

Nuclear Quantum Many-Body Problem

A Saclay Lecture Series' Course

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Chapter 1

Many-body quantum mechanics

This chapter is first devoted to recall the basic principles of quantum mechanics and the necessity to add two principles when addressing a many-body system of identical particles. The notions are then applied to the cases of one- and two-nucleon systems, where relevant operators and one- and two-body bases are discussed.

1.1 One-body quantum system

1.1.1 Basic postulates

Let us first focus on quantum systems made of one particle of mass m , spin $1/2$ and isospin $1/2$.

1. Allowed physical states are represented by vectors of a Hilbert space \mathcal{H}_1 , i.e. $|\psi\rangle \in \mathcal{H}_1$, which can be decomposed as the tensor product of three Hilbert spaces associated with different dynamical variables

$$\mathcal{H}_1 \equiv \mathcal{H}_{1_{\text{space}}} \otimes \mathcal{H}_{1_{\text{spin}}} \otimes \mathcal{H}_{1_{\text{isospin}}} , \quad (1.1)$$

where $\mathcal{H}_{1_{\text{space}}}$ relates to spatial coordinates, $\mathcal{H}_{1_{\text{spin}}}$ to spin coordinates and $\mathcal{H}_{1_{\text{isospin}}}$ to isospin coordinates.

2. An observable \mathcal{A} is represented by a self-adjoint operator \hat{A} acting on \mathcal{H}_1 . In particular, operators \hat{r} , \hat{p} , \hat{s} , and $\hat{\tau}$ denoting respectively position, momentum, spin and isospin observables form an irreducible set, namely they constitute a minimal set such that no operator commutes with all of them at the same time. This implies that any physical observable \mathcal{A} can be described by an operator that is a function of these four operators, i.e. $\hat{A} \equiv \hat{A}(\hat{r}, \hat{p}, \hat{s}, \hat{\tau})$.

Different operators act on different spaces: \hat{r} and \hat{p} act on $\mathcal{H}_{1_{\text{space}}}$, \hat{s} acts on $\mathcal{H}_{1_{\text{spin}}}$ and $\hat{\tau}$ acts on $\mathcal{H}_{1_{\text{isospin}}}$. Operators \hat{r} and \hat{p} are related to their classical counterpart by the principle of "associated operators"

$$\vec{r}_{cl} \rightarrow \hat{r} \equiv \vec{r} \times , \quad (1.2a)$$

$$\vec{p}_{cl} \rightarrow \hat{p} \equiv -i\hbar \vec{\nabla} , \quad (1.2b)$$

where the operators are here given in the position basis of $\mathcal{H}_{1_{\text{space}}}$, i.e. in the basis of eigenvectors of \hat{r} defined through

$$\hat{r} |\vec{r}\rangle = \vec{r} |\vec{r}\rangle . \quad (1.3)$$

The fact that Eq. (1.2) provides the operators in this particular basis of interest is more correctly formalized as

$$\langle \vec{r} | \hat{r} | \vec{r}' \rangle \equiv \delta(\vec{r} - \vec{r}') \vec{r} , \quad (1.4a)$$

$$\langle \vec{r} | \hat{p} | \vec{r}' \rangle \equiv -i\hbar \delta(\vec{r} - \vec{r}') \vec{\nabla} . \quad (1.4b)$$

The principle of associated operators does not applies to \hat{s} and $\hat{\tau}$ that are purely quantum operators expressed in terms of Pauli matrices via

$$\hat{s} \equiv \frac{\hbar}{2} \hat{\sigma}_{spin} , \quad (1.5a)$$

$$\hat{\tau} \equiv \frac{\hbar}{2} \hat{\sigma}_{isospin} , \quad (1.5b)$$

where the Pauli matrices write as

$$\hat{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.6)$$

in the eigenbasis of $\hat{\sigma}_z$. Fundamental operators obey the following commutation rules¹

