$$\boxed{\text{lecture 8}}$$

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}dr^2 + r^2 d\Omega^2$$

$r > 2M$

(v, r, θ, ϕ)

$r < 2M.$

(u, r, θ, ϕ)

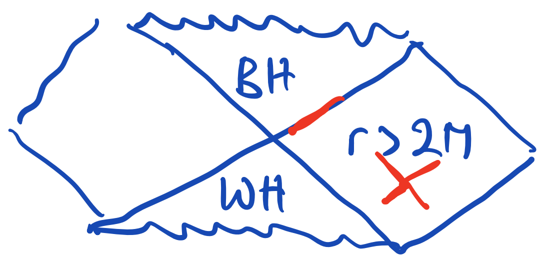
- Can we draw a space-time diagram

which shows both the black-hole & white-hole regions ?

→ Yes !

- But it requires some "machinery" to do so.

↳ Conformal / Penrose diagrams.

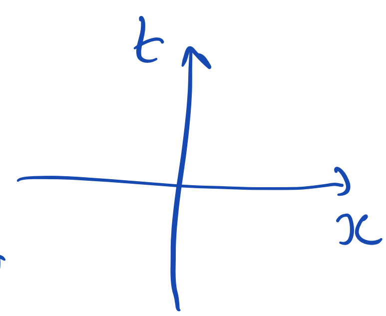


Idea of a conformal diagram:

- Want a way of representing ∀ of space-time on a finite piece of paper

- This is not the case of Minkowski s.t for eg. since $-\infty < t < \infty$

$-\infty < x < \infty$



⇒ would need an ∞ big piece of paper.

-

$f(x \to \pm\infty)$: finite numbers

eg: $\arctan(x \to \pm\infty) = \pm\frac{\pi}{2}$

- A conformal transf is (1) a change of variable (not a coord transf) s.t the new variables have a finite domain (2) Also impose that in the new variable light cones are at 45°.

- Conformal transf^{n}:

  - Start with a physical metric describing physical space-time $g_{\mu\nu}$. (eg. Sch. Minkowski)

$$\rightarrow \quad \tilde{g}_{\mu\nu}(x) = \underbrace{\Lambda^2(x)}_{\text{smooth, non-zero } f^{n}.} g_{\mu\nu}(x)$$

$$\rightarrow \quad \boxed{d\tilde{s}^2 = \Lambda^2(x)\, ds^2}$$

change of scale.

