_0)$ the interaction-picture time-evolution operator, defined by the formula :
$$|\psi(t)\rangle_I = U_I(t, t_0)|\psi(t_0)\rangle_I$$
.

1. Show that the evolution equation in the interaction picture takes the form
   $$i\frac{d}{dt}|\psi\rangle_I = H_I|\psi\rangle_I \qquad (1)$$
   where $H_I(t) = e^{iH_0 t} H_{int} e^{-iH_0 t}$.

2. Show that, given any two states $|i(t)\rangle$ and $|f(t)\rangle$, their scalar product is independent of the representation, i.e. :

$$_S\langle f(t)|i(t)\rangle_S \;=\; _I\langle f(t)|i(t)\rangle_I$$

3. S-matrix elements are defined in the Schrödinger picture by

$$_S\langle f(t=+\infty)|i(t=+\infty)\rangle_S =_S\langle f(t=+\infty)|S|i(t=-\infty)\rangle_S$$

where the $S$-operator on the r.h.s. is the full time-evolution operator :

$$S = U(+\infty,-\infty), \qquad U(t_2,t_1) \equiv e^{-iH(t_2-t_1)}.$$

Show that, going to the interaction picture, one obtains :

$$_S\langle f(t=+\infty)|i(t=+\infty)\rangle_S = \langle f|S_I|i\rangle$$

with the S-operator in the interaction picture given by

$$S_I = U_I(+\infty,-\infty),$$

In the equations above, we assume that interactions are switched off as $t \to +\pm\infty$ (therfore the Interaction-picture states become time-independent Heisenberg picture states of the free theory, denoted simply by $|i\rangle$ and $|f\rangle$).

4. Using equation (1), show that the operator evolution equation (with respect to $t$) for $U_I(t,t_0)$ is :

$$i\frac{d}{dt}U_I(t,t_0) = H_I(t)U_I(t,t_0).$$

5. Show that the solution is given by

$$U_I(t,t_0) = T\exp\left(-i\int_{t_0}^{t} dt' H_I(t')\right) \qquad \text{(Dyson's equation)} \tag{2}$$

where $T$ denotes time-ordering of operator products,

$$TO(t_1)O(t_2) \equiv \begin{cases} O(t_1)O(t_2) & t_1 > t_2 \\ O(t_2)O(t_1) & t_1 < t_2 \end{cases}$$

# 3 Phase space element in relativistic collisions

*This exercise generalizes the expression for the cross-section seen in class to the relativistic case. You can skip it at first and go straight to ex. 4, as the expression needed are summarized there.*

The differential cross section in a two-particle to two-particle collision with incoming momenta $\vec{p_1}, \vec{p_2}$ and outgoing momenta $\vec{p_3}, \vec{p_4}$ is given by

$$d\sigma = \frac{1}{TF}dP \tag{3}$$

where $T$ is the duration of the experiment, $F$ is the incoming flux,

$$F = \frac{1}{V}|\vec{v}_2 - \vec{v}_1|$$

(where we assumed the wave-functions are normalised to 1, i.e. 1 particle per volume $V$), and $dP$ is the differential scattering probability given by [1]

$$dP = \int_{\vec{p}_3 \to d\Omega} \frac{|\langle \vec{p}_3, \vec{p}_4 | S - 1 | \vec{p}_1, \vec{p}_2 \rangle|^2}{\langle \vec{p}_1, \vec{p}_2 | \vec{p}_1, \vec{p}_2 \rangle \langle \vec{p}_3, \vec{p}_4 | \vec{p}_3, \vec{p}_4 \rangle} d\Pi_3 d\Pi_4 \tag{4}$$

where $d\Pi_{3,4}$ are the phase-space elements of the final states,

$$d\Pi = n_{\vec{p}}\, d^3 p \tag{5}$$

( where $n_{\vec{p}}$ is the momentum density of states) and the integral is over the final states such that one of the particles reaches the detector in a given solid angle element $d\Omega$.

Recall the normalisation condition

$$\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 2\omega_{\vec{p}} \delta^{(3)}(\vec{p} - \vec{q}), \tag{6}$$

and the identyties (in finite space-time volume V, T)

$$(2\pi)^3 \delta^{(3)}(\vec{p} = 0) = V, \quad 2\pi\delta(p_0 = 0) = T, \tag{7}$$

to regularise the delta-functions at zero. Also, remember that the total number of particles with a given momentum $\vec{p}$ is

$$N = \langle \vec{p} | \vec{p} \rangle \tag{8}$$

and that the density of 1-particle states $n_{\vec{p}}$ can be obtained by writing the completeness relation of the identity operator times $N$ :

$$N\,\mathbf{1} = \int d^3 p \; n_{\vec{p}} |\vec{p}\rangle\langle\vec{p}| \tag{9}$$