$$[\hat{r}_i, \hat{p}_j] = i\hbar \delta_{ij} , \quad (1.7a)$$

$$[\hat{s}_i, \hat{s}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{s}_k , \quad (1.7b)$$

$$[\hat{\tau}_i, \hat{\tau}_j] = i\hbar \sum_k \epsilon_{ijk} \hat{\tau}_k , \quad (1.7c)$$

$$[\hat{r}_i, \hat{r}_j] = 0 , \quad (1.7d)$$

$$[\hat{p}_i, \hat{p}_j] = 0 , \quad (1.7e)$$

$$[\hat{r}_i, \hat{s}_j] = 0 , \quad (1.7f)$$

$$[\hat{r}_i, \hat{\tau}_j] = 0 , \quad (1.7g)$$

$$[\hat{p}_i, \hat{s}_j] = 0 , \quad (1.7h)$$

$$[\hat{p}_i, \hat{\tau}_j] = 0 , \quad (1.7i)$$

$$[\hat{s}_i, \hat{\tau}_j] = 0 , \quad (1.7j)$$

where, e.g., $i = x, y, z$. Therefore $\{\hat{r}, \hat{s}_z, \hat{\tau}_z\}$ forms a set of commuting observables and can be simultaneously diagonalized by the eigenbasis $\{|\vec{r}\sigma\tau\rangle \equiv |\vec{r}\rangle \otimes |\sigma\rangle \otimes |\tau\rangle\}$ such as

$$\hat{r}_i |\vec{r}\sigma\tau\rangle = r_i |\vec{r}\sigma\tau\rangle , \quad (1.8a)$$

$$\hat{s}_z |\vec{r}\sigma\tau\rangle = \sigma |\vec{r}\sigma\tau\rangle , \quad (1.8b)$$

$$\hat{\tau}_z |\vec{r}\sigma\tau\rangle = \tau |\vec{r}\sigma\tau\rangle , \quad (1.8c)$$

where $r_i \in]-\infty, +\infty[$, $\sigma = \pm\hbar/2$ and $\tau = \pm\hbar/2$ are eigenvalues of \hat{r}_i , \hat{s}_z and $\hat{\tau}_z$, respectively. To be coherent with the usual convention in particle physics, we assign protons and neutrons to be eigenvector of $\hat{\tau}_z$ with eigenvalue $\tau = \hbar/2$ and $\tau = -\hbar/2$, respectively. As for the spin and isospin part, a shorthand notation is used as in fact $|\sigma\rangle \equiv |1/2\sigma\rangle$ and $|\tau\rangle \equiv |1/2\tau\rangle$.

3. The dynamical evolution of a state is governed by the time-dependent Schrödinger equation²

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = h |\psi\rangle , \quad (1.9)$$

¹The Kronecker symbol δ_{ij} is equal to one if both indices are the same and zero otherwise. The Levi-Civita symbol ϵ_{ijk} is zero if at least two of its three indices are identical, otherwise it is equal to the signature of the permutation that maps $\{i, j, k\}$ onto $\{1, 2, 3\}$.

²From here on, hats on operators are omitted.

where h is the Hamilton operator acting on \mathcal{H}_1

$$h = t + v = \frac{\vec{p}^2}{2m} + v(\vec{r}, \vec{p}, \vec{s}, \vec{\tau}) , \quad (1.10)$$

and where t and v denote the kinetic and potential energy of the particle, respectively.

4. The probability interpretation of the wave function stipulates that

$$|\langle \vec{r}\sigma\tau | \psi \rangle|^2 \equiv |\psi(\vec{r}\sigma\tau)|^2 \quad (1.11)$$

corresponds to the probability to find a nucleon at position \vec{r} with spin projection σ and isospin projection τ .