$$ds^2 = \frac{1}{\Lambda^2(x)} d\tilde{s}^2$$

Minkowski

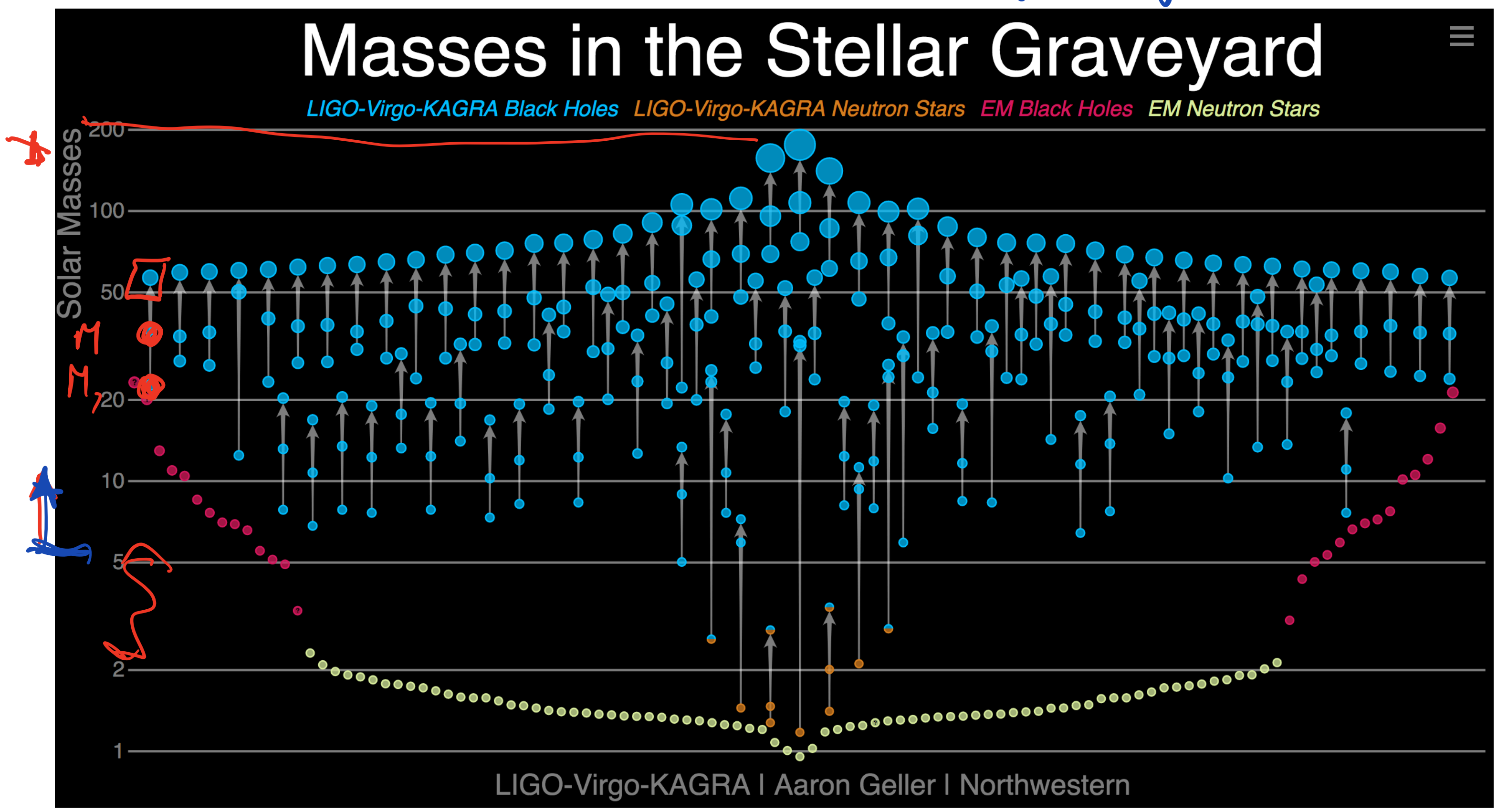
new line element

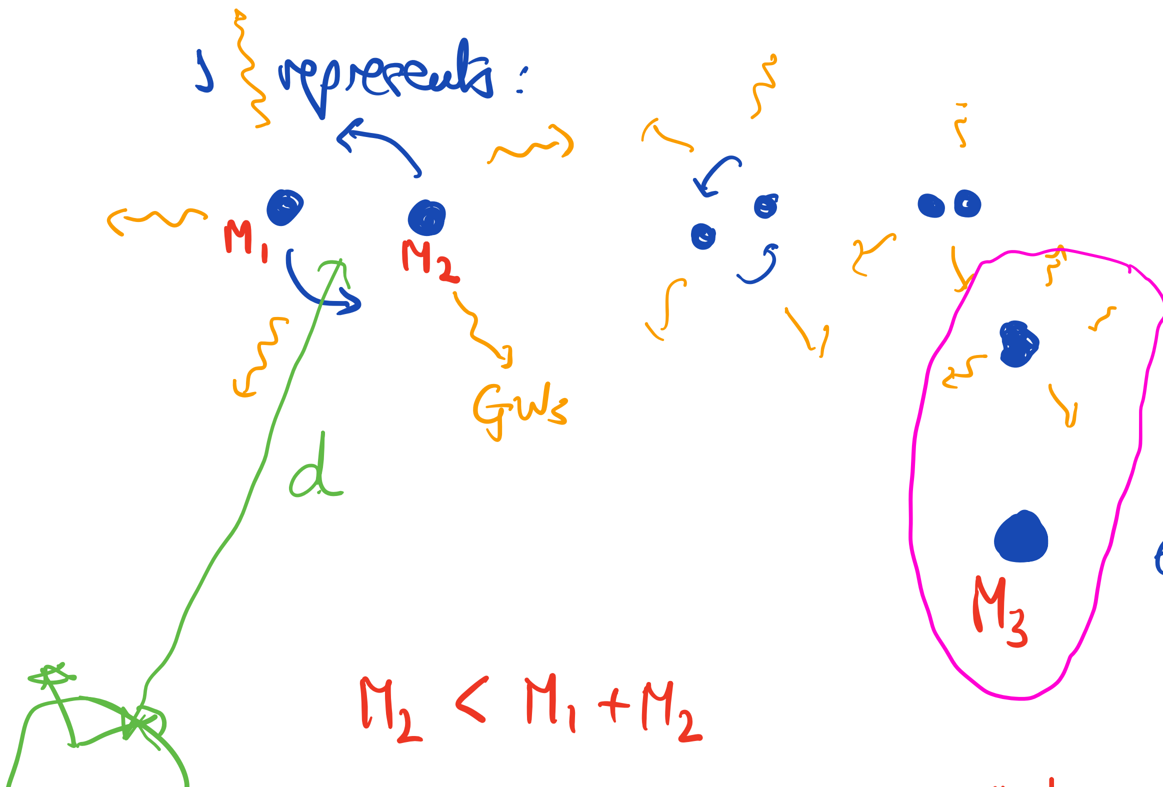
  - Please read the set of notes "conformal diags" and "Sch in Kruskal coords" to see how to draw the conformal diag for (i) Minkowski (ii) Sch.

