Université Paris Cité

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GENERAL RELATIVTY
NPAC

TD 4

1 Linearised Einstein equations and GWs

Decompose the metric into the flat Minkowski metric, plus a small perturbation :

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

with $|h_{\mu\nu}| \ll 1$.

We restrict ourselves to coordinates in which $\eta_{\mu\nu}$ takes its canonical form $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. Write down the following quantities to *linear order* in the perturbation :

1. The inverse metric $g^{\mu\nu}$. If your expression contains $h^{\mu\nu}$, explain how this is obtained from $h_{\alpha\beta}$ (i.e. what metric do you use to raise the indices ?)
2. The Christoffel symbol $\Gamma^{\rho}_{\mu\nu}$.
3. The Riemann tensor $R_{\mu\nu\rho\sigma}$.
4. The Ricci tensor $R_{\alpha\beta}$.
5. The Ricci scalar R.
6. The Einstein tensor $G_{\alpha\beta}$.
7. Does your Einstein tensor satisfy $\partial^{\mu} G_{\mu\nu} = 0$? Why *should* it satisfy this ?

Now consider a gauge/coordinate transformation

$$x^{\mu} \to x'^{\mu} = x^{\mu} - \xi^{\mu}(x^{\nu}) \tag{1}$$

8. Determine how $h_{\alpha\beta}$ transforms under this transformation.
9. Same question for the Riemann tensor $R_{\mu\nu\rho\sigma}$

Action for the linearised Einstein equation :

10. Show that the Einstein tensor of part 6 can be obtained by varying the following Lagrangian $\mathcal{L}$ with respect to $h_{\mu\nu}$:

$$\mathcal{L} = \frac{1}{2}\left[(\partial_{\mu}h^{\mu\nu})(\partial_{\nu}h) - (\partial_{\mu}h^{\rho\sigma})(\partial_{\rho}h^{\mu}{}_{\sigma}) + \frac{1}{2}\eta^{\mu\nu}(\partial_{\mu}h^{\rho\sigma})(\partial_{\nu}h_{\rho\sigma}) - \frac{1}{2}\eta^{\mu\nu}(\partial_{\mu}h)(\partial_{\nu}h)\right], \tag{2}$$

where $h = h^{\alpha}{}_{\alpha}$.

11. (If you are feeling energetic :) This action can also be obtained from the Einstein Hilbert action, derived in TD2 (see the equation in a box under equation (13) in TD2), but where now R must be expanded to *second* order in the metric perturbation. Show that the second order expansion of the EH action is indeed identical to (2).

Trace-reversed perturbation

12. Write down the Einstein equations in terms of the trace-reversed perturbation defined in lectures :

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} \tag{3}$$

13. Show that under the gauge transformation above, Eq. (1),

$$\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} + \xi_{\mu,\nu} + \xi_{\nu,\mu} - \xi^{\rho}{}_{,\rho}\eta_{\mu\nu} \tag{4}$$

14. Show that it is always possible to impose the Lorentz gauge, $\partial^\mu \bar{h}_{\mu\nu} = 0$. Namely, show that if $\bar{h}_{\mu\nu}$ does not satisfy the Lorentz gauge, then one can find a gauge transformation $\bar{h}_{\mu\nu} \to \bar{h}'_{\mu\nu}$ such that the corresponding $\bar{h}'_{\mu\nu}$ does.
15. Show that *in the absence of matter*, the linearised Einstein equations become

$$\Box \bar{h}_{\mu\nu} = 0 \tag{5}$$

These equations are very similar to Maxwells equations in empty space : the only difference is that the perturbations are associated with a metric tensor (2 indices). Convince yourself that equation (5) is nothing other than the wave equation. Show that a solution for a wave travelling in the z-direction, is

$$\bar{h}_{\mu\nu} = H_{\mu\nu}e^{ik_\alpha x^\alpha} \tag{6}$$

where $H_{\mu\nu}$ is the polarisation tensor and

$$k^\mu = (\omega, 0, 0, \omega) \tag{7}$$

with $k^2 = 0$. What is the speed of propagation of the gravitational wave ?

2 Gravitational waves [Exam 2018]

As explained in lectures, GWs are studied by considering a metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where the perturbation $h_{\mu\nu}(t, \vec{x})$ is small. One works to linear order in $h_{\mu\nu}$.

1. Show that in the transverse and tracless gauge (recalled in the formula sheet)

$$\partial_0 h^{00}_{\mathrm{TT}} = 0, \qquad \partial_i h^{ij}_{TT} = 0,$$

where $(i, j = 1, 2, 3)$. Deduce that $h^{00}_{\mathrm{TT}} = 0$, and hence that $h^{0\mu}_{\mathrm{TT}} = 0$.
2. The metric describing Minkowski space and a GW is thus given by

$$ds^2 = -dt^2 + (\delta_{ij} + h^{\mathrm{TT}}_{ij})dx^i dx^j$$

Show that $R^\mu{}_{00\nu} = \frac{1}{2}\partial_t^2 (h_{\mathrm{TT}})^\mu{}_\nu$.
3. The invariant length between two free static particles A and B, situated at coordinate positions x^i_A and x^i_B is

$$L = \sqrt{g_{ij}\Delta x^i \Delta x^j}$$

where $\Delta x^i = x^i_B - x^i_A \equiv L_0 n^i$, the unit vector n^i satisfies $\delta_{ij}n^i n^j = 1$, and $L_0 = \sqrt{\delta_{ij}\Delta x^i \Delta x^j}$. Show that when the GW passes, the relative variation in distance between A and B is given by

$$\frac{\delta L}{L_0} = \frac{1}{2}h^{\mathrm{TT}}_{ij} n^i n^j$$

3 Electromagnetism and the TT gauge [from D.Langlois book]

The aim of this exercise is to understand the TT gauge, using electromagnetism as a helpful example.

1. The electromagnetic Lagrangian $L \propto \sqrt{-g}F_{\mu\nu}F^{\mu\nu}$ is invariant under the $U(1)$ gauge transformations $A_\mu \to A_\mu + \partial_\mu \chi$. Use this invariance to show that one can always choose the Lorentz gauge $\partial_\mu A^\mu = 0$.
2. Write down Maxwells equations (in the vacuum) in the Lorentz gauge. Show that there exists a residual gauge freedom, and use it to fix $A_0 = 0$. (Note that the solution of the wave eqaution $\partial_\mu \partial^\mu f = 0$ with initial conditions $f = 0$ and $\partial_t f = 0$ on a hypersurface of t=constant, is $f = 0$.)
3. Using the above, show that for gravitational waves propagating in empty space, one can impose the TT gauge.

4 Gravitational waves

Consider a non-relativistic system with one degree of freedom, namely a mass μ that performs harmonic oscillations along the z axis : $z_0(t) = A\cos\omega_s t$, with $A\omega_s \ll 1$ and $\omega_s > 0$. (In practise the system could consist of 2 masses connected by a massless spring, and $z_0(t)$ is the relative coordinate of the centre-of-mass system.)

i) The mass density is given by $\rho(t, \vec{x}) = \mu\delta(x)\delta(y)\delta(z - z_0(t))$. Determine $\bar{h}_{ij}(t, \vec{x})$ at a distance $|\vec{x}| = R$ far from the source.

ii) Calculate h_{ij}^{TT} for a wave propagating in the direction $\vec{x} = R\vec{n}$ with $\vec{n} = (0, \sin\theta, \cos\theta)$. Comment on the θ-dependence of your result.