Université Paris Cité 2022-2023

D.A. Steer

GENERAL RELATIVTY

NPAC

TD 3

1 Stress-energy tensor

1. Show that the covariant conservation of the stress-energy tensor, $\nabla_\mu T^{\mu\nu} = 0$ can be rewritten in the equivalent form

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^{\mu\nu}) + \Gamma^\nu{}_{\sigma\mu}T^{\mu\sigma} \tag{1}$$

2. Consider a perfect fluid consisting of dust, so $P = 0$, and hence $T^{\mu\nu} = \rho u^\mu u^\nu$. Starting from $\nabla_\mu T^{\mu\nu} = 0$ (and contracting with u_ν), deduce that $\nabla_\mu(\rho u^\mu) = 0$. Also deduce that the conservation of stress-energy implies that u^μ must satisfy the geodesic equation.
3. Now consider the conservation equation in Minkowski space, where it reduces to $\partial_\mu T^{\mu\nu} = 0$, with $T^{\mu\nu} = (\rho + P/c^2)u^\mu u^\mu + P\eta^{\mu\nu}$ (now inserting factors of c). We want to show that, in the Newtonian limit, the conservation equation reduces to the well known equations of motion and continuity equation.
   (a) Show that $u_\nu \partial_\mu T^{\mu\nu} = 0$ leads to the relativistic continuity equation

$$\partial_\mu(\rho u^\mu) + (P/c^2)\partial_\mu u^\mu = 0 \tag{2}$$

   (b) Show that the following equation of motion is also satisfied :

$$(\rho + P/c^2)(\partial_\mu u^\nu)u^\mu = -(\eta^{\mu\nu} + u^\mu u^\nu/c^2)\partial_\mu P \tag{3}$$

   (c) Now take the non-relativistic limit $u/c \ll 1$, and assume "weak" pressure namely $P/c^2 \ll \rho$. Show that $u^\mu \simeq (c, \vec{u})$ (assuming $\gamma_u \simeq 1$), and that the continuity equation reduces to $\partial_\mu(\rho u^\mu) \simeq 0$. Show also that the equation of motion reduces to the usual Euler equation for a parfect fluid, namely

$$\rho\left(\frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}\right)\vec{u} \simeq -\vec{\nabla}P \tag{4}$$

2 Solutions of Einsteins equations

1. Compute explicitly the non-vanishing Christoffel symbols and components of the Ricci tensor for the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

2. Identify the change of coordinates $\bar{t} = \bar{t}(t, r)$, $\bar{r} = \bar{r}(t, r)$ which transfroms the de Sitter metric :

$$ds^2 = -dt^2 + e^{2Ht}(dr^2 + r^2 d\Omega^2)$$

(which we will discuss further in the section on cosmology) with H=constant, into the following form

$$ds^2 = -\left(1 - \frac{\bar{r}^2}{R_H^2}\right) d\bar{t}^2 + \left(1 - \frac{\bar{r}^2}{R_H^2}\right)^{-1} d\bar{r}^2 + \bar{r}^2 d\Omega^2$$

where $R_H = H^{-1}$. The first metric is known as the flat form of the de Sitter metric, and the second one as the static form of the de Sitter metric.
[Hint : $e^{Ht} = e^{H\bar{t}}\sqrt{1 - H^2\bar{r}^2}$ and $re^{Ht} = \bar{r}$.]

3 [Exam 2018] Conformally flat metrics

In two dimensions (time together with one spatial direction), a general line-element can be written locally as

$$ds^2 = \Omega^2(t, x)(-dt^2 + dx^2)$$

where $\Omega(t, x)$ is an arbitrary non-vanishing function of t and x. The factor $\Omega(t, x)$ which multiplies the Minkowski metric, is known as the *conformal factor*, and the above metric is said to be *conformally flat*.

i) Write down the Lagrangian from which one determines the equation of motion for time-like geodesics. Together with the geodesic equation, use it to determine the Christoffel symbols for this metric.

ii) Verify your calculations by determining the Christoffel symbols by direct calculation from the metric.

iii) Using the symmetries of the Riemann tensor, determine the number N of independent components of the Riemann tensor in 2 dimensions. Calculate these N components.

4 Schwarzchild metric in different coordinate systems

In some problems it is useful to use alternative, non-standard, coordinates for the Schwarzschild metric. Here are two examples.