[Non-examinable].

$\boxed{\text{GWs}}$

BH observed through their
↓ GW emission by the
Ligo-Virgo collab.





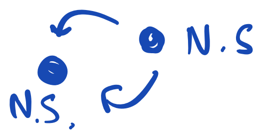
Remember:
an isolated BH
cannot emit
GWs, since
the sph. symm
sol^n was static

equil^m final state
is sph. symm

$M_2 < M_1 + M_2$

Since energy is emitted in GWs during
the process of M_1 & M_2 merging to form M_3.

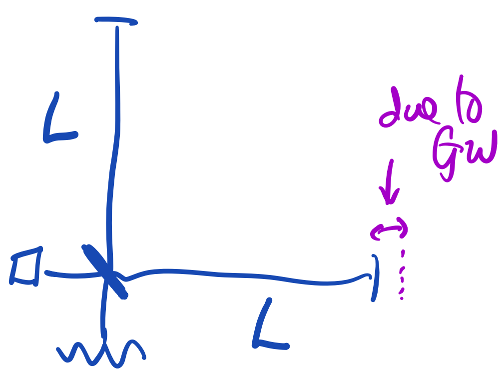
- below $5M_\odot$, the objects could be Neutron stars:



. In this case one has "tidal" effects due to the matter making up the neutron stars → influences the GW emission

→ & leads to effects which can be looked for experimentally.

Ligo-Virgo, as we will see, is an experiment consisting essentially of a Michelson-Morely interferometer.



As a GW passes, it changes the length of the arms from

$L \to L + \delta L$

$\Rightarrow$ leads to a modified interference pattern.

δL depends on the GW emitted, & hence (for binary BH systems) on M_1 & M_2.

What is L for Ligo-Virgo? $L \sim 3$ km.

$$f \sim \frac{1}{L} \to 1\,\text{Hz}$$

Frequency an experiment is sensitive to depends on the length L of the arms.

What is L for the future LISA detector

$L \sim$ millions of km.

$f \sim \frac{1}{L} \longrightarrow$ milliHz.

As we will see, that means LIGO $\Rightarrow$ sensitive to BH masses $1 < \frac{M}{M_\odot} <$ few hundred

remnants of stars which have run out of nuclear fuel $\rightarrow$ formed BH.

LISA $\Rightarrow$ ~~sensitive~~ sensitive to much heavier BHs including $M \sim 10^9 M_\odot$

their formation is an active area of research today.

To calculate the GW emission, we will work in an approximation scheme known as "linearized GR".

i.e. expand E. eq^{n}s about some backgd space-time $\bar{g}_{\mu\nu}(x)$.

