Solution of the mid-term exam, November 2021

1. In a passive rotation, the vector field $\vec{V}$ transforms as:

$$V_i'(\vec{x}') = R_{ij}\, V_j(\vec{x}) \qquad \text{with} \quad x_i' = R_{ij}\, x_j$$

2. $$S = \int d^3x \; \partial_i A_j(x)\, \partial_i A_j(x) = S[\vec{A}(x)]$$

For the new field

$$S[\vec{A}'(\vec{x}')] = \int d^3\vec{x}' \; \partial_i' A_j'(x')\, \partial_i' A_j'(x') \quad \left(\text{with } \partial_i' = \frac{\partial}{\partial x_i'}\right)$$

$$d^3\vec{x}' = d^3\vec{x} \quad \text{because} \quad \det R = 1$$

$$\Rightarrow \quad S[A_i'(x')] = \int d^3x \; R_{ik}\partial_k (R_{jl} A_l(x))\, R_{im}\partial_m (R_{jn} A_n(x))$$

$$= \int d^3x \; ({}^tR_{ki} R_{im})({}^tR_{lj} R_{jn})\, \partial_k A_l(x)\, \partial_m A_n(x)$$

$$= \int d^3x \; \partial_m A_n(x)\, \partial_m A_n(x)$$

since ${}^tR_{ki} R_{im} = ({}^tRR)_{km} = \delta_{km}$ because $R \in SO(3)$

3.a $$A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} A^{\nu}(x) \qquad \text{with} \quad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

3.b $$\eta_{\mu\nu} \Lambda^{\mu}{}_{\rho} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$$

$$\Leftrightarrow \sum_{\mu,\nu} ({}^t\Lambda)^{\rho}{}_{\mu}\, \eta_{\mu\nu} \Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}$$

$$\Rightarrow \quad \det({}^t\Lambda\, \eta\, \Lambda) = \det \eta$$

$$\Leftrightarrow \quad (\det \Lambda)^2 \det \eta = \det \eta$$

$$\Rightarrow \quad (\det \Lambda)^2 = 1 \quad \Rightarrow \quad |\det \Lambda| = 1$$

for a Lorentz boost that can be obtained by composition of infinitesimal Lorentz boosts that are infinitesimally close to the identity $\det \Lambda = +1$

4. $$S = \int d^4x \; \partial_\mu A_\nu(x)\, \partial^\mu A^\nu(x)$$

for the new field

$$S' = \int d^4x' \; \partial'_\mu A'_\nu(x')\, \partial'^\mu A'^\nu(x') \qquad \left(\text{with } \partial'_\mu = \frac{\partial}{\partial x'^\mu}\right)$$

$d^4x' = d^4x$ for Lorentz boosts

$$S' = \int d^4x \; \Lambda_\mu{}^\rho \partial_\rho\left(\Lambda_\nu{}^\sigma A_\sigma(x)\right) \Lambda^\mu{}_\tau \partial^\tau \left(\Lambda^\nu{}_\delta A^\delta(x)\right)$$

$$= \int d^4x \; \Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma \Lambda^\mu{}_\tau \Lambda^\nu{}_\delta \, \partial_\rho A_\sigma \partial^\tau A^\delta$$

$$Q = \Lambda_\mu{}^\rho \Lambda^\mu{}_\tau = \eta_{\mu\nu} \Lambda^\nu{}_\sigma \eta^{\sigma\rho} \Lambda^\mu{}_\tau$$

using $\eta_{\mu\nu} \Lambda^\nu{}_\sigma \Lambda^\mu{}_\tau = \eta_{\sigma\tau}$

$$\Rightarrow Q = \eta_{\sigma\tau} \eta^{\sigma\rho} = \delta^\rho_\tau$$

$$\Rightarrow S' = \int d^4x \; \delta^\rho_\tau \delta^\sigma_\delta \, \partial_\rho A_\sigma \partial^\tau A^\delta$$

$$= \int d^4x \; \partial_\tau A_\delta \partial^\tau A^\delta$$

$$= S$$ $\Rightarrow S$ is invariant under Lorentz transformations

5. Under translations

$$A'_\mu(x') = A_\mu(x) \qquad \text{for } x'^\mu = x^\mu + a^\mu$$

$$S' = \int d^4x' \; \partial'_\mu A'_\nu(x')\, \partial'^\mu A'^\nu(x')$$

$d^4x' = d^4x$ and $\partial'^\mu = \partial^\mu$

$$\Rightarrow S' = \int d^4x \; \partial_\mu A_\nu(x) \partial^\mu A^\nu(x)$$

$$= S$$

6: $Z(x)$ is a spinor field $\Rightarrow Z'_\alpha(x') = U_{\alpha\beta} Z_\beta(x)$