1. **Isotropic coordinates**

Let X, Y, Z be new coordinates related to r, θ, ϕ (and hence $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$) by

$$r = R\left(1 + \frac{m}{2R}\right)^2$$

and

$$X = Rr^{-1}x\,, \qquad Y = Rr^{-1}y\,, \qquad Z = Rr^{-1}z$$

In terms of these coordinates, show that the Schwarzschild metric becomes

$$ds^2 = -\left(\frac{2R - m}{2R + m}\right)^2 dt^2 + \left(1 + \frac{m}{2R}\right)^4 (dX^2 + dY^2 + dZ^2)$$

Hence you can show that spaces with t=constant in the Schwarzschild metric are conformal to Euclidean space.

2. **Regge-Wheeler/tortoise coordinates**
   In the region $r > 2m$, define the tortoise radial coordinate by

   $$\rho = r + 2m \log(r - 2m)$$

   Write down the Schwarzschild metric in (t, ρ, θ, ϕ) coordinates. Show that it takes a form in which the time-like sections θ =constant, ϕ =constant are conformal to 2-dimensional Minkowski space.

5 Spherically symmetric metrics

Spherical symmetry implies that any line element can be written in the form

$$ds^2 = -C(t,r)dt^2 + D(t,r)dr^2 + 2E(t,r)drdt + F(t,r)r^2 d\Omega^2.$$

To write this in standard form, make the following change of coordinates :

1. Go from (t, r) to (t, r') coordinates where $r'^2 = F(t,r)r^2$ (and invertiblity is assumed)
2. Now label r' as r again. Show that the cross term can be removed by setting $dt' = \eta(t,r)[C(t,r)dt - E(t,r)dr]$
3. Finally, show that the resulting metric takes the form (on labelling t' as t again)

   $$ds^2 = -B(t,r)dt^2 + A(t,r)dr^2 + r^2 d\Omega^2$$

   and find the link between A, B and the other variables. This is the most useful form in which to write a general spherically symmetric metric.

6 Shapiro Effect [From D.Langlois book]

The aim is to calculate the time taken by a light-ray (or radar signal) to go from the earth (E) to a planet (P) and back again.

1. Consider a light signal propagating in the Schwarschild metric. Denote by r_0 the minimum radius of the trajectory of the light ray. Determine dt/dr as a function of r as well as the constants M, r_0, and the conserved angular momentum L and energy E.
2. Now expand the RHS of your expression in a perturbative expansion in powers of m/r (i.e. assume $m \ll r$). Use this to calculate the time taken to go from the earth to the planet and back again, as a function of r_0 ,r_E and r_P (where this last two are the radial coordinates of the earth and the planet).

7 Kerr metric and Killing vectors [From D.Langlois book]

The Kerr metric is given by

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta d\phi^2 \tag{5}$$

where $0 \le a \le M$ and

$$\begin{aligned} \Delta &= r^2 - 2Mr + a^2 \\ \rho^2 &= r^2 + a^2\cos^2\theta \end{aligned}$$

1. What can you say about the $a \to 0$ limit ?
2. Consider circular trajectories in the equatorial plane of the Kerr BH ($\theta = \pi/2$).
   (a) Show that the orbital period measured by an asymptotic observer is

$$T_\infty(r) = \frac{2\pi}{1 - \frac{2M}{r}} \left(\sqrt{\Delta} - \frac{2Ma}{r} \right),$$

   for a trajectory that is in *corotation* with the blackhole ($\dot{\phi} > 0$). Also find the ortibal period for a trajectory in *anti-rotation*.
   (b) Now focus on the case of corotation. Show that the radial coordinate $r_{\min}$ corresponding to the minimal period satisfies the equation

$$r(r-2M)(r-3M) - 2Ma^2 + 2Ma\sqrt{\Delta} = 0 \tag{6}$$

3. Now consider *lightlike geodesics* in the plane $\theta = \pi/2$.
   (a) Identify the two Killing vectors of the metric (5).
   (b) From them, determine the two conserved quantities.
   (c) Deduce that the geodesics satisfy

$$\dot{r}^2 = \alpha^2 \left[1 + \frac{a^2 - b^2}{r^2} + \frac{2M(a-b)^2}{r^3} \right]$$

   where α and b need to be expressed in terms of the two constants of motion.
   (d) For a circular geodesic, determine b as a function of the radius r_{circ} of the geodesic. Show that r_{circ} is a solution of (6).

