M2 NPAC 2022/2023

QFT - part 2 problem set n°3

Green's functions and propagators

Useful formulae

1. Dirac delta :
$$\int_{-\infty}^{+\infty} dy\, e^{iyz} = 2\pi\delta(z); \qquad \frac{d}{dz}\theta(z) = \delta(z)$$
2. Cauchy's theorem :
$$\oint dz \frac{f(z)}{z-w} = 2n\pi i f(w)$$
where n is the number of times the integration contour goes around w counter-clockwise.
3. Oscillator expansion of a free Klein-Gordon real scalar field operator with mass m :
$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2\omega_{\vec{p}}}} \left[\hat{a}_{\vec{p}} e^{-i(\omega_{\vec{p}}t - \vec{p}\cdot\vec{x})} + \hat{a}^\dagger_{\vec{p}} e^{i(\omega_{\vec{p}}t - \vec{p}\cdot\vec{x})}\right]$$
where x is shorthand for the four-vector $x^\mu \equiv (t, \vec{x})$ and
$$\omega_{\vec{p}} \equiv \sqrt{|\vec{p}|^2 + m^2}.$$
The creation-annihilation operators satisfy the commutation rule :
$$\left[\hat{a}_{\vec{p}}, \hat{a}^\dagger_{\vec{q}}\right] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\hat{\mathbf{1}}.$$

1. KG equation coupled to an external source

Consider the equation for a real scalar field with an external source :
$$(\Box + m^2)\phi(\vec{x}, t) = J(\vec{x}, t). \tag{1}$$

1. Assuming there is an instant t_0 such that $J(\vec{x}, t < t_0) = 0$, write, in terms of an appropriate Green's function, the solution $\phi(\vec{x}, t)$ which satisfies $\phi(\vec{x}, t < t_0) = 0$. Which is the correctn Green's function in this case ?
2. Find the explicit solution $\phi(\vec{x}, t)$ in the particular case :
$$J(\vec{x}, t) = j_0\, \theta(t)\, e^{-\mu t}, \qquad j_0, \mu > 0, \qquad \theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{2}$$
3. Recall the solution for the classical Klein-Gordon (1) in the absence of sources :
$$\phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} \left(a(\vec{k})\, e^{i\vec{k}\cdot\vec{x} - i\omega_k t} + a^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x} + i\omega_k t}\right) \tag{3}$$
where $\omega_k \equiv \sqrt{\vec{k}^2 + m^2}$. Show that, for $t \to \infty$, the solution found in the previous point reduces to a solution of the free equation with coefficients :
$$a(\vec{k}) = -\frac{(2\pi)^3 j_0}{(m + i\mu)\sqrt{2m}}\, \delta^3(\vec{k}). \tag{4}$$

2. Green's function for Helmoltz equation

Consider the following static equation in d=3 space dimensions (no time) :

$$(-\nabla^2 + m^2)\phi(\vec{x}) = 0 \tag{5}$$

where $\phi(\vec{x})$ is a scalar field (with respect rotations in 3d). We want to construct the associated Green's function, $G(\vec{x})$, i.e. the solution of

$$(-\nabla^2 + m^2)G(\vec{x}) = \delta^{(3)}(\vec{x}). \tag{6}$$

1. Writing equation (6) in Fourier space, show that :

$$G(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}}}{k^2 + m^2}. \tag{7}$$

   Is the Green's function unique ? Do we need to go through the contour-deforming procedure in this case ? Why is that the case ?

2. Perform the 3d momentum integral explicitly and show that :

$$G(\vec{x}) = \frac{e^{-m|\vec{x}|}}{4\pi|\vec{x}|} \tag{8}$$

   *[Hint : choose spherical coordinate *wisely*, then perform the integral over angles, then use a contour integral for the final integration over $|\vec{k}|$.]*

