Midterm Exam

November 3 2020 - 3h

1 Exercise 1

Consider the theory of a real scalar field, with action

$$S = \int d^4x \left[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - g\phi^p\right] \tag{1}$$

where $g > 0$ and $p \geq 3$ is an integer. We consider the following transformation, which for any $\lambda > 0$ acts on both space-time coordinates and fields:

$$x^\mu \to (x')^\mu = \lambda x^\mu, \qquad \phi(x) \to \phi'(x') = \lambda^{-D_\phi}\phi(x) \qquad (equivalently:\ \phi'(x) = \lambda^{-D_\phi}\phi(\lambda^{-1}x)). \tag{2}$$

where D_ϕ is a fixed real number.

1.1 Conditions for invariance

1. Find the value of D_ϕ such that the two-derivative term in S is invariant under the transformation (**??**).
2. For the value of D_ϕ found above, find the values of m and p such that (**??**) is a symmetry of the action.

1.2 Noether current

We suppose from now on that $D_\phi = 1$, $m = 0$ and $p = 4$.

3. Show that the action of an *infinitesimal* transformation of the type (**??**) on the coordinates and field is :

$$x'^\mu = x^\mu + \epsilon x^\mu, \qquad \phi'(x) = \phi(x) - \epsilon x^\mu\partial_\mu\phi(x) - \epsilon\phi(x), \qquad \epsilon \ll 1. \tag{3}$$

4. Write the corresponding infinitesimal transformation of the Lagrangian density L.
5. Construct the conserved Noether current J_μ associated to the transformation (**??**).
6. Recall the expression for the energy-momentum tensor $T_{\mu\nu}$. What is the relation between J_μ and $T_{\mu\nu}$?

2 Exercise 2

We consider the Hilbert space $\mathcal{H}$ of a quantum Klein-Gordon field $\hat{\phi}(x)$. We want to look for a state $|\Phi\rangle \in \mathcal{H}$ such that

$$\langle\Phi|\hat{\phi}(x^\mu)|\Phi\rangle = \phi(x^\mu) \tag{4}$$

where $\phi(x^\mu)$ is a given, but arbitrary, solution (which we assume not to be identically zero) of the classical Klein-Gordon equation,

$$\Box\phi(x^\mu) + m^2\phi(x^\mu) = 0. \tag{5}$$

We recall the mode expansion for the quantum field (be careful it is not exactly the same normalization as in the lectures: there is a $1/\sqrt{2\omega_{\vec{q}}}$ instead of $1/(2\omega_{\vec{q}})$ and, consistently, the commutation relation $[\hat{a}(\vec{q}), \hat{a}^\dagger(\vec{q}')]$ does not involve $\omega_{\vec{q}}$; going from one normalization to the other amounts to redefining $\hat{a}(\vec{q}) \to \sqrt{2\omega_{\vec{q}}}\,\hat{a}(\vec{q})$):

$$\hat{\phi}(x^\mu) = \int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_{\vec{q}}}}\left[\hat{a}(\vec{q})e^{-iqx} + \hat{a}^\dagger(\vec{q})e^{iqx}\right], \qquad qx \equiv \omega_q t - \vec{q}\cdot\vec{x}, \quad \omega_q = \sqrt{|\vec{q}|^2 + m^2}, \tag{6}$$

and the commutation relations:

$$[\hat{a}(\vec{q}), \hat{a}^\dagger(\vec{q}')] = (2\pi)^3\delta^{(3)}(\vec{q} - \vec{q}'). \tag{7}$$

1. Take $|\Phi\rangle$ to be an eigenstate of $\hat{a}(\vec{q})$:

$$\hat{a}(\vec{q})|\Phi\rangle = \alpha(\vec{q})|\Phi\rangle. \tag{8}$$

   Show that it has the property (**??**), with a function $\phi(x^\mu)$ given in terms of $\alpha(\vec{q})$.

2. Show that (**??**) cannot be satisfied if $|\Phi\rangle$ is a state containing a finite number of particles. *[Recall that a generic n-particles state is given (up to a normalization factor) by:*

$$|n\rangle = \hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)\dots\hat{a}^\dagger(\vec{k}_n)|0\rangle. \tag{9}$$

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3. Consider now the following state:

$$|\Phi\rangle = \mathcal{N}\sum_{n=0}^{+\infty} c_n\left(\int\frac{d^3k}{(2\pi)^3}z(\vec{k})\hat{a}^\dagger(\vec{k})\right)^n|0\rangle. \tag{10}$$

   where $\mathcal{N}$ is a normalization coefficient which (for now) will be left undetermined. Determine the function $z(\vec{q})$ and the coefficients c_n such that the state defined by (**??**) has the desired property (**??**).