5. The postulate of reduction of the wave-packet following a measure states that, given an observable associated with the self-adjoint operator A characterized by

$$A|\phi_i\rangle = A_i|\phi_i\rangle , \quad (1.12)$$

and a state reading as

$$|\psi\rangle = \sum_i c_i |\phi_i\rangle \quad (1.13)$$

at the time of the measurement, the value A_k is measured with a probability $|c_k|^2$ and the state collapses to $|\psi\rangle = |\phi_k\rangle$ as a result of the measurement.

1.1.2 Bases of \mathcal{H}_1

In actual many-body calculations, the first practical step is to make a choice of basis $\mathcal{B}_1 \equiv \{|\mu\rangle\}$ of \mathcal{H}_1 , where the index μ embodies the set of quantum numbers characterizing single-particle basis states. Usually, there is an obvious choice to make, i.e. there exists a one-body hermitian operator h whose orthonormal eigenbasis obtained by solving the equation

$$h|\mu\rangle = \epsilon_\mu |\mu\rangle , \quad (1.14)$$

is best suited to tackle the problem of interest.

Examples.

1. The hamiltonian of a nucleon in a translationally invariant system reads

$$h = \frac{\vec{p}^2}{2m} + V(\vec{p}) , \quad (1.15)$$

such that

$$|\mu\rangle \equiv |\vec{p}\sigma\tau\rangle , \quad (1.16)$$

with

$$\begin{aligned} \langle \vec{r}\sigma'\tau' | \vec{p}\sigma\tau \rangle &\equiv \varphi_{\vec{p}}(\vec{r}) \delta_{\sigma\sigma'} \delta_{\tau\tau'} \\ &= \frac{1}{(2\pi\hbar)^{2/3}} e^{\frac{i}{\hbar}\vec{p}\cdot\vec{r}} \delta_{\sigma\sigma'} \delta_{\tau\tau'} . \end{aligned} \quad (1.17)$$

In the case where the potential is a (possibly null) constant $V(\vec{p}) = V_0$, the states $|\mu\rangle$ are actually eigenvectors of the present hamiltonian.

2. The hamiltonian of a one-dimensional harmonic oscillator reads as

$$h = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2, \quad (1.18)$$

such that

$$|\mu\rangle \equiv |n\sigma\tau\rangle \quad (1.19)$$

with

$$\begin{aligned} \langle x\sigma'\tau' | n\sigma\tau \rangle &\equiv \varphi_n(x) \delta_{\sigma\sigma'} \delta_{\tau\tau'} \\ &= \frac{1}{\sqrt{2^n n!}} \left(\frac{m\omega}{\pi\hbar} \right) H_n \left(\sqrt{\frac{m\omega}{\hbar}} x \right) e^{-\frac{m\omega}{2\hbar} x^2} \delta_{\sigma\sigma'} \delta_{\tau\tau'}, \end{aligned} \quad (1.20)$$

where H_n denotes the Hermite polynomials.

1.2 Many-body quantum system

1.2.1 Extension of basic postulates

Having reviewed basic postulates of quantum mechanics for one-particle systems, let us extend the discussion to a N -particle system, where the N particles are labeled from 1 to N .

1. The space of allowed states \mathcal{H}_N is given by the tensor product of the one-particle Hilbert spaces associated with each of the N particles involved

$$\mathcal{H}_N(1, 2, \dots, N) \equiv \mathcal{H}_1(1) \otimes \mathcal{H}_1(2) \otimes \dots \otimes \mathcal{H}_1(N), \quad (1.21)$$

where $\mathcal{H}_1(i)$ denotes the Hilbert space characterizing particle i .

2. The irreducible set of operators collects the irreducible sets of each of the N particles

$$\vec{r}_1, \vec{p}_1, \vec{s}_1, \vec{\tau}_1; \dots; \vec{r}_N, \vec{p}_N, \vec{s}_N, \vec{\tau}_N. \quad (1.22)$$

The commutation rules of the operators extend naturally Eq. (1.7) at the price of adding that operators associated with different particles commute.

