KRUSKAL COORDS, white holes etc.

- Starting pt: Sch metric in Sch. coords

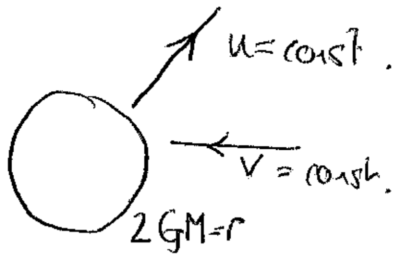
$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2 \quad — (1)$$

- On null geodesics $ds^2=0$: $\quad dt = \pm \dfrac{dr}{\left(1-\frac{2GM}{r}\right)} \equiv \pm dr_* \quad — (2)$

$$\Rightarrow \quad r_* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right| \quad — (3)$$

which can be inverted to give $r(r_*)$, where <u>for the moment we assume (for invertability) that $r > 2GM$.</u>

- let $\left.\begin{array}{l} v = t + r_* \\ u = t - r_* \end{array}\right\}$ — (4)



$\Rightarrow$ v const $\Rightarrow$ $t + r_* = $ const, so from (2)

$\Rightarrow dt = -dr_*$ — this is an ingoing null geodesic.

$u = $ const, $dt - dr_* = 0 \Rightarrow dt = +dr_* \Rightarrow$ outgoing null geodesic

- So, for $r > 2m$,

$$ds^2 \overset{(1)}{=} +\left(1-\frac{2GM}{r}\right)\left[-dt^2 + \frac{dr^2}{\left(1-\frac{2GM}{r}\right)}\right] + r^2 d\Omega^2$$

$$\overset{(2)}{=} \left(1-\frac{2GM}{r(r_*)}\right)\left[-dt^2 + dr_*^2\right] + r^2(r_*)\, d\Omega^2$$

$$\overset{(4)}{=} \left(1-\frac{2GM}{r(u,v)}\right)\left[-du\,dv\right] + r^2(u,v)\, d\Omega^2 \quad — (5)$$

where $r(r_*)$ from (3)

Since $du\,dv = dt^2 - dr_*^2$, and from (4) also, $r_* = \dfrac{v-u}{2}$ — (6)

From ⑤, subs in ③ $\Rightarrow$ $\frac{v-u}{2} - r = 2GM \ln\left|\frac{r}{2GM} - 1\right|$ ⑪

⑥′ $\longrightarrow$ $e^{\frac{v-u}{4GM}} e^{-r/2GM} = \frac{r}{2GM} - 1$ $(r > 2GM)$

$\Rightarrow$ $r = 2GM + 2GM e^{-r/2GM} e^{(v-u)/4GM}$

(for $r > 2GM$)

and

$\Rightarrow \frac{r}{2GM}\left(1 - \frac{2GM}{r}\right) = e^{\frac{v-u}{4GM}} e^{-r/2GM}$

$\Rightarrow \boxed{\left(1 - \frac{2GM}{r}\right) = \frac{2GM}{r} e^{\left(\frac{v-u}{4GM}\right)} e^{-r/2GM}}$ — ⑦ $(r > 2GM)$

- As $r \longrightarrow 2GM$ from above, $r_* \longrightarrow -\infty$ $\Rightarrow$ $v \rightarrow -\infty$ and $u \rightarrow +\infty$.

So $r = 2GM$ is "infinitely far away").

- Change coords to put these pts to a finite coord value:

$$\left.\begin{aligned} v' &= e^{v/4GM} \\ u' &= -e^{-u/4GM} \end{aligned}\right\} \text{ — ⑧}$$

$r \to 2M \Rightarrow v' \to 0$ or $u' \to 0$

$$\Rightarrow \left.\begin{aligned} dv' &= \frac{1}{4GM} e^{v/4GM} dv \\ du' &= \frac{1}{4GM} e^{-u/4GM} du \end{aligned}\right\} \Rightarrow dv' du' = \frac{1}{16G^2M^2} e^{\frac{v-u}{4GM}} du\, dv$$

So $du\, dv = 16 G^2M^2 e^{\frac{u-v}{4GM}} du' dv'$ — ⑨

- Subs. ⑨ & ⑦ into ⑤ $\Rightarrow$

$$ds^2 = -2 \cdot \frac{16 G^3M^3}{r} e^{-r/2GM} du' dv' + r^2(u', v')\, d\Omega^2$$

— ⑩

- Note: nothing special happens at $r = 2GM$.