$O(\epsilon^0)$ $O(\epsilon)$

$$\rightarrow \boxed{g_{\mu\nu}(x) = \bar{g}_{\mu\nu}(x) + h_{\mu\nu}(x) + \dots}$$

where $h_{\mu\nu} \ll 1$ small.

- perturbative expansion in powers of $h_{\mu\nu}$, working to linear order only
- $\bar{g}_{\mu\nu}$ is unaffected by $h_{\mu\nu}$.

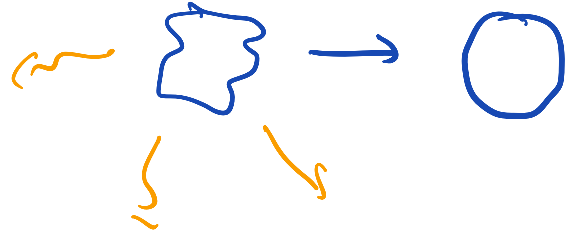
→ What should one take for the bkgd space-time? Answer will depend on what physical system you're trying to describe.

- $\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu}$ Minkowski

  sufficient to understand GW emission from 2 very well separated BH.
  Certainly cannot describe the merger of 2 bhs.
  - provided they are not too far & can neglect the expansion of the universe.

- $\bar{g}_{\mu\nu}(x) =$ Sch. (or $\dots$) + Kerr

  Used to understand how a perturbed BH relaxes to a sph. sym black hole by emitting GWs.



["Black hole perturbation theory"]
(only very recently extended beyond linear order).

- $\bar{g}_{\mu\nu}$ = cosmological background
  (FRLW metric $ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2$)

  a: scale factor telling us how the universe expands [Minkowski $a = 1$].

  – required to understand GWs from well separated BH on large "cosmological" scales

What you need for LISA, and also for some LIGO BBH

We will stick here with a Minkowski bkgd $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$.

<u>Idea</u>:

- $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

- Inverse metric: $g^{\mu\nu} = \eta^{\mu\nu} + \alpha h^{\mu\nu}$

  where α is such that $\underline{g^{\mu\nu} g_{\nu\beta}} = \delta^{\mu}{}_{\beta}$ to linear order.

$$\left[(\eta^{\cdot\cdot} + \alpha h^{\cdot\cdot})(\eta_{\cdot\cdot} + h_{\cdot\cdot}) \right.$$

To linear order

$$= \underbrace{\eta^{\cdot\cdot}\eta_{\cdot\cdot}} + \eta^{\cdot\cdot}h_{\cdot\cdot} + \alpha h^{\cdot\cdot}\eta_{\cdot\cdot} + \cancel{X} = \delta^{\cdot}{}_{\cdot}$$

$$= \delta^{\cdot}{}_{\cdot} + \eta^{\cdot\cdot}h_{\cdot\cdot} + \alpha h^{\cdot\cdot}\eta_{\cdot\cdot} = \delta^{\cdot}{}_{\cdot}$$

$$\delta^{\mu}{}_{\beta} + \eta^{\mu\nu}h_{\nu\beta} + \alpha h^{\mu\nu}\eta_{\nu\beta} = \delta^{\mu}{}_{\beta}$$

$\Rightarrow$ need $\alpha = -1$. Note that to linear order $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$

$\Rightarrow$ $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ to linear order

- Substitute in $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ & linearise.

LHS

$$G_{\mu\nu} = G_{\mu\nu}[\bar{g}] = G_{\mu\nu}[\eta + h]$$

$$\overset{\text{Taylor exp}}{=} \underset{O(\varepsilon^0)}{G^{(0)}_{\mu\nu}} + \underset{O(\varepsilon^1)}{G^{(1)}_{\mu\nu}} \cdots\cdots$$

RHS

$$T_{\mu\nu} = T_{\mu\nu}[\bar{g}] = T_{\mu\nu}[\eta + h]$$

$$= T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu} + \cdots$$

What can we say about $G^{(0)}_{\mu\nu}$? $G^{(0)}_{\mu\nu} = 0$ as we've taken a Minkowski bkgd

$\Rightarrow$ - - - - - - - - - $T^{(0)}_{\mu\nu}$? $T^{(0)}_{\mu\nu} = 0$

So $T_{\mu\nu} = \underline{T^{(1)}_{\mu\nu}}$ is first order.

