Université Paris Cité

2022-2023
D.A. Steer

GENERAL RELATIVTY

NPAC

TD 1

1 General coordinate transformations in Minkowski space

1. Start from Minkowski coordinates $\xi^\mu = (t, x, y, z)$ with metric $\eta_{\mu\nu}$. On transforming to general curvilinear coordinates x^μ, the metric tensor and Christoffel symbols are defined by

$$g_{\mu\nu}(x) = \eta_{\alpha\beta}\frac{\partial\xi^\alpha}{\partial x^\mu}\frac{\partial\xi^\beta}{\partial x^\nu} \quad (1)$$

$$\Gamma^\mu{}_{\nu\lambda}(x) = \frac{\partial x^\mu}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\nu\partial x^\lambda} \quad (2)$$

Show that

$$\Gamma^\mu{}_{\nu\lambda} = \frac{1}{2}g^{\mu\kappa}(\partial_\nu g_{\kappa\lambda} + \partial_\lambda g_{\kappa\nu} - \partial_\kappa g_{\nu\lambda}). \quad (3)$$

What are the symmetries of $\Gamma^\mu{}_{\nu\lambda}$?

2. Show that under a coordinate transformation $x^\mu \to x'^\mu$ (assumed invertible),

$$\frac{d^2x'^\alpha}{d\tau^2} + \Gamma'^\alpha{}_{\beta\gamma}\frac{dx'^\beta}{d\tau}\frac{dx'^\gamma}{d\tau} = \left(\frac{\partial x'^\alpha}{\partial x^\mu}\right)\left[\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau}\right]. \quad (4)$$

Hence show that if the geodesic equation holds in one set of coordinates, it holds in another.

3. Determine how the Christoffel symbols transform under a coordinate transformation $x^\mu \to x'^\mu$.

2 Geodesic equation

1. Consider a time-like curve $C(\lambda)$, parametrised by a parameter λ, on a space-time with metric $g_{\mu\nu}$. What is the sign of $g_{\alpha\beta}(x)\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$ on this curve ? Obtain the geodesic equation by minimising the proper-time between two points $p_0 = C(\lambda_0)$ and $p_1 = C(\lambda_1)$:

$$S_0[x] = -m\int_{p_0}^{p_1} d\tau = -m\int_{p_0}^{p_1}\frac{d\tau}{d\lambda}d\lambda = -m\int_{p_0}^{p_1} d\lambda\sqrt{-g_{\alpha\beta}(x)\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}}. \quad (5)$$

(Here τ is the proper-time.) In the last step choose $\lambda = \tau$ to express the geodesic equation in terms of x^μ and $\dot{x}^\mu = dx^\mu/d\tau$.

2. Show that the same geodesic equation is obtained from the action

$$S_1[x] = \int d\tau\mathcal{L}[x^\mu, \dot{x}^\mu] \quad (6)$$

where

$$\mathcal{L} = g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{7}$$

Note : For massless particles, proper-time does not exist. The geodesic equation is expressed in terms of a parameter λ along the light-like geodesic, satisfying

$$g_{\alpha\beta}(x)\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0. \tag{8}$$

The geodesic equation then reads

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\nu\lambda}\frac{dx^\nu}{d\lambda}\frac{dx^\lambda}{d\lambda} = 0 \tag{9}$$

3. Consider a space-time metric of the form

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2 \tag{10}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. This is the general form of a static spherically symmetric metric, which we will use later in the course to describe the gravitational field of a star. Write down the corresponding Lagrangian $\mathcal{L}$. Show that t is a cyclic variable, and determine its equation of motion. From this, show that

$$\Gamma^t_{rt} = \Gamma^t_{tr} = \frac{A'}{2A}, \qquad \Gamma^t_{\mu\nu} = 0 \text{ otherwise.} \tag{11}$$

From the (r, θ, ϕ) equation determine the remaining Christoffel symbols. Verify you calculations by determining the Christoffel symbols directly from the metric.