- Finally let $T = \frac{1}{2}(v' + u')$, $R = \frac{1}{2}(v' - u')$ $\Rightarrow -dT^2 + dR^2 = -du'dv'$ } — (11)

  & subs in (10)

$$\Rightarrow \boxed{ds^2 = -\frac{32G^3M^3}{r(T,R)} e^{-r(T,R)/2GM} \left(-dT^2 + dR^2\right) + r^2(T,R)\,d\Omega^2} \quad \text{— (12)}$$

- From eq^n (11), $+T^2 - R^2 = u'v' \overset{(8)}{=} -e^{\frac{v-u}{4GM}}$

$$\overset{(6')}{=} -\left(\frac{r}{2GM} - 1\right) e^{r/2GM}$$

$$\boxed{T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}} \quad \text{— (13)}$$

$\Rightarrow$ <u>lines of constant r</u> $\Rightarrow T^2 - R^2 = \text{const.}$ — (14) <u>hyperbolae</u>

& for $r > 2GM$ $T^2 - R^2 = -\alpha^2$ for real α.

If $r = 2GM$, $T^2 = R^2 \Rightarrow T = \pm R$ — (15)

- Also from (11) & (8) & (4)

$$T = \frac{1}{2}\left(e^{v/4GM} - e^{-u/4GM}\right)$$

$$\overset{(4)}{=} \frac{1}{2}\left(e^{t/4GM} - e^{-t/4GM}\right) e^{r_*/4GM}$$

$$T = \operatorname{sh}\left(\frac{t}{4GM}\right) e^{r_*/4GM}$$

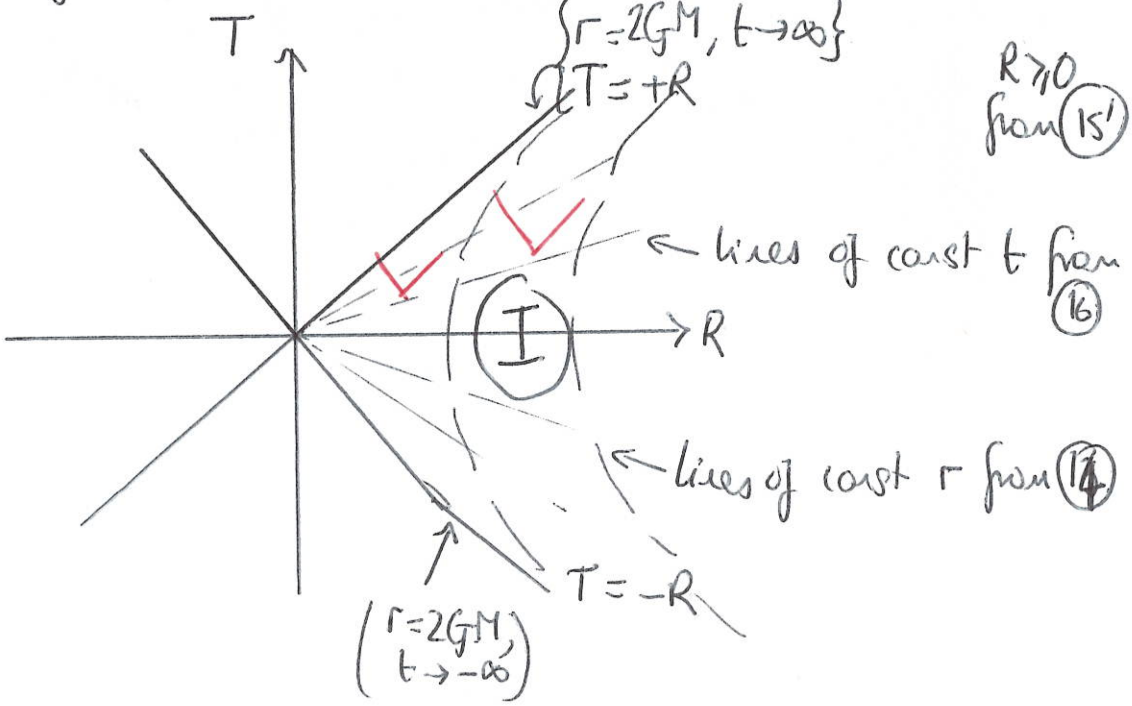
& idem $R = \operatorname{ch}\left(\frac{t}{4GM}\right) e^{r_*/4GM} \Rightarrow \boxed{R \geq 0}$ — (15')