1. Show that the differential cross section can be written as

$$d\sigma = \frac{1}{4E_1 E_2|} \frac{1}{|\vec{v}_1 - \vec{v}_2|} \int_{\vec{p}_3 \to d\Omega} |\mathcal{A}_{i\to f}|^2 d\Pi_{LIPS}, \tag{10}$$

where the amplitude $\mathcal{A}_{i\to f}$ is defined by extracting a momentum-conservation *delta*-function from the S- matrix element [2],

$$\langle f | S - 1 | i \rangle = \mathcal{A}_{i\to f} (2\pi)^4 \delta^{(4)} \left( \sum p_f - \sum p_i \right), \tag{11}$$

and the Lorentz Invariant Phase Space element is given by :

$$\Pi_{LIPS} = (2\pi)^4 \delta^{(4)}(p_3 + p_4 - p_1 - p_2) \frac{d^3 p_3}{(2\pi)^3 2\omega_{p_3}} \frac{d^3 p_4}{(2\pi)^3 2\omega_{p_4}} \tag{12}$$

2. For a collision in the center of mass frame, and when all particles have the same mass show that, by performing the integral over the particle 3 and 4 momenta (such that at least one particle is detected in the solid angle $d\Omega$) and by using relativistic kinematics (i.e. $p^2 = m^2$ for all 4-momenta) one obtains :

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{|\mathcal{A}_{i\to f}|^2}{E_{CM}^2} \tag{13}$$

where $E_{CM}$ is the total center-of-mass energy of the collision.

1. We subtract the identity to eliminate the process where no interaction occurs.
2. It is customary to subtract the identity from $S$. If the initial and final states are not the same, this does not make any difference.

# 4 Two-particle scattering in $\phi^4$ theory

Recall that for relativistic scattering, the differential cross section can be written as

$$d\sigma = \frac{1}{4E_1E_2|}\frac{1}{|\vec{v}_1 - \vec{v}_2|}\int_{\vec{p}_3 \to d\Omega} |\mathcal{A}_{i\to f}|^2 d\Pi_{LIPS}, \tag{14}$$

where the amplitude $\mathcal{A}_{i\to f}$ is defined by extracting a momentum-conservation *delta*-function from the S- matrix element,

$$\langle f|S-1|i\rangle = \mathcal{A}_{i\to f}(2\pi)^4\delta^{(4)}\left(\sum p_f - \sum p_i\right), \tag{15}$$

and the Lorentz Invariant Phase Space element is given by :

$$\Pi_{LIPS} = (2\pi)^4\delta^{(4)}(p_3 + p_4 - p_1 - p_2)\frac{d^3p_3}{(2\pi)^3 2\omega_{p_3}}\frac{d^3p_4}{(2\pi)^3 2\omega_{p_4}} \tag{16}$$

Finally, recall that for a collision in the center of mass frame, and when all particles have the same mass, one obtains :

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{|\mathcal{A}_{i\to f}|^2}{E_{CM}^2} \tag{17}$$

where $E_{CM}$ is the total center-of-mass energy of the collision.

## 4.1 Warm up : one-dimensional harmonic oscillator

Consider a non-relativistic quantum harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}m\omega^2x^2 + \frac{1}{2m}p^2 = \omega\left(a^\dagger a + \frac{1}{2}\right)$$

with

$$x = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger), \quad p = \sqrt{\frac{m\omega}{2}}i(a - a^\dagger), \qquad [a, a^\dagger] = 1$$

and eigenstates $|n\rangle$ satisfying

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad a|n\rangle = \sqrt{n}|n-1\rangle \qquad (a|0\rangle = 0).$$

1. Compute the expectation values

$$\langle 0|x^2|0\rangle, \qquad \langle 0|x^4|0\rangle$$

2. Compute the matrix element

$$\langle 0|x^4|2\rangle$$

## 4.2 $2 \to 2$ S-Matrix element in $\phi^4$ theory

We now consider $\phi^4$ theory, with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

1. Compute the S-matrix element for $2 \to 2$ scattering, to the lowest-order in $\lambda$,

$$\langle p_1', p_2' | S - 1 | p_1, p_2 \rangle.$$

   Recall Dyson's formula for the S-matrix :

$$S = T \exp\left(-i \int dt H_I(t)\right) \tag{18}$$

   as well as the free field mode expansion in terms of creation and annihilation operators,

$$\phi(x) = \phi^+(x) + \phi^-(x), \quad \phi^+(x) \equiv \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} a_{\vec{p}}\, e^{-ipx}, \quad \phi^-(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2\omega_p}} a_{\vec{p}}^\dagger e^{ipx},$$

   and the commutation relation

$$[a_{\vec{p}}, a_{\vec{q}}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}).$$

2. Verify that the result found above has the form (15), and give the scattering amplitude $\mathcal{A}_{i \to f}$ to lowest order in $\lambda$.
3. Using (17), obtain the differential cross-section for $2 \to 2$ scattering in $\phi^4$ theory in the center-of-mass frame for the incoming particles.

*Congratulations ! You have just computed your first QFT observable.*