3. The dynamical Schrödinger equation reads as

$$i\hbar \frac{\partial}{\partial t} |\Psi\rangle = H |\Psi\rangle \quad (1.23)$$

where H denotes the N -body Hamiltonian

$$H = T + V = \sum_{i=1}^N \frac{\vec{p}_i^2}{2m} + V(\vec{r}_1, \vec{p}_1, \vec{s}_1, \vec{\tau}_1; \dots; \vec{r}_N, \vec{p}_N, \vec{s}_N, \vec{\tau}_N), \quad (1.24)$$

where T and V characterize kinetic- and potential-energy operators, respectively.

4. The probability interpretation of the wave function becomes

$$|(1 : \vec{r}_1 \sigma_1 \tau_1; \dots; N : \vec{r}_N \sigma_N \tau_N | \Psi \rangle|^2 = |\Psi(\vec{r}_1 \sigma_1 \tau_1; \dots; \vec{r}_N \sigma_N \tau_N)|^2 \quad (1.25)$$

and corresponds to the probability to find particles $i \in \llbracket 1, N \rrbracket$ at position \vec{r}_i with spin projection σ_i and isospin projection τ_i .

5. The postulate of reduction of the wave packet after a measurement extends naturally from the one-body case.

1.2.2 Bases of \mathcal{H}_N

Considering a basis $\mathcal{B}_1 = \{|\alpha_1\rangle\}$ of the one-body Hilbert space $\mathcal{H}_1(1)$, the direct product basis of \mathcal{H}_N reads as

$$\mathcal{B}_N \equiv \{|1 : \alpha_1; \dots; N : \alpha_N\rangle \equiv |1 : \alpha_1\rangle \otimes \dots \otimes |N : \alpha_N\rangle\}, \quad (1.26)$$

where $|1 : \alpha_1; \dots; N : \alpha_N\rangle$ is the direct product of single-particle basis state $|\alpha_1\rangle$ for particle 1, \dots , and of single-particle basis state $|\alpha_N\rangle$ for particle N .

Example. In general, one can expand $|\Psi\rangle \in \mathcal{H}_N$ as a linear combination of direct-product basis states, i.e.

$$|\Psi\rangle = \sum_{\alpha_1 \dots \alpha_N} C_{\alpha_1 \dots \alpha_N} |1 : \alpha_1\rangle \otimes \dots \otimes |N : \alpha_N\rangle \quad (1.27)$$

where $C_{\alpha_1 \dots \alpha_N}$ denotes the coefficients of the linear combination. The particular case where a single term appears in the sum

$$|\Psi\rangle = |1 : \alpha_1\rangle \otimes \dots \otimes |N : \alpha_N\rangle \quad (1.28)$$

characterizes a (direct) product state that can be interpreted as the mere conjunction of N one-body states. In general, it is however *not* possible to write $|\Psi\rangle$ as a product of one-body states, i.e. $|\Psi\rangle$ is said to be entangled.

1.2.3 Operators

In Eq. (1.24), T is the sum of individual kinetic-energy operators and, as such, is said to be a one-body operator. However, there exists operators that act on more than one particle at a time. As such, a k -body operator ($k \leq N$) is defined as an operator that can be decomposed as a sum of operators each acting non trivially³ on k -body Hilbert sub-spaces of \mathcal{H}_N

$$\mathcal{H}_k(i_1, \dots, i_k) = \mathcal{H}_1(i_1) \otimes \mathcal{H}_1(i_2) \otimes \dots \otimes \mathcal{H}_1(i_k), \quad (1.29)$$

with $(i_1, \dots, i_k) \in \llbracket 1, N \rrbracket^k$.

In Eq. (1.24) for example, the potential-energy operator V reads as a genuine N -body operator as it can be decomposed as a sum of one term (itself) acting on \mathcal{H}_N , i.e. it acts non trivially on each of the $\mathcal{H}_1(i)$ at once. Most approximations, however, consist in considering the potential energy to be a two-body operator or a sum of two- and three-body operators, etc. Let us now characterize operators more precisely from the formal viewpoint.