Therefore an implication of this assumption is the following:

$$\boxed{\nabla_\mu T^{\mu\nu} = 0}$$

(see also TD 3, ex 1)

$$\partial_\mu T^{\mu\nu}_{(1)} + \cancel{\left[\Gamma_{(1)} T_{(1)} + \Gamma_{(1)} T_{(1)}\right]} = 0$$

What is the order of Γ?

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \underbrace{g^{\mu\nu}}_{(\eta - h)} \underbrace{\left(\partial_\alpha \underbrace{g_{\nu\beta}}_{(\eta + h)} + \partial_\beta g_{\nu\alpha} - \partial_\nu g_{\alpha\beta} \right)}_{\partial(\eta+h) = \cancel{\partial\eta} + \partial h}$$

$O(\varepsilon^1)$

$\Rightarrow \Gamma^{(1)}$, ie Γ is of order $O(\varepsilon^1)$

Therefore to linear order $\nabla_\mu T^{\mu\nu} = 0$ becomes

$$\partial_\mu T^{\mu\nu}_{(1)} = 0$$

exactly the special relativistic continuity eq^n & e of m for the fluid.

in other words, to linear order the grav field has NO effect on the dynamics of the matter producing $h_{\mu\nu}$.

$\Rightarrow$ we can totally ignore to linear order the effect of GR on the dynamics of the matter fields. $\rightarrow$ standard S.R dynamics of matter is relativistic

$\rightarrow$ or newtonian dynamics of matter is non-relativistic.

For our 2 BHs which are far apart, if they are also moving slowly ($v \ll 1$), then their trajectories can be taken to be Keplerian.

- we will also treat the BH as point objects, and neglect spin.

- Back to $G_{\mu\nu}[\eta + h] = T_{\mu\nu}[\eta + h]\left(\frac{8\pi G}{c^4}\right)$

& we want to write this to 1st order.

Calc^{n} of $G^{(1)}_{\mu\nu}$ is totally mechanical & a bit boring & long...

$\rightarrow$ $\Gamma^{\mu\,(1)}_{\alpha\beta}$

$\rightarrow$ ${}^{(1)}R^{\mu}{}_{\alpha\beta\gamma} = \underbrace{\partial\Gamma - \partial\Gamma}\ + \underbrace{\cancel{\Gamma\Gamma - \Gamma\Gamma}}$

$\rightarrow$ ${}^{(1)}R_{\alpha\beta}$

$\rightarrow {}^{(1)}G_{\alpha\beta}$

The answer ~~is~~ slightly simpler to write in terms

$$\boxed{\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h} \quad — (1)$$

$h = \eta^{\mu\nu} h_{\mu\nu}$

called the "trace-reversed" perturbation,

$$\eta^{\mu\nu}\bar{h}_{\mu\nu} = \bar{h} = \eta^{\mu\nu}h_{\mu\nu} - \tfrac{1}{2}\overbrace{\eta^{\mu\nu}\eta_{\mu\nu}}^{4} h$$

$$= h - \frac{1}{2}\cdot 4 \cdot h$$

$$= -h$$

$$G^{(1)}_{\mu\nu} = \frac{8\pi G}{c^4} T^{(1)}_{\mu\nu}$$

$$\boxed{\Box \bar{h}_{\mu\nu} - \cancel{\partial_\nu \partial_\rho \bar{h}^\rho{}_\mu} - \cancel{\partial_\mu \partial_\nu \bar{h}^\rho{}_\nu} + \cancel{\eta_{\mu\nu}\partial_\rho\partial_\sigma \bar{h}^{\rho\sigma}} = -\frac{16\pi G}{c^4} T_{\mu\nu}}$$

- We can simplify this because we still have the freedom to do coordinate transformations:

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu(x)$$

with ξ^μ of order ϵ^1 $\quad (\xi^\mu \ll 1)$

This will change the value of $h^{\mu\nu}$, and we can use it to work in a coord system in which

$$\boxed{\partial_\rho \bar{h}^\rho{}_\mu = 0}$$

Lorentz gauge.