8 [Exam 2018] Another coordinate transformation

1. The space-time geometry around a static spherically symmetric object of mass M can be described by a line-element of the form

$$ds^2 = -F(r)dt^2 + H(r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{7}$$

(Notice that the function $H(r)$ also multiplies the part in $r^2 d\Omega^2$.) In the limit of a weak gravitational field, an approximate expression for the functions F and H is the following :

$$F(r) = 1 - 2\frac{GM}{r} + 2\beta\left(\frac{2GM}{r}\right)^2 + \dots \tag{8}$$

$$H(r) = 1 + 2\gamma\frac{GM}{r} + \dots, \tag{9}$$

where β and γ are parameters (constants) called ‘post-newtonian parameters’.
   i) Explain in a few words the origin of the first non-trivial term in $F(r)$. Will these approximate expressions for F and H be a good description for the metric around a neutron star ? And for the sun ?
   ii) Using a known static spherically symmetric metric of your choice, and after carrying out the necessary coordinate transformations, determine the values of β and γ in General Relativity.
   [In modified gravity theories, different values of β and γ are allowed.]

9 [Exam 2018] Geodesics in modified gravity

For this exercise, we work with the metric (7) where the functions $F(r)$ and $G(r)$ are given in (8) and (9) respectively.

1. Consider two observers at fixed positions : the first is at $r = r_1$, and the second is at $r = r_2 > r_1$. The first observer sends two successive light signals to the second observer. The proper time interval between the two light signals is $\delta\tau_1$ for the first observer, and $\delta\tau_2$ for the second observer.
   — The first light signal is emitted at coordinate time t_1 and arrives at r_2 are coordinate time t_2. The second light signal is emitted at coordinate time $t_1 + \delta t_1$ and arrives at r_2 are coordinate time $t_2 + \delta t_2$. Show that $\delta t_1 = \delta t_2$.
   — The frequency ν of the electromagnetic signals is related to the proper time interval by $\nu = 1/\delta\tau$. Show that the ratio ν_2/ν_1 (where ν_2=received frequency, and ν_1=emitted frequency) is given by $\sqrt{F(r_1)/F(r_2)}$. Would this frequency shift be visible on earth (say with the two observers separated by a few hundred meters) ?
2. Identify all the Killing vectors of the metric (7). Write down the conserved quantities along geodesics describing the dynamics of free particles, and give them a physical interpretation.
3. Study light-like geodesics in the space-time (7), in the plane $\theta = \pi/2$. Find the first order equation for $r(\phi)$. In terms of the variable $u = b/r$, where the constant b should be determined, show that this equation takes the form
$$\left(\frac{du}{d\phi}\right)^2 + u^2 = \frac{H}{F}. \tag{10}$$
4. Working to first order in GM/r, write the explicit solution of (10) in terms of the post-newtonian parameter γ. [The solution of $(du/d\phi)^2 + u^2 = 1 + 2\alpha u$, for some constant α, is $u = \sqrt{1+\alpha^2}\sin(\phi - \bar{\phi}) + \alpha$.]
5. Hence deduce the angle $\Delta\phi$ through which a light ray is deflected.

10 [Exam 2019] Stress energy tensor and Einstein equations

1. The action for electromagnetism is given by
$$S_m = -\frac{1}{4}\int d^4x\sqrt{-g}F_{\mu\nu}F^{\mu\nu}$$
where the Maxwell field $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Show that the stress energy tensor associated with the electromagnetic field is given by
$$T_{\alpha\beta} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\alpha\beta}} = F_{\sigma\alpha}F^{\sigma}{}_{\beta} - \frac{1}{4}g_{\alpha\beta}F^{\mu\nu}F_{\mu\nu}.$$
[Hint : you may use (though bonus for proving it) the result derived in TD, namely that $\delta g = g g^{\alpha\beta}\delta g_{\alpha\beta}$.]
2. Show that in the presence of an electromagnetic field, Einstein's equations can be written as
$$R_{\mu\nu} = 8\pi G T_{\mu\nu}$$
3. For a pure electric field $F^{0i} = E^i$, $F^{ij} = 0$, write the different components of $T^\mu{}_\nu$ (you may answer this question either using general arguments or by calculation).