M2 NPAC 2022/2023

QFT **problem set n°5**

QED Cross Sections

The goal of this exercise is to calculate the unpolarized differential cross section for two simple QED processes, at tree level, in the center of mass frame. The result will be expressed as a function of the center of mass energy E_{CM} and the scattering angle θ (i.e. the angle between the outgoing particles and the incoming direction. The latter may be taken to be the z direction).

Recall that, for $2 \to 2$ scattering, the differential cross section in the center of mass is related to the amplitude by (cfr. TD3)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 E_{CM}^2} \frac{|\vec{p}_f|}{|\vec{p}_i|} |\mathcal{A}|^2 \tag{1}$$

where $\mathcal{A}$ is the scattering amplitude and $\vec{p}_{i,f}$ are the initial and final momenta of one of the particles.

Recall that :

$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \quad ,$$
$$\mathrm{Tr}\,(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4\,(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \quad .$$

where the trace is over the spinor indices.

Recall also the spin sum rules :

$$\bar{u}^s(p)u^{s'}(p) = 2m\delta^{ss'}, \quad \bar{v}^s(p)v^{s'}(p) = -2m\delta^{ss'}, \quad \bar{v}^s(p)u^{s'}(p) = \bar{u}^s(p)v^{s'}(p) = 0$$

$$\sum_{s=1}^{2} u^s_\alpha(p)\bar{u}^s_\beta(p) = \not{p}_{\alpha\beta} + m\,\delta_{\alpha\beta}, \qquad \sum_{s=1}^{2} v^s_\alpha(p)\bar{v}^s_\beta(p) = \not{p}_{\alpha\beta} - m\,\delta_{\alpha\beta}$$

1 e^+e^- (Bhabha) Scattering

Consider the process

$$e^+e^- \longrightarrow e^+e^-$$

We want to compute the *unpolarized* cross section (i.e. averaged over initial spins and summed over final spins).

1. Draw the tree-level Feynman diagrams which contribute to this process *(Hint : there are two of them : one in the s-channel, one in the t-channel).*
2. Find scattering amplitude associated to each of diagram. What is their relative sign ?
3. Compute the square of the amplitude using the spin sum rules, and the corresponding differential cross section using equation (1). Show that, in the high-energy limit $E_{cm} \gg m_e$, one finds :

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{2s}\left[u^2\left(\frac{1}{s}+\frac{1}{t}\right)^2 + \left(\frac{t}{s}\right)^2 + \left(\frac{s}{t}\right)^2\right]. \tag{2}$$

where s, t, u are the Mandelstam variables *(Notice that, if we ignore the electron mass, then $s + t + u = 0$).*

4. Rewrite equation (2) in terms of $\cos\theta$ and the center-of-mass energy.

2 Pair annihilation into photons

Consider the process

$$e^+e^- \longrightarrow \gamma\,\gamma$$

in the center-of-mass frame. We want to compute the *unpolarized* cross section.

1. Draw the tree-level Feynman diagrams which contribute to this process *(Hint : there are two of them : one in the s-channel, one in the t-channel).*
2. Find scattering amplitude associated to each of diagram. What is their relative sign ?
3. Prove the *photon polarisation sum* rules :

$$\sum_{i=1}^{2} (\epsilon^i_\mu)^* \epsilon^i_\nu = -g_{\mu\nu} + \frac{1}{2E^2}(p_\mu \bar{p}_\nu + \bar{p}_\mu p_\nu) \tag{3}$$

where $p_\mu = (E, \vec{p})$, $\bar{p}_\mu = (E, -\vec{p})$, ϵ^i with $i = 1, 2$ are two transverse polarizations (i.e. orthogonal to both p_μ and $\bar{p}_\mu$ (Notice that if p_μ is a null vector, so is $\bar{p}_\mu$. For example, $p_\mu = (E, 0, 0, E)$, $\bar{p}_\mu = (E, 0, 0, -E)$, $\epsilon^1_\mu = (0, 1, 0, 0)$, $\epsilon^2_\mu = (0, 0, 1, 0)$.)

4. Show that the amplitude vanishes whenever the polarisation is along p_μ or $\bar{p}_\mu$. Deduce that we can substitute

$$\sum_{i=1}^{2} (\epsilon^i_\mu)^* \epsilon^i_\nu \to -g_{\mu\nu}$$

when squaring the amplitude.

5. Compute the square of the amplitude using the result above *(make sure you first add the contribution from the two diagrams, then square)* and show that the corresponding differential cross section is :

$$\frac{d\sigma}{d\cos\theta} = \frac{2\pi\alpha^2}{s}\frac{E}{p}\left[\frac{E^2 + m^2 + p^2\cos^2\theta}{m^2 + p^2\sin^2\theta} - \frac{2m^4}{(m^2 + p^2\sin^2\theta)^2}\right] \tag{4}$$

6. Show that, in the high-energy limit $E \gg m$, and for finite θ (i.e. for $\theta \gtrsim m/p$) equation (4) becomes

$$\frac{d\sigma}{d\cos\theta} \simeq \frac{2\pi\alpha^2}{s}\left(\frac{1 + \cos^2\theta}{\sin^2\theta}\right)^2 \tag{5}$$