with $x'_i = R_{ij} x_j$ and $U \in SU(2)$: $R = R(\vec{\theta})$ and $U = U(\vec{\theta})$

$$V(x) = Z^\dagger(x)\, \sigma_i\, \partial_i Z(x)$$

$$V'(x') = Z'^\dagger(x')\, \sigma_i\, \partial'_i Z'(x')$$

$$= Z'^\dagger_\alpha(x')\, (\sigma_i)_{\alpha\beta}\, \partial'_i Z'_\beta(x')$$

$$= Z^\dagger_\delta(x)\, U^\dagger_{\delta\alpha}\, (\sigma_i)_{\alpha\beta}\, R_{ij}\, \partial_j\, U_{\beta\sigma} Z_\sigma(x)$$

$$= R_{ij}\, Z^\dagger_\delta(x)\, U^\dagger_{\delta\alpha} (\sigma_i)_{\alpha\beta} U_{\beta\sigma}\, \partial_j Z_\sigma(x)$$

Using $U^\dagger \sigma_i U = R_{ij}\sigma_j$ we get

$$V'(x') = R_{ij}\, R_{ik}\, Z^\dagger_\delta(x)\, (\sigma_k)_{\delta\sigma}\, \partial_j Z_\sigma(x)$$

$$= ({}^tR R)_{jk}\, Z^\dagger(x)\, \sigma_k\, \partial_j Z(x)$$

$$= Z^\dagger(x)\, \sigma_j\, \partial_j Z(x)$$

$$= V(x) \qquad \Rightarrow \quad V \text{ is a scalar field}$$

7.a There are two types of Lorentz spinors with two components:

* χ_L that transforms with $M_1(\vec{\theta}, \vec{\varphi})$
* χ_R " " $M_2(\vec{\theta}, \vec{\varphi})$

7.b The generalization is obtained by defining

$$V(x) = \chi_L^\dagger(x)\, \sigma^\mu\, \partial_\mu \chi_L(x)$$

with $\sigma^\mu = (\mathbb{1}, -\vec{\sigma}) = (\mathbb{1}, -\sigma_i)$

It is obviously invariant under rotations from 6 and the proof that it is also under Lorentz boosts has been done during the lectures: take an infinitesimal boost along $\hat{n}$ and compute:

$$(\chi_L^\dagger \sigma^\mu \chi_L)' = \begin{cases} \mu=0 : (\chi_L^\dagger \chi_L)' = \chi_L^\dagger (1\!\!1 + \sigma_1 d\phi)\chi_L \\ \mu = i : (\chi_L^\dagger \sigma_i \chi_L)' = \chi_L^\dagger (1\!\!1 + \sigma_1 d\phi)\sigma_i (1\!\!1 + \sigma_1 d\phi)\chi_L \end{cases}$$

$$= \begin{pmatrix} 1 & d\phi & & \\ d\phi & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \chi_L^\dagger \sigma^\mu \chi_L$$

$\Rightarrow$ $V^\mu = \chi_L^\dagger \sigma^\mu \chi_L$ is a 4-vector $\Rightarrow$ $V'^\mu = \Lambda^\mu{}_\nu \chi_L^\dagger \sigma^\nu \chi_L$

$$\Rightarrow V' = \Lambda^\mu{}_\nu \Lambda_\mu{}^\rho \chi_L^\dagger \sigma^\nu \partial_\rho \chi_L$$

$$= \delta^\rho_\nu \chi_L^\dagger \sigma^\nu \partial_\rho \chi_L = \chi_L^\dagger \sigma^\mu \partial_\mu \chi_L = V$$

