There are different ways to obtain the results of exercice 11 (iii).

Either we act with $\mathcal{J}_1^2$, $\mathcal{J}_2^2$ and $\mathcal{J}_3^2$ on $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ or we compute directly these operators and then act on the above vectors. Let's do both :

$$\begin{aligned}
\mathcal{J}_1^2|\uparrow\downarrow\rangle &= \mathcal{J}_1\left(\tfrac{\sigma_1}{2}\otimes \mathrm{I} + \mathrm{I}\otimes\tfrac{\sigma_1}{2}\right)|\uparrow\downarrow\rangle \\
&= \mathcal{J}_1\left(\tfrac{\sigma_1}{2}|\uparrow\rangle\otimes \mathrm{I}|\downarrow\rangle + \mathrm{I}|\uparrow\rangle\otimes\tfrac{\sigma_1}{2}|\downarrow\rangle\right) \\
&= \tfrac{1}{2}\mathcal{J}_1\left(|\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle\right) \\
&= \tfrac{1}{2}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)
\end{aligned} \tag{1}$$

We can repeat the same calculation for $\mathcal{J}_2^2$ and $\mathcal{J}_3^2$ and for $|\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle, |\uparrow\uparrow\rangle$ and thus obtain what $\vec{\mathcal{J}}^2$ is. The second strategy consists in computing directly $\mathcal{J}_i^2$:

$$\begin{aligned}
\mathcal{J}_1^2 &= \left(\tfrac{\sigma_1}{2}\otimes \mathrm{I} + \mathrm{I}\otimes\tfrac{\sigma_1}{2}\right)\cdot\left(\tfrac{\sigma_1}{2}\otimes \mathrm{I} + \mathrm{I}\otimes\tfrac{\sigma_1}{2}\right) \\
&= \tfrac{\sigma_1}{2}\otimes \mathrm{I}\cdot\tfrac{\sigma_1}{2}\otimes \mathrm{I} + \mathrm{I}\otimes\tfrac{\sigma_1}{2}\cdot \mathrm{I}\otimes\tfrac{\sigma_1}{2} + \tfrac{\sigma_1}{2}\otimes \mathrm{I}\cdot \mathrm{I}\otimes\tfrac{\sigma_1}{2} + \mathrm{I}\otimes\tfrac{\sigma_1}{2}\cdot\tfrac{\sigma_1}{2}\otimes \mathrm{I} \\
&= \tfrac{\sigma_1}{2}\cdot\tfrac{\sigma_1}{2}\otimes \mathrm{I}\cdot \mathrm{I} + \mathrm{I}\cdot \mathrm{I}\otimes\tfrac{\sigma_1}{2}\cdot\tfrac{\sigma_1}{2} + \tfrac{\sigma_1}{2}\cdot \mathrm{I}\otimes \mathrm{I}\cdot\tfrac{\sigma_1}{2} + \mathrm{I}\cdot\tfrac{\sigma_1}{2}\otimes\tfrac{\sigma_1}{2}\cdot \mathrm{I} \\
&= \tfrac{3}{2}\mathrm{I}\otimes \mathrm{I} + \tfrac{1}{2}\left(\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2 + \sigma_3\otimes\sigma_3\right) \\
&= \tfrac{1}{2}\left(\mathrm{I}\otimes \mathrm{I} + \sigma_1\otimes\sigma_1\right)
\end{aligned} \tag{2}$$

from which we obtain

$$\vec{\mathcal{J}}^2 = \frac{3}{2}\mathrm{I}\otimes \mathrm{I} + \frac{1}{2}\left(\sigma_1\otimes\sigma_1 + \sigma_2\otimes\sigma_2 + \sigma_3\otimes\sigma_3\right).$$

Now acting for instance on $|\uparrow\downarrow\rangle$, we find :

$$\begin{aligned}
\vec{\mathcal{J}}^2|\uparrow\downarrow\rangle &= \tfrac{3}{2}|\uparrow\downarrow\rangle + \tfrac{1}{2}\left(|\downarrow\uparrow\rangle + |\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle\right) \\
&= |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle.
\end{aligned} \tag{3}$$