$\Rightarrow T/R = \operatorname{th}\left(\frac{t}{4GM}\right)$ — (16) $\Rightarrow$ <u>lines of const t are straight lines</u> with gradient between -1 & $+1$

(iv)

- And as $t \to \pm\infty$, $T \to \pm R$, from (16), which thus coincides with the horizon at $r = 2GM$ from (15) [just as we expected from studying photon/ptle geodesics]

- So, for $r > 2M$, which we assumed from after (3) till now, we arrive at (12), (13), which on a (T, R) coord diag is the region



& light cones are at 45°. We call this region region (I)

- But as it stands, (12) & (13) ALSO are well defined $\forall r$.

Just as in the example of TD 1, we can extend the coords (T, R) beyond their orig. defn [which came from (t, r), then (t, r^*) with $r > 2GM$, then (u, v) then (u', v') then (T, R)].

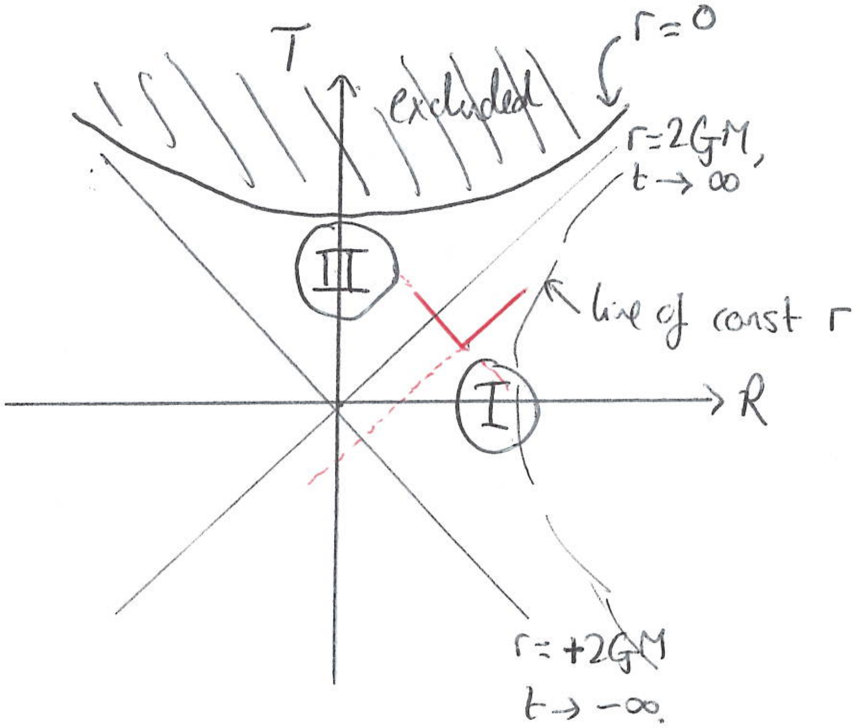
- Infact metric (12), (13) fine for $r < 2GM$. Just have to replace $1 - \frac{2GM}{r}$ in previous pages to $\left|1 - \frac{2GM}{r}\right|$ so as to have access to $\forall r$ from $0 < r \leq \infty$. So it's a qn of signs

Infact for $r < 2GM$ (region II), one has replace (8) by region I $\quad v'_{I} = e^{v/4GM}$ $\qquad u'_{I} = -e^{-u/4GM}$ (so $R>0$ from ⑩)

region II $\quad v'_{II} = e^{v/4GM}$ $\qquad u'_{II} = e^{-u/4GM}$. (so $T>0$ from ⑪)

Then one recovers ⑫ as well as ⑬, valid now for $r<2GM$.