$\Rightarrow$ V is a scalar

Exercise 2

2.1 $\mathcal{L} = i\bar\psi \gamma^\mu \partial_\mu \psi - m\bar\psi\psi$

$$\frac{\partial \mathcal{L}}{\partial \bar\psi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \bar\psi} = i\gamma^\mu \partial_\mu \psi - m\psi \; ; \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar\psi)} = 0$$

$$\Rightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0$$

Let us notice that the Euler-Lagrange for ψ leads to

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\bar\psi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar\psi\gamma^\mu$$

$$\Rightarrow -m\bar\psi - i\partial_\mu \bar\psi \gamma^\mu = 0$$

Take the hermitic conjugate of this equation:

$$m(\psi^\dagger \gamma^0)^\dagger - i\gamma^{\mu\dagger}(\partial_\mu \psi^\dagger \gamma^0)^\dagger = 0$$

$$\Rightarrow m\gamma^0\psi - i\gamma^0\gamma^\mu\gamma^0\gamma^0 \partial_\mu \psi = 0$$

where we have used $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$ and $(\gamma^0)^2 = 1\!\!1$

$$\Rightarrow \gamma^0(m\psi - i\gamma^\mu \partial_\mu \psi) = 0$$

$\Rightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0$ $\Rightarrow$ the two Euler-Lagrange equations are hermitic conjugate

2.2 We assume that $\psi'_\alpha(x') = S_{\alpha\beta}(\Lambda)\,\psi_\beta(x)$

and $\psi'_\alpha(x)$ satisfies also the Dirac equation

$$\Rightarrow \quad (i\gamma^\mu \partial'_\mu - m)\,\psi'(x') = 0$$

$$\Rightarrow \quad (i\gamma^\mu \Lambda_\mu{}^\rho \partial_\rho - m)\, S\psi(x) = 0$$

by multiplying this equation by S^{-1} and comparing with the Dirac equation for $\psi(x)$ we get:

$$i S^{-1}\gamma^\mu S \Lambda_\mu{}^\rho \partial_\rho \psi = i\gamma^\nu \partial_\nu \psi$$

$$\Rightarrow \quad S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu \Rightarrow (S^{-1})_{\alpha\beta}(\gamma^\mu)_{\beta\delta} S_{\delta\rho} = \Lambda^\mu{}_\nu (\gamma^\nu)_{\alpha\rho}$$

3.a $$S = \mathbb{1} - \frac{i}{4}\varepsilon_{\mu\nu}\sigma^{\mu\nu}$$

$$\Rightarrow \quad S_{\alpha\beta} = \delta_{\alpha\beta} - \frac{i}{4}\varepsilon_{\mu\nu}(\sigma^{\mu\nu})_{\alpha\beta}$$

For each couple of indices μ,ν, $\sigma^{\mu\nu}$ is a 4×4 matrix : $(\sigma^{\mu\nu})_{\alpha\beta}$ with $\alpha,\beta = 1-4$.

3.b The set of parameters $\varepsilon_{\mu\nu}$ is antisymmetric so that the symmetric part in $\sigma^{\mu\nu}$ would play no role $\Rightarrow$ we can choose it antisymmetric

3.c. $$\sigma^{\mu\nu} = \alpha[\gamma^\mu, \gamma^\nu]$$

$$\Rightarrow \quad (\mathbb{1} + \frac{i}{4}\varepsilon_{\tau\delta}\sigma^{\tau\delta})\,\gamma^\mu\,(\mathbb{1} - \frac{i}{4}\varepsilon_{\tau\delta}\sigma^{\tau\delta}) = \Lambda^\mu{}_\nu\gamma^\nu = \gamma^\mu + \varepsilon^\mu{}_\nu\gamma^\nu$$

We therefore compute

$$A^{\tau\delta\mu} = \sigma^{\tau\delta}\gamma^\mu - \gamma^\mu\sigma^{\tau\delta} = \alpha[\gamma^\tau,\gamma^\delta]\gamma^\mu - \alpha\gamma^\mu[\gamma^\tau,\gamma^\delta]$$

$$= \alpha\left(\gamma^\tau\gamma^\delta\gamma^\mu - \gamma^\delta\gamma^\tau\gamma^\mu - \gamma^\mu\gamma^\tau\gamma^\delta + \gamma^\mu\gamma^\delta\gamma^\tau\right)$$

$$\gamma^\tau \gamma^\delta \gamma^\mu = -\gamma^\tau \gamma^\mu \gamma^\delta + 2\eta^{\mu\delta}\gamma^\tau$$
$$= \gamma^\mu \gamma^\tau \gamma^\delta + 2\eta^{\mu\delta}\gamma^\tau - 2\eta^{\mu\tau}\gamma^\delta$$