One-body operator

A one-body operator F is defined via its action on a basis of \mathcal{H}_N with $N \geq 1$ ⁴

$$\begin{aligned} F : \quad \mathcal{H}_N &\rightarrow \mathcal{H}_N \\ |1 : \alpha; 2 : \beta; \dots\rangle &\rightarrow F|1 : \alpha; 2 : \beta; \dots\rangle \end{aligned}$$

such that

$$F \equiv \sum_{i=1}^N f(i), \quad (1.30)$$

where the sum runs over the N particles and where the operator $f(i)$ acts on $\mathcal{H}_1(i)$, i.e.

$$\begin{aligned} f(i) : \quad \mathcal{H}_1(i) &\rightarrow \mathcal{H}_1(i) \\ |i : \alpha\rangle &\rightarrow f(i)|i : \alpha\rangle. \end{aligned}$$

³Rigorously speaking, any operator defined on \mathcal{H}_N acts on all particles at once. However, if the operator is proportional to the identity operator $\mathbb{1}_1(i)$ on $\mathcal{H}_1(i)$, it is said to be acting trivially on particle i .

⁴The action of a one-body operator on the particle vacuum provides the null vector, i.e. that $F|0\rangle = 0$.

Two-body operator

A two-body operator G is defined via its action on a basis of \mathcal{H}_N with $N \geq 2$ ⁵

$$\begin{aligned} G : \quad \mathcal{H}_N &\rightarrow \mathcal{H}_N \\ |1 : \alpha; 2 : \beta; \dots\rangle &\rightarrow G|1 : \alpha; 2 : \beta; \dots\rangle \end{aligned}$$

such that

$$G \equiv \frac{1}{2} \sum_{i \neq j=1}^N g(i, j), \quad (1.31)$$

where the sums run over the N particles and where the operator $g(i, j)$ acts non-trivially on $\mathcal{H}_2(i, j)$ at once, i.e.

$$\begin{aligned} g(i, j) : \quad \mathcal{H}_2(i, j) &\rightarrow \mathcal{H}_2(i, j) \\ |i : \alpha; j : \beta\rangle &\rightarrow g(i, j)|i : \alpha; j : \beta\rangle. \end{aligned}$$

k -body operator

A k -body operator K is defined via its action on a basis of \mathcal{H}_N with $N \geq k$ ⁶

$$\begin{aligned} K : \quad \mathcal{H}_N &\rightarrow \mathcal{H}_N \\ |1 : \alpha; 2 : \beta; \dots\rangle &\rightarrow K|1 : \alpha; 2 : \beta; \dots\rangle \end{aligned}$$

such that

$$K \equiv \frac{1}{k!} \sum_{i \neq j \neq \dots l=1}^N k(i, j, \dots l), \quad (1.32)$$

with $(i, j, \dots l) \in \llbracket 1, N \rrbracket^k$ such that the operator $k(i, j, \dots l)$ acts non-trivially on $\mathcal{H}_k(i, j, \dots l)$ at once, i.e.

$$\begin{aligned} k(i, j, \dots l) : \quad \mathcal{H}_k(i, j, \dots l) &\rightarrow \mathcal{H}_k(i, j, \dots l) \\ |i : \alpha; j : \beta; \dots; l : \delta\rangle &\rightarrow k(i, j, \dots l)|i : \alpha; j : \beta; \dots; l : \delta\rangle. \end{aligned}$$

1.2.4 Operator representations

Practically speaking, a k -body operator is defined in a chosen representation, i.e. through the given of the set of its matrix elements in a given basis \mathcal{B}_k of \mathcal{H}_k .