3. Propagators, Wightman Functions and Green functions

We define the functions $D_\pm$ (called the positive and negative frequency *Wightman functions*) by :

$$D_\pm(x) = \int \frac{d^3p}{(2\pi)^3} \frac{e^{\mp i(\omega_p t - \vec{p}\cdot\vec{x})}}{2\omega_p}$$

where $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$. They have the equivalent expressions in terms of 4d integrals :

$$D_\pm = i \oint_{C_\pm} \frac{d\omega}{2\pi} \int \frac{d^3p}{2\pi^3} \frac{e^{-i(\omega t - \vec{p}\cdot\vec{x})}}{p^2 - m^2} \qquad p^2 \equiv \omega^2 - |\vec{p}|^2 \tag{9}$$

where $C_\pm$ are close contours around $\pm\omega_p$, respectively. We also have the relations :

$$G_F(x) = G_A(x) + iD_+(x), \qquad \text{and} \qquad G_F(x) = G_R(x) + iD_-(x) \tag{10}$$

where G_A and G_R are the advanced and retarded Green's functions. It follows that :

$$G_F(x - x') = i\Big(\theta(t - t')D_+(x - x') + \theta(t' - t)D_-(x - x')\Big). \tag{11}$$

1. Using the oscillator expansion of the scalar field operators given on page 1, show by explicit calculation that

$$\langle 0|\hat{\phi}(x)\hat{\phi}(x')|0\rangle = D_+(x - x') \tag{12}$$

   Then use equations (11) and (12) to obtain :

$$\Delta_F(x - x') \equiv -iG_F(x - x') = \langle 0|T\Big(\hat{\phi}(x)\hat{\phi}(x')\Big)|0\rangle.$$

   where T denotes time-ordering.
   This gives the fundamental result that the Feynman Green's function coincides with the time-ordered two-point correlator.

2. Similarly, show that

$$G_R(x, x') = i\theta(t - t') \langle 0| \left[\hat{\phi}(x), \hat{\phi}(x')\right] |0\rangle. \tag{13}$$

and give a similar expression for $G_A(x, x')$.

3. Show that the Fourier transform (defined below) of Feynman's propagator has the following expression :

$$\tilde{\Delta}_F(p_1, p_2) \equiv \int d^4x_1 d^4x_2 e^{-ip_1x_1} e^{-ip_1x_1} \Delta_F(x_1 - x_2) = (2\pi)^4 \delta^{(4)}(p_1 + p_2) \Delta_F(p_1) \tag{14}$$

where we have defined :

$$\Delta_F(p) \equiv \frac{i}{p^2 - m^2 + i\epsilon} \tag{15}$$

4. Given a 4-point function $\Delta(x_1, x_2, x_3, x_4)$, we define its momentum space expression :

$$\tilde{\Delta}_F(p_1, p_2, p_3, p_4) \equiv \int d^4x_1 d^4x_2 d^4x_3 d^4x_4 e^{-ip_1x_1} e^{-ip_1x_1} e^{-ip_3x_3} e^{-ip_4x_4} \Delta(x_1, x_2, x_3, x_4) \tag{16}$$

Give the position space expressions, then the momentum-space expressions of the 4-point diagrams in figure 1 *(forget symmetry factors)* :
In particular, make the momentum-conservation δ-functions explicit.

4. Wick's theorem

1. Given the relation

$$T\hat{\phi}(x_1)\hat{\phi}(x_2) =: \hat{\phi}(x_1)\hat{\phi}(x_2) : + \Delta_F(x_1 - x_2)\hat{\mathbf{1}}, \tag{17}$$

use it to prove Wick's theorem for the product of 3 fields and for that of 4 fields.

2. Prove Wick's theorem by induction :

$$T\hat{\phi}(x_1) \ldots \hat{\phi}(x_n) =: \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) : + \sum : \text{all possible contractions} :$$

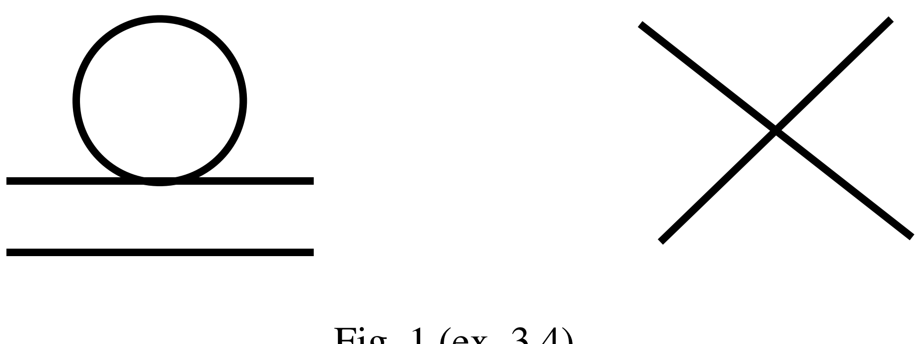

Fig. 1 (ex. 3.4)