CONFORMAL DIAGRAMS

(C1)

(Carter-Penrose diags).

- In the continuation of what we've done, the aim of <u>conformal diagrams</u> is to have a diagram that <u>captures the causal structure</u> of the space-time.

<u>ie</u> conformal diagram is simply an ordinary space-time diagram for a metric on which we've done a particularly clever set of coord transf^ns, such that

1. light cones are always at 45°.
2. "infinity" is only at a finite coord value away so that the entire space-time is immediately apparent.

- In doing our transformations from Sch (t, r) coords to Kruskal (T, R) coords, we have satisfied ①, but not ②.

- In fact the same is true for Minkowski space

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad (*) \quad \text{(in polar coords)}$$

! Here we satisfy ①, but not ② since

$$-\infty < t < \infty$$

$$\& \quad 0 \leq r < \infty.$$

The good changes of variable to write minkowski satisfying ① and ② are: (think of analogies with different steps for Sch) (C2)

Step a) : use null coords $u = t - r$, $v = t + r$ $\Big\}$ $-\infty < u < \infty$, $-\infty < v < \infty$

where $v = u + 2r \geq u$

b) Finite values : $U = \arctan u$, $V = \arctan v$ $\Big\}$ $-\pi/2 \leq U \leq \pi/2$, $-\pi/2 \leq V \leq \pi/2$

c) Set $T = V + U$, $R = V - U$ $\qquad U \leq V$

so $0 \leq R \leq \pi$, $|T| + R < \pi$, $0 < |T| < \pi$ $\Rightarrow$ $\Leftarrow$ & $T + R = 2V < \pi$, $T - R = 2U > -\pi$

- Doing the change of coords explicitly starting from ⊛ gives

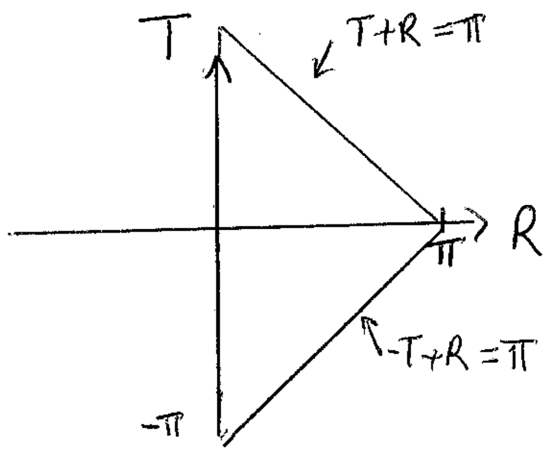
$$ds^2 = \omega^{-2}(T, R)\left(-dT^2 + dR^2 + \sin^2 R \, d\Omega^2\right)$$

where $\omega = \cos T + \cos R$.

- Hence the minkowski ~~metric~~ line element ds^2 is a "conformal factor" ω^{-2} times the "unphysical" line-element $d\tilde{s}^2$:

$$ds^2 = \omega^{-2} d\tilde{s}^2$$

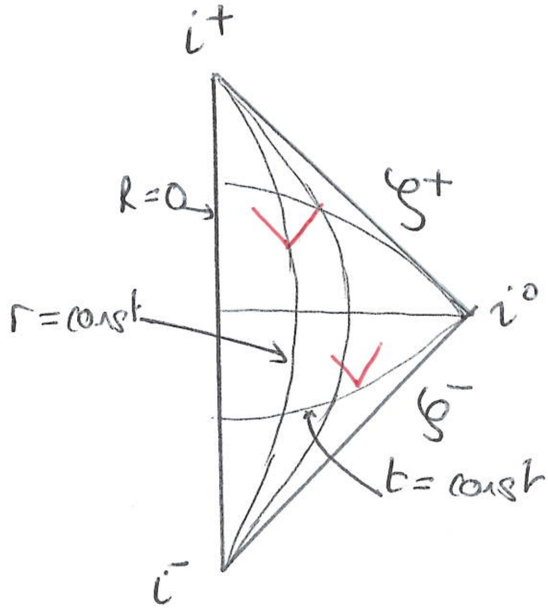
- The conformal diagram is the space-time diagram of $d\tilde{s}^2$:



where as shown above, $0 \leq R \leq \pi$

- $r = 0 \Rightarrow R = 0$ (as $u = v \Rightarrow U = V$)

- Then lines of const t & r are as shown: with light cones at 45°



conformal diag of Minkowski

i^+ : future timelike infinity ($T = \pi$, $R = 0$ $\Rightarrow U = V = \pi/2 \Rightarrow t \to \infty$).

i^- : past time-like infinity ($T = -\pi$, $R = 0 \Rightarrow U = V = -\pi/2 \Rightarrow t \to -\infty$).

i^0 : spatial infinity ($T = 0$, $R = \pi \Rightarrow V = -U$ & $V = \pi/2$, $U = -\pi/2 \Rightarrow r \to +\infty$)

$\mathscr{I}^+$: future null infinity ($T = \pi - R \Rightarrow 2V \to \pi \Rightarrow V \to \pi/2 \Rightarrow v \to \infty \Rightarrow t + r \to \infty$, $t - r$ finite).
at 45°

$\mathscr{I}^-$: past null infinity. ($t + r \to -\infty$, $t - r$ finite).

$\Rightarrow$ all time-like geodesics begin at i^- and finish at i^+

[eg start at $r = 5$, at $t \to -\infty$. Then you rec. finish at $t \to \infty$ at i^+.].

- All light-like geodesics have $t \pm r = \text{const}$. Start at $\mathscr{I}^-$ and finish at $\mathscr{I}^+$.

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Schwarzchild.

- From u' & v' coords, define (sim to above) ↓ page II, eqn (8)

$$u'' = \arctan\left(\frac{u'}{\sqrt{2GM}}\right)$$

$$v'' = \arctan\left(\frac{v'}{\sqrt{2GM}}\right)$$

where we had $u'v' = -\left(\frac{r}{2GM} - 1\right) e^{r/2GM}$

with ranges

$$-\pi/2 < v'' < \pi/2$$

$$-\pi/2 < u'' < \pi/2$$

$$-\pi/2 < v'' + u'' < \pi/2$$

- & again define $T = U'' + V''$, $R = V'' - U''$.