One-body operator

A one-body operator $f(1)$ acting on $\mathcal{H}_1(1)$ is typically defined by the given of its complete set of its matrix elements in basis, e.g., $\mathcal{B}_1(1) = \{|1 : \vec{r}\sigma\tau\rangle\}$

$$[f(1), \mathcal{B}_1(1)] \rightarrow \langle 1 : \vec{r}\sigma\tau | f(1) | 1 : \vec{r}'\sigma'\tau' \rangle.$$

With this definition at hand, one may be interested in representing the operator in a different basis $\mathcal{B}_{1'}(1) = \{|1 : \mu\rangle\}$, i.e. in computing the set of matrix elements $\langle 1 : \alpha | f(1) | 1 : \beta \rangle$. Typical basis of interest are the momentum basis or the harmonic oscillator basis introduced

⁵The action of a two-body operator on states of \mathcal{H}_0 or \mathcal{H}_1 provides the null vector.

⁶The action of a k -body operator on states of \mathcal{H}_N with $N < k$ provides the null vector.

in Sec. 1.1.2. To do so, one inserts twice the completeness relation on $\mathcal{H}_1(1)$ associated with basis $\mathcal{B}_1(1) = \{|1 : \vec{r}\sigma\tau\rangle\}$

$$\mathbb{1}_1 \equiv \int d\vec{r} \sum_{\sigma\tau} |1 : \vec{r}\sigma\tau\rangle \langle 1 : \vec{r}\sigma\tau|, \quad (1.33)$$

to obtain the wanted matrix elements in terms of the originally given ones

$$\begin{aligned} \langle 1 : \alpha | f(1) | 1 : \beta \rangle &= \iint d\vec{r}_1 d\vec{r}'_1 \sum_{\sigma\tau} \sum_{\sigma'\tau'} \langle 1 : \alpha | \vec{r}_1 \sigma \tau \rangle \langle 1 : \vec{r}_1 \sigma \tau | f(1) | \vec{r}'_1 \sigma' \tau' \rangle \langle 1 : \vec{r}'_1 \sigma' \tau' | 1 : \beta \rangle \\ &= \iint d\vec{r}_1 d\vec{r}'_1 \sum_{\sigma\tau} \sum_{\sigma'\tau'} \varphi_\alpha^*(\vec{r}_1 \sigma \tau) f(\vec{r}_1, \sigma, \tau; \vec{r}'_1 \sigma' \tau') \varphi_\beta(\vec{r}'_1 \sigma' \tau'), \end{aligned} \quad (1.34)$$

where the notation $f(\vec{r}, \sigma, \tau; \vec{r}' \sigma' \tau') \equiv \langle 1 : \vec{r}\sigma\tau | f(1) | 1 : \vec{r}'\sigma'\tau' \rangle$ has been introduced.

Quite often, the operator acts non-trivially only on spatial coordinates, i.e. the operator is spin- and isospin-independent. In this situation, the operator acting on \mathcal{H}_1 writes formally as

$$f(1) = f_{\text{space}}(1) \otimes \mathbb{1}_{\text{spin}}(1) \otimes \mathbb{1}_{\text{isospin}}(1), \quad (1.35)$$

such that its matrix elements in basis $\mathcal{B}_1(1) = \{|1 : \vec{r}\sigma\tau\rangle\}$ reduce to

$$\begin{aligned} \langle 1 : \vec{r}_1 \sigma \tau | f(1) | \vec{r}'_1 \sigma' \tau' \rangle &\equiv \langle 1 : \vec{r}_1 | f_{\text{space}}(1) | 1 : \vec{r}'_1 \rangle \langle 1 : \sigma | 1 : \sigma' \rangle \langle 1 : \tau | 1 : \tau' \rangle \\ &\equiv f(\vec{r}_1; \vec{r}'_1) \delta_{\sigma\sigma'} \delta_{\tau\tau'}. \end{aligned} \quad (1.36)$$

Consequently, the matrix elements in basis $\mathcal{B}_{1'}(1) = \{|1 : \mu\rangle\}$ are obtained as

$$\langle 1 : \alpha | f(1) | 1 : \beta \rangle = \iint d\vec{r}_1 d\vec{r}'_1 \sum_{\sigma\tau} \varphi_\alpha^*(\vec{r}_1 \sigma \tau) f(\vec{r}_1; \vec{r}'_1) \varphi_\beta(\vec{r}'_1 \sigma \tau). \quad (1.37)$$