3 Rindler coordinates

Rindler coordinates (ρ, ψ) are defined in terms of Minkowski coordinates (t, x) by

$$t = \rho \sinh\psi \qquad x = \rho\cosh\psi \tag{12}$$

1. Write down the metric in Rindler coordinates, and determine all the non-vanishing Christoffel symbols.
2. Write down the ρ and ψ components of the geodesic equation, together with the definition of proper-time expressed in Rindler coordinates.
3. Show that a first integral of the ψ-geodesic equation is $\rho^2\dot{\psi} = K$ where K is a positive integration constant. Now show that ρ satisfies

$$\dot{\rho}^2 - \frac{K^2}{\rho^2} + 1 = 0 \tag{13}$$

4. The trajectories of the geodesics in space-time are of the form $\rho(\psi)$. Eliminate τ to find an equation for $d\rho/d\psi$. Verify that its solution is

$$\rho^{-1} = \frac{1}{K}\cosh(\psi - \psi_0) \tag{14}$$

where ψ_0 is an integration constant. Show that this corresponds to rectilinear motion of the form $x = x_0 + vt$ where (t, x) are Minkowski coordinates.

5. Show that the proper-time for an observer is given by

$$\tau = \int d\psi \sqrt{\rho^2 - \left(\frac{d\rho}{d\psi}\right)^2} \tag{15}$$

6. Consider two observers : $\mathcal{O}$ who is intertial and fixed at $x = x_0 > 0$; and $\mathcal{O}'$ who has constant acceleration (that is, in her instantaneous rest frame, the acceleration is constant). Show that the trajectory of $\mathcal{O}'$ is given by

$$\rho = \rho_0 \,, \qquad x^2 - t^2 = \rho_0^2 \tag{16}$$

   and determine her acceleration in terms of ρ_0. Draw the world-lines of $\mathcal{O}$ and $\mathcal{O}'$ on the space-time diagrams (t, x) and then (ψ, ρ). Indicate x_0 and ρ_0 on each of your diagrams.

7. Use (15) to deterime the proper-time of $\mathcal{O}$ and $\mathcal{O}'$ as a function of ψ.

8. Now introduce a constant ψ_0 such that $x_0 \equiv \rho_0 \cosh \psi_0$. Calculate the proper-time which has elapsed between the two instances at which the observers meet (namely $\psi = \pm\psi_0$). Show that

$$\frac{\Delta\tau_{\mathcal{O}}}{\Delta\tau_{\mathcal{O}'}} = \frac{\sinh \psi_0}{\psi_0} > 1 \tag{17}$$

   Which is larger ?

9. Show that the trajectories $\rho(\psi)$ of light rays are given by

$$\rho = \rho_* e^{\pm(\psi - \psi_*)} \tag{18}$$

   where ψ_* and ρ_* are the coordinates of a point on the light-ray.

4 Extension of the Rindler metric

Here we consider the metric

$$ds^2 = -x^2 dt^2 + dx^2, \qquad -\infty < t < +\infty, \qquad x > 0. \tag{19}$$

Notice that, in this coordinate system, the metric is singular at $x = 0$.

1. Write down the equation giving trajectories $t(x)$ of light-rays in this metric. Express them in terms of the new coordinates

$$u \equiv t - \ln(x) \qquad v \equiv t + \ln(x)$$

2. Write the metric in this new coordinate system (u, v).

3. Now carry out a further change of variables

$$U = -e^{-u}, \qquad V = e^{v}$$

   Write down the metric in these variables. Same question for the change of variables

$$T = \frac{1}{2}(U + V)\,, \qquad X = \frac{1}{2}(U - V)$$

   Identify this new metric. In what range are the coordinates T and X defined ? What can you say about the singularity at $x = 0$ in the metric (19) ? Convince yourself that it is just a "coordinate singularity", namely due to an inadapted choice of coordinates, and that the metric written in another set of coordinates is perfectly well defined.