$$\Rightarrow A^{\tau\delta\mu} = \alpha\big(\gamma^\mu\gamma^\tau\gamma^\delta + 2\eta^{\mu\delta}\gamma^\tau - 2\eta^{\mu\tau}\gamma^\delta$$
$$- \gamma^\mu\gamma^\delta\gamma^\tau - 2\eta^{\mu\tau}\gamma^\delta + 2\eta^{\mu\delta}\gamma^\tau$$
$$- \gamma^\mu\gamma^\tau\gamma^\delta + \gamma^\mu\gamma^\delta\gamma^\tau \big)$$
$$= 4\alpha\left(\eta^{\mu\delta}\gamma^\tau - \eta^{\mu\tau}\gamma^\delta \right)$$

$$\Rightarrow \frac{i}{4}\,\varepsilon_{\tau\delta}\,A^{\tau\delta\mu} = i\alpha\,\varepsilon_{\tau\delta}\left(\eta^{\mu\delta}\gamma^\tau - \eta^{\mu\tau}\gamma^\delta\right)$$
$$= 2i\alpha\,\varepsilon_{\tau\delta}\,\eta^{\mu\delta}\gamma^\tau$$
$$= -2i\alpha\,\varepsilon_{\delta\tau}\,\eta^{\mu\delta}\gamma^\tau$$
$$= -2i\alpha\,\varepsilon^\mu{}_\nu\,\gamma^\nu$$

$$\Rightarrow \alpha = \frac{i}{2} \qquad \Rightarrow \sigma^{\mu\nu} = \frac{i}{2}\left[\gamma^\mu, \gamma^\nu\right]$$

2.4 $$\sigma_{23} = \frac{i}{2}\left[\gamma^2, \gamma^3\right] = \frac{i}{2}\left[\begin{pmatrix} & \sigma_2 \\ -\sigma_2 & \end{pmatrix}, \begin{pmatrix} & \sigma_3 \\ -\sigma_3 & \end{pmatrix}\right]$$
$$= \frac{i}{2}\begin{pmatrix} -[\sigma_2,\sigma_3] & 0 \\ 0 & -[\sigma_2,\sigma_3] \end{pmatrix}$$
$$= \frac{1}{2}\,2i\,(-i)\begin{pmatrix} \sigma_1 & \\ & \sigma_1 \end{pmatrix}$$
$$= 2\begin{pmatrix} \frac{\sigma_1}{2} & \\ & \frac{\sigma_1}{2} \end{pmatrix}$$
(there was a mismatch of a factor 2 with the text!)

$\Rightarrow \sigma_{23} = 2S_1$ (and not S_1).

In a rotation around $\hat{x}$:

$$S = \mathbb{1} - \frac{i}{4}\left(\varepsilon_{23}\sigma^{23} + \varepsilon_{32}\sigma^{32}\right) = \mathbb{1} - \frac{i}{2}\varepsilon_{23}\sigma^{23} = \mathbb{1} + \frac{i}{2}d\theta_1\begin{pmatrix} \frac{\sigma_1}{2} & \\ & \frac{\sigma_1}{2} \end{pmatrix}$$

which is the expected rotation on $\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$.

For a boost:

$$\sigma^{01} = \frac{i}{2}[\gamma^0,\gamma^1] = \frac{i}{2}\left[\begin{pmatrix} & 1\!\!1 \\ 1\!\!1 & \end{pmatrix}, \begin{pmatrix} & \sigma_1 \\ -\sigma_1 & \end{pmatrix}\right]$$

$$= i\begin{pmatrix} -\sigma_1 & \\ & \sigma_1 \end{pmatrix}$$

and more generally

$$\sigma^{0i} = 2i\begin{pmatrix} -\frac{\sigma_i}{2} & \\ & \frac{\sigma_i}{2} \end{pmatrix}$$

$\Rightarrow$ for a boost along $\hat{x}$:

$$S = 1\!\!1 - \frac{i}{4}(\varepsilon_{01}\sigma^{01} + \varepsilon_{10}\sigma^{10})$$

$$= 1\!\!1 - \frac{i}{2}\varepsilon_{01}\sigma^{01}$$

$$= 1\!\!1 + \frac{i}{2}d\phi_1 \, 2i\begin{pmatrix} -\frac{\sigma_1}{2} & \\ & \frac{\sigma_1}{2} \end{pmatrix}$$

$$= 1\!\!1 + d\phi_1\begin{pmatrix} \frac{\sigma_1}{2} & \\ & \frac{\sigma_1}{2} \end{pmatrix}$$

which is the expected transformation for $\begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix}$.