4. With the coefficients c_n determined as above, write the state $|\Phi\rangle$ in a compact form by performing the sum over n.

5. Determine the normalization factor $\mathcal{N}$ by requiring $\langle\Phi|\Phi\rangle = 1$ (assume the vacuum is normalized to unity, $\langle 0|0\rangle = 1$). *[Hint: recall that when the commutator of two operators A and B is a number (i.e. a multiple of the identity operator) then $e^A e^B = e^{A+B}e^{[A,B]/2}$.]*

6. Consider now is a Dirac fermion field $\psi(x)$. Recall the mode expansion and anticommmutation relations which replace equations (**??**-**??**). Show that in this case no such state as (**??**) exists which satisfies the analog of equation (**??**) (with the left hand side now satisfying Dirac's equation) or equivalently of equation (**??**) . What is the physical conclusion one can draw from this ?

3 Exercise 3

1. **Scalar**
   Let $\phi(x)$ be a real Klein-Gordon field of mass m. Define the function:

$$D(x,y) = \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle. \tag{11}$$

   By expanding $\phi(x)$ in momentum modes (see below), show that:

$$D(x,y) = \int \frac{d^3q}{(2\pi)^3} \frac{e^{-iq(x-y)}}{2\omega_q.} \tag{12}$$

   [Recall the mode expansion:

$$\hat{\phi}(x) = \int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_{\vec{q}}}} \left[\hat{a}(\vec{q})e^{-iqx} + \hat{a}^\dagger(\vec{q})e^{iqx}\right], \qquad qx \equiv \omega_q t - \vec{q}\cdot\vec{x},\ \ \omega_q = \sqrt{|\vec{q}|^2 + m^2}, \tag{13}$$

   and commutation relations:

$$[\hat{a}(\vec{q}), \hat{a}^\dagger(\vec{q}')] = (2\pi)^3\delta^{(3)}(\vec{q}-\vec{q}'). \tag{14}$$

   (be careful it is not exactly the same normalization as in the lectures: there is a $1/\sqrt{2\omega_{\vec{q}}}$ instead of $1/(2\omega_{\vec{q}})$ and, consistently, the commutation relation $[\hat{a}(\vec{q}), \hat{a}^\dagger(\vec{q}')]$ does not involve $\omega_{\vec{q}}$; going from one normalization to the other amounts to redefining $\hat{a}(\vec{q}) \to \sqrt{2\omega_{\vec{q}}}\,\hat{a}(\vec{q})$)]

2. **Spinor**
   Consider now a Dirac spinor field $\psi(x)$, and the function:

$$D_{\alpha\beta}(x,y) = \langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle \tag{15}$$

   Show that:

$$D_{\alpha\beta}(x,y) = (-i\partial\!\!\!/ + m\mathbf{1})_{\alpha\beta} \int \frac{d^3q}{(2\pi)^3} \frac{e^{-iq(x-y)}}{2\omega_q}. \tag{16}$$

   [Recall the mode expansion (same remark as above for the normalization of $\hat{b}_s(\vec{q})$ and $\hat{c}_s^\dagger(\vec{q})$):

$$\psi_\alpha(x) = \sum_{s=1}^{2} \int \frac{d^3q}{(2\pi)^3\sqrt{2\omega_{\vec{q}}}} \left[\hat{b}_s(\vec{q})u_{s,\alpha}(\vec{q})e^{-iqx} + \hat{c}_s^\dagger(\vec{q})v_{s,\alpha}(\vec{q})e^{iqx}\right], \tag{17}$$

   where s runs over the two spin polarizations and $u_s(p)$ and $v_s(p)$ are positive- and negative-frequency spinors, solutions of the momentum-space Dirac equation.
   The anti-commutation relations are:

$$\{\hat{b}_s(\vec{q}), \hat{b}_{s'}^\dagger(\vec{q}')\} = (2\pi)^3\delta^{(3)}(\vec{q}-\vec{q}')\delta_{ss'}, \quad \{\hat{c}_s(\vec{q}), \hat{c}_{s'}^\dagger(\vec{q}')\} = (2\pi)^3\delta^{(3)}(\vec{q}-\vec{q}')\delta_{ss'}, \tag{18}$$

   with all other anti-commutators vanishing. You may use the identities:

$$\sum_{s=1}^{2} u_{s,\alpha}(\vec{p})\bar{u}_{s,\beta}(\vec{p}) = (p\!\!\!/ + m\mathbf{1})_{\alpha\beta}, \qquad \sum_{s=1}^{2} v_{s,\alpha}(\vec{p})\bar{v}_{s,\beta}(\vec{p}) = (p\!\!\!/ - m\mathbf{1})_{\alpha\beta}. \tag{19}$$

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