Other black holes

1) Electrically charged BH [Reisser-Nordström]

- not particularly relevant to realistic astrophys. situations :- in real world, would quickly neutralise by interacting with matter in its vicinity.

- Still assume sph. symm.
- But not in vacuum : there's an EM field that acts as a source of energy : $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$

- From $\mathcal{L}_{EM} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, $S_{EM} \propto \int d^4x \sqrt{-g}\, F_{\mu\nu} F^{\mu\nu}$ — ①

where $F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$

& $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$

- So Maxwell's eq^{n}s are,

$$\left.\begin{aligned} &\nabla^\mu F_{\mu\nu} = 0 \\ \& \ &\nabla_{[\mu} F_{\nu\rho]} = 0 \end{aligned}\right\} \quad —②$$

- From ①, can get the stress energy tensor defined by

$$T^{EM}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{EM}}{\delta g^{\mu\nu}}$$ (see TD 2), ex 6.

exercise

$$\Rightarrow \quad T^{EM}_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

- Assume again most gen sph. symm metric

$$ds^2 = -e^{2\Phi(t,r)}dt^2 + e^{2\Psi(t,r)}dr^2 + r^2 d\Omega^2$$

~~&~~ similar procedure to Sch case gives solution [EXERCISE] to $G_{\mu\nu} = 8\pi G T^{EM}_{\mu\nu}$

$$\boxed{ds^2 = -\Delta\, dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2}$$

where $$\boxed{\Delta = 1 - \frac{2GM}{r} + \frac{G(Q^2+P^2)}{r^2}}$$

- M = const, interpreted as mass of the B.hole
- Q = total electric charge
- P = total magnetic charge.

- Since no isolated mag. charges ever seen, usually set $P=0$.

- Electromag. fields associated with sol^{n}:

$$\boxed{\begin{aligned} E_r &= F_{rt} = \frac{Q}{r^2} \\ B_r &= \frac{F_{\theta\phi}}{r^2\sin^2\theta} = P/r^2 \end{aligned}}$$

- Again $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \to \infty$ at $r=0$.

given comment above.

- Horizon structure more complicated. ~~S~~etting $P=0$, Δ can vanish, $\Delta=0$, at two, one or no values of r.

- In case $\underline{GM^2 < Q^2}$ then Δ ~~has~~ no roots.

  Singularity at $r=0$ not hidden behind a horizon.

  This is called a naked singularity

- In the case $\underline{GM^2 > Q^2}$, then Δ has two roots, $r_\pm$

$$r_\pm = GM \pm \sqrt{G^2M^2 - GQ^2}$$

$\Rightarrow$ 2 coord sing at r_+ & r_-. In both cases they can be removed by a coord transf^n as in Sch.

- If you fall into the BH, r_+ acts just like $2GM$ in Sch b.h: at this radius r switches to becoming time-like & you nec. move in dir^n of decreasing r.

- But as r decreases you arrive at r_-, where again r becomes spacelike! So r can again increase (or continue decreasing). In first case you go back through $r = r_+$ etc.

- Special "extreme" case $GM^2 = Q^2$, one root.
  (& unstable)

2) Rotating BH. (Kerr)

- give up on Sph symm → straight away a much more complicated problem.
- look for sol^n's with axial symm, ie rotational symm about one axis only (called the angular momentum axis)
- They will be charac by 2 param, M & J

  ↑ angular mmt.

- clearly sch can't have L^r mmt as sph. symm.
- Since J has dim^n's of mass^2, conventionally define

$$a = J/M$$

- In vacuum $T_{\mu\nu} = 0$, sol^n is

$$\boxed{\begin{aligned} ds^2 = & -\left(\frac{\Delta - a^2\sin^2\theta}{\Sigma}\right)dt^2 - \frac{2a\sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma}\,dt\,d\phi \\ & + \left[\frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\right]\sin^2\theta\,d\phi^2 + \frac{\Sigma}{\Delta}\,dr^2 \\ & + \Sigma\,d\theta^2 \end{aligned}}$$

where $\Sigma = r^2 + a^2\cos^2\theta$
$\Delta = r^2 + a^2 - 2mr$

↑ Kerr in Boyer–Lindquist coords.

- Charged rot. BH exists also.

Note: - surfaces of $t = $ const & $r = $ const do not have metric of a 2sphere (ie not of form $r^2 d\Omega^2$)

- $a = 0 \Rightarrow$ Sch. sol^n.

- off diag term in $dt\,d\phi$, which vanishes if $a=0$.

- $M = 0 \Rightarrow$ flat space-time but NOT in ordinary polar coords, but rather ellipsoidal coords (related to cartesian coords (x, y, z) by

$$x = (r^2+a^2)^{1/2} \sin\theta \cos\phi$$
$$y = (r^2+a^2)^{1/2} \sin\theta \sin\phi$$
$$z = r\cos\theta$$
)

- Horizon when $g^{rr} = 0$. Since $g^{rr} = \frac{\Delta}{\rho^2}$ & $\rho^2 > 0 \Rightarrow$

$$\Delta(r) = 0 = r^2 + a^2 - 2Mr$$

$\Rightarrow$ 3 possibilities
$GM > a \rightarrow$ 2 sol^{n}s $r_\pm = GM \pm \sqrt{G^2M^2 - a^2}$
$GM = a \rightarrow$ extreme, as in RN: unstable
$GM < a \rightarrow$ as in R.N: naked sing.

- First case: $r_+ =$ outer horizon
$r_- =$ inner horizon

  - Also there's a surface on which $g_{tt} = 0 \Rightarrow \Delta = a^2\sin^2\theta$ within this surface $g_{tt} < 0$!
  - Between this surface & r_+ is called the ergosphere.
  - Studying geodesics is interesting: you can't be stationary here, but must move in dir^n of rotation of BH (ϕ-dir^n)

- Calculating $R_{\rho\sigma\mu\nu} R^{\rho\sigma\mu\nu}$: this divergent not at $r=0$ but at $\underline{\rho = 0}$.

$$\Rightarrow \quad r^2 + a^2 \cos^2\theta = 0$$

$$\Rightarrow \quad r = 0 \quad \& \quad \theta = \pi/2 .$$