$$\Rightarrow \boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T^{(1)}_{\mu\nu}}$$

$$\Box = \partial^\alpha \partial_\alpha$$

In the vacuum, $T^{(1)}_{\mu\nu} = 0$, this reduces to

$$\Box \bar{h}_{\mu\nu} = 0 \Rightarrow (c^2\partial_t^2 - \vec{\nabla}^2)\bar{h}_{\mu\nu} = 0$$

$\rightarrow$ wave eq^n, with waves at the speed c

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu$$

$$g_{\mu\nu} \rightarrow \boxed{g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}}$$

$$x^\mu = x'^\mu - \xi^\mu$$

$$\Rightarrow \frac{\partial x^\alpha}{\partial x'^\mu} = \frac{\partial}{\partial x'^\mu}(x'^\alpha - \xi^\alpha)$$

$$= \delta^\alpha{}_\mu - \frac{\partial \xi^\alpha}{\partial x'^\mu}$$

$$= \delta^\alpha{}_\mu - \partial_\mu \xi^\alpha$$

$$\eta_{\mu\nu} + h'_{\mu\nu} = \left(\delta^\alpha{}_\mu - \partial_\mu \xi^\alpha\right)\left(\delta^\beta{}_\nu - \partial_\nu \xi^\beta\right)\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right)$$

$$\cancel{\eta_{\mu\nu}} + h'_{\mu\nu} = \cancel{\eta_{\mu\nu}} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + h_{\mu\nu} + O(\varepsilon^2)$$

$$\Rightarrow \quad h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

from ① $\Rightarrow$

$$\boxed{\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \left(\partial_\mu \xi_\nu + \partial_\nu \xi_\mu\right) + \eta_{\mu\nu}\left(\partial_\alpha \xi^\alpha\right)}$$

How can we use this to set $\partial_\mu \bar{h}^\mu{}_\nu = 0$??

non-primed

Let's suppose that in the "old" set of coords,

$$\partial_\mu \bar{h}^{\mu\nu}_{old} \neq 0$$

' coords

In the "new" coords

$$\partial^\mu \bar{h}^{new}_{\mu\nu} = \underbrace{\partial^\mu \bar{h}^{old}_{\mu\nu} - \partial^\mu \partial_\mu \xi_\nu}_{} - \cancel{\partial^\mu \partial_\nu \xi_\mu} + \cancel{\partial^\mu \eta_{\mu\nu}\left(\partial_\alpha \xi^\alpha\right)}$$

$$\partial_\nu \partial_\alpha \xi^\alpha = \partial_\alpha \partial_\nu \xi^\alpha = \cancel{\partial^\alpha \partial_\nu \xi_\alpha}$$

So we can set $\partial^\mu \bar{h}^{new}_{\mu\nu} = 0$ provided we choose our ξ^μ to satisfy

$$\partial^\mu \bar{h}^{old}_{\mu\nu} = \Box \xi_\nu$$

Infact, the Lorentz gauge does not fix the coord system completely, since there is still the freedom to choose sol^{n}s of the homogenous eq^n

$$\Box \xi_\nu^{homog} = 0 \qquad 4 \text{ eq}^n\text{s}$$

in other words, we can still impose 4 more conditions on $\bar{h}_{\mu\nu}$.
(beyond the Lorentz gauge)

• <u>Impose</u> the "transverse & traceless conditions"

$$\partial^i \bar{h}_{ij} = 0 \qquad \bar{h}^\mu{}_\mu = 0$$

$(i = 1,2,3)$

3 1

$$\bar{h}_{ij} \sim e^{i(\omega t - \vec{k}\cdot\vec{x})} \epsilon_{ij}$$

$$\partial^i \bar{h}_{ij} \to k^i \bar{h}_{ij} = 0$$ hence origin of name "transverse"

We have imposed: $\underset{\text{Lorentz gauge}}{4} + \underset{\text{transverse traceless}}{4} = 8$

conditions on the 10 components of $\bar{h}_{\mu\nu}$

$\Rightarrow$ left with $10 - 8 = 2$ <u>degrees of freedom</u>

Generally, to find the GWs emitted from some source:

we (a) solve $\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$

(b) impose the transverse & traceless condition to find the 2 physical degrees of freedom.