5. $h(\vec{p}) = \dfrac{\vec{p}\cdot\vec{S}}{|\vec{p}|}$

Let us first compute the eigenvalues of $\dfrac{\vec{p}\cdot\vec{\sigma}}{|\vec{p}|}$:

$$\frac{\vec{p}\cdot\vec{\sigma}}{|\vec{p}|}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\Rightarrow \frac{1}{\vec{p}^2}(\vec{p}\cdot\vec{\sigma})^2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \lambda^2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

but $(\vec{p}\cdot\vec{\sigma})^2 = p_i p_j \sigma_i \sigma_j$

$$= p_i p_j (\delta_{ij} 1\!\!1 + i\varepsilon_{ijk}\sigma_k)$$

$$= \vec{p}^{\,2} 1\!\!1$$

$$\Rightarrow \frac{(\vec{p}\cdot\vec{\sigma})^2}{|\vec{p}|^2} = 1\!\!1 \quad \Rightarrow \quad \lambda = \pm 1$$

$\Rightarrow$ the eigenvalues of $h(\vec{p}) = \frac{\vec{p}}{|\vec{p}|} \cdot \frac{\vec{\sigma}}{2}$ are $\pm \frac{1}{2}$

6. $\frac{\mathbb{1}+\gamma^5}{2} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix}$ $\qquad \frac{\mathbb{1}-\gamma^5}{2} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = \begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$

They are obviously projectors.

$$\gamma^5 \begin{pmatrix} 0 \\ \chi_R \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_R \end{pmatrix}$$

$$\gamma^5 \begin{pmatrix} \chi_L \\ 0 \end{pmatrix} = -\begin{pmatrix} \chi_L \\ 0 \end{pmatrix}$$

7. $(i\gamma^\mu \partial_\mu - m) \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0$

$$\Rightarrow \left\{ i \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} \partial_0 + i \begin{pmatrix} & \sigma_i \\ -\sigma_i & \end{pmatrix} \partial_i - m \begin{pmatrix} \mathbb{1} & \\ & \mathbb{1} \end{pmatrix} \right\} \begin{pmatrix} \chi_L \\ \chi_R \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} i(\partial_0 + \sigma_i \partial_i)\chi_R - m\chi_L = 0 \\ i(\partial_0 - \sigma_i \partial_i)\chi_L - m\chi_R = 0 \end{cases}$$

Obviously, when $m=0$, the chiral components satisfy decoupled equations.

Exercise 3

1. Call $R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ $\Rightarrow$ $(\psi'_i(x)) = R_{ij}\psi_j(x)$

$\lightning$ $\psi_1(x)$ and $\psi_2(x)$ are 4-components bi-spinors. Thus, the above relation means

$$[\psi'_i(x)]_\alpha = R_{ij}[\psi_j(x)]_\alpha$$

or $\psi'_1(x) = \cos\theta\, \psi_1(x) + \sin\theta\, \psi_2(x) \Rightarrow [\psi'_1(x)]_\alpha = \cos\theta [\psi_1(x)]_\alpha + \sin\theta [\psi_2(x)]_\alpha$

$$\psi'_i = R_{ij}\psi_j \quad \Rightarrow \quad \psi_i'^{+} = R_{ij}\psi_j^{+} \quad \Rightarrow \quad \bar{\psi}_i'(x) = R_{ij}\bar{\psi}_j(x)$$

$$\Rightarrow \mathcal{L}_{kin}(\psi') = i\,\bar{\psi}_i'(x)\gamma^\mu\partial_\mu\psi_i'(x)$$
$$= i\,R_{ij}\bar{\psi}_j(x)\gamma^\mu\partial_\mu\left(R_{ik}\psi_k(x)\right)$$
$$= i\,R_{ij}R_{ik}\,\bar{\psi}_j(x)\gamma^\mu\partial_\mu\psi_k(x)$$
$$= i\,\bar{\psi}_j'(x)\,\gamma^\mu\partial_\mu\psi_j(x)$$

we can see that neither γ^μ nor ∂_μ plays any role in this calculation. Therefore, exactly the same calculation holds for $\bar{\psi}_i\psi_i$:

