M2 NPAC 2022/2023

QFT **Problem set n°6**

1 Massive Spin-1

The dynamics of a massive spin-one field is described by the Lagrangian :

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \tag{1}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.$$

1.1 Field equations

1. Show that the action (1) is not invariant under the gauge transformation
$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$$
2. Write down the Euler-Lagrange equations
3. Show that, for $m \neq 0$, the EL equations imply that A_μ must satisfy
$$\partial^\mu A_\mu = 0. \tag{2}$$
Why is this *not* the same as a gauge-condition ?
4. Using the EL equations, show that A_0 is not a dynamical degree of freedom but it can be eliminated at each instant of time (i.e. by solving an equation with no time-derivatives on A_0) in terms of the spatial components A_i.

1.2 Propagator

We now want to write down the propagator for the massive spin-1 theory.

5. Show that, up to total derivatives, the Lagrangian (1) can be rewritten as
$$\mathcal{L} = \frac{1}{2}A^\mu \left(\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu\right) A^\nu + \frac{1}{2}m^2 A_\mu A^\mu \tag{3}$$
6. Write the momentum-space version of the quadratic kinetic operator (including the mass term) above, and use it to find the propagator of the theory in momentum space, $G_{\mu\nu}(p)$ *(Hint : by Lorentz-invariance, $G_{\mu\nu}(p)$ can only be the sum of two types of terms : $G_{\mu\nu}(p) = A(p,m)g_{\mu\nu} + B(p,m)p_\mu p\nu$ where A and B are functions of p and m to be determined.)*
7. Show that, on-shell, the propagator satisfies $p^\mu G_{\mu\nu}(p) = 0$, in accordance with the constraint equation (2).

1.3 Recovering Gauge invariance

We can make the theory of a massive spin-1 gauge-invariant by introducing an appropriate auxiliary scalar field $\pi(x)$, such that under a gauge transformation :

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x), \qquad \pi(x) \to \pi(x) + \alpha(x).$$

8. Which gauge-invariant Lagrangian for the fields $A_\mu(x)$ and $\pi(x)$ is equivalent to the original Lagrangian (1) ?
9. Discuss the differences and similarities between this procedure and the Higgs mechanism for giving mass to A_μ.

2 Spontaneous Symmetry Breaking

Consider a theory with an $SU(2)$ gauge symmetry, with gauge coupling g, spontaneously broken by a scalar multiplet in two different cases :

— a doublet of complex scalars

$$\phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

in the fundamental representation of $SU(2)$, with v.e.v :

$$\phi^\dagger \phi = v^2$$

— a triplet of real scalars

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$$

in the adjoint representation of $SU(2)$ (a.k.a. a 3-dimensional vector of $SO(3)$) with v.e.v. :

$${}^t\Phi\,\Phi = v^2$$

In each case :

1. Determine the unbroken symmetry group ;
2. Define the the covariant derivative of the scalar field, and write explicitly its matrix form ;
3. Write the scalar field kinetic term and, by setting the scalar field to its vacuum value, determine the masses of the gauge fields.

Recall that the (hermitian) generators of $SU(2)$ are given by :

— Fundamental representation :

$$\tau^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

— Adjoint (3-dimensional vector) representation :

$$T^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad T^2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$