- So one can complete the (T,R) diag of page IV by:



- So we see that future directed ~~from~~ light-like geodesics from region ① necessarily arrive in region ⑪ and hit $r=0$.

- We can now study ⑫ in the 2 other quadrants of the (T,R) diagram, which are clearly beyond the ~~vali~~ range in which the (t,r) coordinates were well defined.

- If we follow past-directed null rays, we arrive in region III, which can be described by $T<0$, so from (11): (VI)

region III

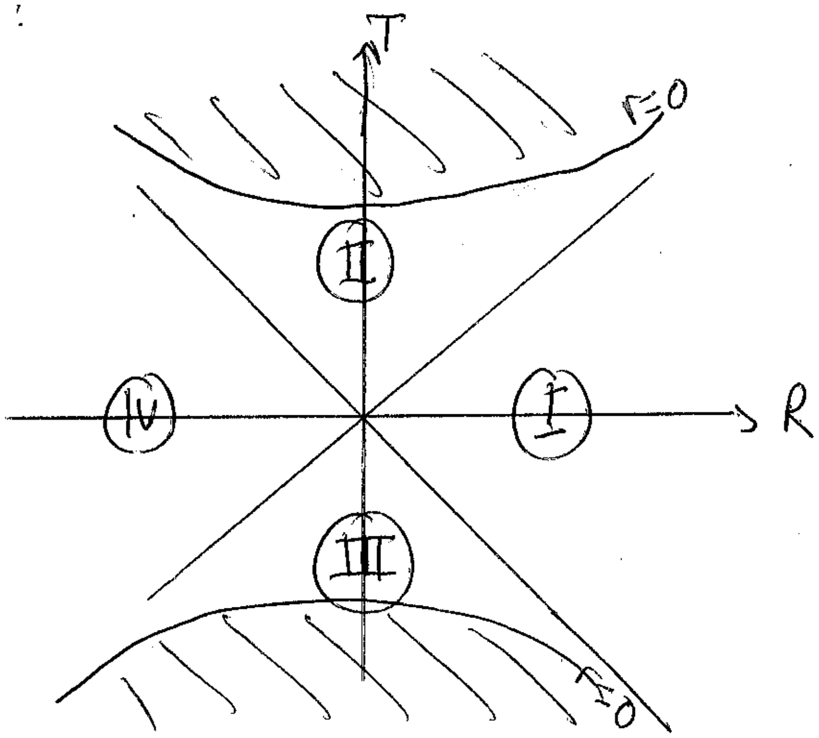
$$v'_{III} = -e^{v/4MG} \qquad u'_{III} = -e^{-u/4GM}.$$

- and finally region IV has $R<0$, so from (11):

region IV :

$$v'_{IV} = -e^{v/4MG} \qquad u'_{IV} = e^{-u/4MG}.$$

- With these 4 sets of (u', v') we cover all the (T, R) space-time:



KRUSKAL DIAG.

$r = 2GM$ called the event horizon

(note, if you start from region I & fall into region II there's no escape; even changing direction from $R \to -R$, you will hit $r=0$)

- Regions III & IV ~~are~~ not of course described by the orig. (t, r) coords.

- Region III : "white hole" : ~~things~~ has the singularity in the PAST and not in the future!

  $\left(\begin{array}{c} T \to -T \\ \text{of II} \end{array}\right)$

  - often said to describe a "white hole", as everything which emerges from the singularity will be visible to an observer in region I.

- Region IV : can never get to from region I as would need to go faster than speed of light.

- For black holes formed by collapse of stars, only regions I & II are physical.