$$\left(\bar{\psi}_i(x)\psi_i(x)\right)' = \bar{\psi}_i(x)\psi_i(x)$$

and therefore $\mathcal{L}_{int}(\psi') = \mathcal{L}_{int}(\psi)$

2. $$m_1\left(\bar{\psi}_1(x)\psi_1(x)\right)' = m_1\left(\cos\theta\,\bar{\psi}_1 + \sin\theta\,\bar{\psi}_2\right)\left(\cos\theta\,\psi_1 + \sin\theta\,\psi_2\right)$$
$$= m_1\left(\cos^2\theta\,\bar{\psi}_1\psi_1 + \sin^2\theta\,\bar{\psi}_2\psi_2 + \cos\theta\sin\theta\,\bar{\psi}_1\psi_2 + \cos\theta\sin\theta\,\bar{\psi}_2\psi_1\right)$$

$$m_2\left(\bar{\psi}_2(x)\psi_2(x)\right)' = m_2\left(-\sin\theta\,\bar{\psi}_1 + \cos\theta\,\bar{\psi}_2\right)\left(-\sin\theta\,\psi_1 + \cos\theta\,\psi_2\right)$$
$$= m_2\left(\sin^2\theta\,\bar{\psi}_1\psi_1 + \cos^2\theta\,\bar{\psi}_2\psi_2 - \sin\theta\cos\theta\left(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1\right)\right)$$

$$\Rightarrow -\mathcal{L}_{mass}(\psi'_i) = \left(m_1\cos^2\theta + m_2\sin^2\theta\right)\bar{\psi}_1\psi_1 + \left(m_1\sin^2\theta + m_2\cos^2\theta\right)\bar{\psi}_2\psi_2$$
$$+ \left(m_1 - m_2\right)\cos\theta\sin\theta\left(\bar{\psi}_1\psi_2 + \bar{\psi}_2\psi_1\right)$$

$$\mathcal{L}_{mass}(\psi'_i) = \mathcal{L}_{mass}(\psi_i) \quad \forall\psi_i \text{ and } \forall\theta$$

$$\Rightarrow \begin{cases} m_1\cos^2\theta + m_2\sin^2\theta = m_1 \\ m_1\sin^2\theta + m_2\cos^2\theta = m_2 \\ (m_1 - m_2)\sin\theta\cos\theta = 0 \end{cases} \quad \Rightarrow \quad m_1 = m_2$$

3.a $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_1(x))} = i\bar{\psi}\gamma^\mu$

$$\Rightarrow j^\mu(x) = i\bar{\psi}_1\gamma^\mu\psi_2 - i\bar{\psi}_2\gamma^\mu\psi_1$$

Let us compute: $-i\partial_\mu j^\mu$:

$$-i\partial_\mu j^\mu = \partial_\mu\bar{\psi}_1\gamma^\mu\psi_2 + \bar{\psi}_1\gamma^\mu\partial_\mu\psi_2 - \partial_\mu\bar{\psi}_2\gamma^\mu\psi_1 - \bar{\psi}_2\gamma^\mu\partial_\mu\psi_1$$

and we have

$$(i\gamma^\mu\partial_\mu - m)\psi_i = 0 \quad \text{and} \quad i\partial_\mu\bar{\psi}\gamma^\mu + m\bar{\psi} = 0$$

$$\Rightarrow \partial_\mu j^\mu = -m\bar{\psi}_1\psi_2 + m\bar{\psi}_1\psi_2 + m\bar{\psi}_2\psi_1 - m\bar{\psi}_2\psi_1$$

$$= 0$$

3.b When $g \neq 0$, the Euler-Lagrange equations become:

$$i\gamma^\mu\partial_\mu\psi_i - m\psi_i - g(\bar{\psi}_j\psi_j)\psi_i = 0$$

and

$$-i\partial_\mu\bar{\psi}_i\gamma^\mu - m\bar{\psi}_i - g(\bar{\psi}_j\psi_j)\bar{\psi}_i = 0$$

$$\Rightarrow j^\mu = i\bar{\psi}_1\gamma^\mu\psi_2 - i\bar{\psi}_2\gamma^\mu\psi_1$$

$$\partial_\mu j^\mu = (-m\bar{\psi}_1 - g(\bar{\psi}_j\psi_j)\bar{\psi}_1)\psi_2 + \bar{\psi}_1(m\psi_2 + g(\bar{\psi}_j\psi_j)\psi_2)$$
$$+ (m\bar{\psi}_2 + g(\bar{\psi}_j\psi_j)\bar{\psi}_2)\psi_1 \;\; -\bar{\psi}_2(m\psi_1 + g(\bar{\psi}_j\psi_j)\psi_1)$$

$$= 0$$

j^μ is the Noether current of the O(2) symmetry