5 From past exam : Basics

1. Show that the spacetime interval $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ is invariant under coordinate transformations $x^\alpha \to \tilde{x}^\alpha$ if $g_{\alpha\beta}$ are components of a tensor transforming according to the tensor transformation law
$$g_{\alpha\beta} \longrightarrow \tilde{g}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\frac{\partial x^\nu}{\partial \tilde{x}^\beta} g_{\mu\nu}.$$
2. Let V^α be the contravariant components of a vector, and consider an invertible coordinate transformation $x^\delta \to \tilde{x}^\delta$. Write down the transformation law for $\nabla_\beta V^\alpha$, and deduce that Christoffel symbols transform according to
$$\Gamma^\alpha_{\beta\gamma} \longrightarrow \tilde{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\mu_{\rho\sigma}\frac{\partial \tilde{x}^\alpha}{\partial x^\mu}\frac{\partial x^\rho}{\partial \tilde{x}^\beta}\frac{\partial x^\sigma}{\partial \tilde{x}^\gamma} + \frac{\partial^2 x^\sigma}{\partial \tilde{x}^\beta \tilde{x}^\gamma}\frac{\partial \tilde{x}^\alpha}{\partial x^\sigma}.$$
3. Consider a 2-sphere with coordinates (θ, ϕ) and line-element
$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$
Show that lines of constant longitude (ϕ =constant) are geodesics, and that the only line of constant latitude (θ =constant) that is a geodesic is the equator ($\theta = \pi/2$).

6 From Exam : Locally inertial coordinates

1. At a point $x^\alpha_{(0)}$ in some coordinate system x^α, and as seen in lectures, it is always possible to construct a *locally inertial coordinate system* ξ^α. Which quantity should vanish at $x^\alpha_{(0)}$ in this locally inertial coordinate system, and why ?
2. Suppose that the point $x^\alpha_{(0)}$ and in the coordinate system x^α, the Christoffel symbol has the value $\Gamma^\alpha_{(0)\mu\nu}$. Then at $x^\alpha_{(0)}$, the ξ^α are constructed as follows :
$$\xi^\alpha(x) = x^\alpha - x^\alpha_{(0)} + \frac{1}{2}\left(x^\mu - x^\mu_{(0)}\right)\left(x^\nu - x^\nu_{(0)}\right)\Gamma^\alpha_{(0)\mu\nu}. \quad (20)$$
The point $x^\alpha_{(0)}$ in the new coordinates is the origin $\xi^\alpha = 0$. Prove explicitly that, when transformed to the new coordinates, the relevant quantity that should vanish at $\xi^\alpha = 0$ indeed does so.
[Hint : for simplicity, choose the origin of your x^α coordinates such that $x^\alpha_{(0)} = 0$.]
3. In the locally inertial coordinate system ξ^α, show that $\partial_\alpha(g_{\beta\gamma}\xi^\beta\xi^\gamma) = 2g_{\alpha\beta}\xi^\beta$.

7 From past exam : Coordinate transformations

Consider the line element
$$ds^2 = -dt^2 + t^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2)], \qquad t > 0.$$
Carry out the change of coordinates
$$\tilde{t} = t\cosh\chi, \qquad \tilde{r} = t\sinh\chi, \qquad \tilde{\theta} = \theta, \qquad \tilde{\phi} = \phi. \quad (21)$$
Identify the new metric, specifying carefully the allowed ranges of the coordinates $\tilde{t}$ and $\tilde{r}$. What do geodesics look like in this new metric (note : essentially *no* calculation is required to answer this question) ? Conclude that in the original (t, χ, θ, ϕ) coordinate system, geodesics are given by $t = d/(\sinh\chi - v\cosh\chi)$ where v is a constant that can be interpreted as a speed, and d is another constant that can be interpreted as an initial position.