Further considering that the operator is local in space

$$f(\vec{r}_1; \vec{r}'_1) \equiv f(\vec{r}_1) \delta(\vec{r}_1 - \vec{r}'_1), \quad (1.38)$$

the computation of the matrix elements in basis $\mathcal{B}_{1'} = \{|\mu\rangle\}$ reduces to the standard expression involving a single (triple) integral

$$\langle 1 : \alpha | f(1) | 1 : \beta \rangle = \int d\vec{r}_1 \sum_{\sigma\tau} \varphi_\alpha^*(\vec{r}_1 \sigma \tau) f(\vec{r}_1) \varphi_\beta(\vec{r}_1 \sigma \tau). \quad (1.39)$$

Example. An example of spin- and isospin-independent local one-body operator is given by the kinetic energy of a fermion of mass m , whose expression in terms of the momentum operator is

$$t(1) \equiv \frac{\vec{p}^2(1)}{2m} \otimes \mathbb{1}_{\text{spin}}(1) \otimes \mathbb{1}_{\text{isospin}}(1). \quad (1.40)$$

Given the representation of the operator \vec{p} in the position basis given by Eq. (1.4b), the one of $t(1)$ is obtained by inserting a completeness relation in basis $\mathcal{B}_1 = \{|\vec{r}\sigma\tau\rangle\}$ such that

$$\begin{aligned} \langle 1 : \vec{r}\sigma\tau | t(1) | 1 : \vec{r}'\sigma'\tau' \rangle &= \frac{1}{2m} \int d\vec{r}_1 \sum_{\sigma_1\tau_1} \langle 1 : \vec{r}\sigma\tau | \vec{p}(1) | 1 : \vec{r}_1 \sigma_1 \tau_1 \rangle \langle 1 : \vec{r}_1 \sigma_1 \tau_1 | \vec{p}(1) | 1 : \vec{r}'\sigma'\tau' \rangle \\ &= -\frac{\hbar^2}{2m} \int d\vec{r}_1 \sum_{\sigma_1\tau_1} \delta(\vec{r} - \vec{r}_1) \delta_{\sigma\sigma_1} \delta_{\tau\tau_1} \vec{\nabla}_{\vec{r}} \delta(\vec{r}_1 - \vec{r}') \delta_{\sigma_1\sigma'} \delta_{\tau_1\tau'} \vec{\nabla}_{\vec{r}'} \\ &= -\frac{\hbar^2}{2m} \delta(\vec{r} - \vec{r}') \delta_{\sigma\sigma'} \delta_{\tau\tau'} \Delta, \end{aligned} \quad (1.41)$$

which specifies Eqs. (1.38) and (1.38) for the operator of interest. With this representation at hand, the matrix elements of the kinetic energy operator in an arbitrary basis $\mathcal{B}_1 = \{|\mu\rangle\}$ read, according to Eq. (1.39), as

$$\begin{aligned} \langle 1 : \alpha | t(1) | 1 : \beta \rangle &= -\frac{\hbar^2}{2m} \int d\vec{r} \sum_{\sigma\tau} \varphi_\alpha^*(\vec{r}\sigma\tau) \Delta \varphi_\beta(\vec{r}\sigma\tau) \\ &= +\frac{\hbar^2}{2m} \int d\vec{r} \sum_{\sigma\tau} \vec{\nabla} \varphi_\alpha^*(\vec{r}\sigma\tau) \cdot \vec{\nabla} \varphi_\beta(\vec{r}\sigma\tau), \end{aligned} \quad (1.42)$$

where the last line was obtained via an integration by parts.

Two-body operator

A two-body operator $g(1, 2)$ acting on $\mathcal{H}_2(1, 2)$ is typically defined by the given of its complete set of its matrix elements in the direct-product basis $\mathcal{B}_2(1, 2) = \{|1 : \vec{r}_1\sigma_1\tau_1; 2 : \vec{r}_2\sigma_2\tau_2\rangle\}$

$$[g(1, 2), \mathcal{B}_2(1, 2)] \rightarrow \langle 1 : \vec{r}_1\sigma_1\tau_1; 2 : \vec{r}_2\sigma_2\tau_2 | g(1, 2) | 1 : \vec{r}'_1\sigma'_1\tau'_1; 2 : \vec{r}'_2\sigma'_2\tau'_2 \rangle.$$

With this definition at hand, and following the same steps as for one-body operators, matrix elements of $g(1, 2)$ in basis $\mathcal{B}_2(1, 2) = \{|1 : \mu, 2 : \nu\rangle\}$ read as

$$\begin{aligned} \langle 1 : \alpha; 2 : \beta | g(1, 2) | 1 : \gamma; 2 : \delta \rangle &= \iiint d\vec{r}_1 d\vec{r}_2 d\vec{r}'_1 d\vec{r}'_2 \sum_{\substack{\sigma_1\sigma_2 \\ \sigma'_1\sigma'_2}} \sum_{\substack{\tau_1\tau_2 \\ \tau'_1\tau'_2}} \varphi_\alpha^*(\vec{r}_1\sigma_1\tau_1) \varphi_\beta^*(\vec{r}_2\sigma_2\tau_2) \\ &\quad \times g(\vec{r}_1, \sigma_1, \tau_1, \vec{r}_2, \sigma_2, \tau_2; \vec{r}'_1, \sigma'_1, \tau'_1, \vec{r}'_2, \sigma'_2, \tau'_2) \\ &\quad \times \varphi_\gamma(\vec{r}'_1\sigma'_1\tau'_1) \varphi_\delta(\vec{r}'_2\sigma'_2\tau'_2), \end{aligned} \quad (1.43)$$

where the notation

$$g(\vec{r}_1, \sigma_1, \tau_1, \vec{r}_2, \sigma_2, \tau_2; \vec{r}'_1, \sigma'_1, \tau'_1, \vec{r}'_2, \sigma'_2, \tau'_2) \equiv \langle 1 : \vec{r}_1\sigma_1\tau_1; 2 : \vec{r}_2\sigma_2\tau_2 | g(1, 2) | 1 : \vec{r}'_1\sigma'_1\tau'_1; 2 : \vec{r}'_2\sigma'_2\tau'_2 \rangle$$

has been introduced.

Further considering a spin- and isospin-independent local two-body operator

$$g(1, 2) = g_{\text{space}}(1, 2) \otimes \mathbb{1}_{2_{\text{spin}}}(1, 2) \otimes \mathbb{1}_{2_{\text{isospin}}}(1, 2), \quad (1.44)$$

whose matrix elements in basis $\mathcal{B}_2(1, 2) = \{|1 : \vec{r}_1\sigma_1\tau_1; 2 : \vec{r}_2\sigma_2\tau_2\rangle\}$ take the simpler form

$$\begin{aligned} \langle 1 : \vec{r}_1\sigma_1\tau_1; 2 : \vec{r}_2\sigma_2\tau_2 | g(1, 2) | 1 : \vec{r}'_1\sigma'_1\tau'_1; 2 : \vec{r}'_2\sigma'_2\tau'_2 \rangle &\equiv g(\vec{r}_1, \vec{r}_2) \\ &\quad \times \delta(\vec{r}_1 - \vec{r}'_1) \delta_{\sigma_1\sigma'_1} \delta_{\tau_1\tau'_1} \\ &\quad \times \delta(\vec{r}_2 - \vec{r}'_2) \delta_{\sigma_2\sigma'_2} \delta_{\tau_2\tau'_2}, \end{aligned} \quad (1.45)$$

the expression of the matrix elements reduces to

$$\begin{aligned} \langle 1 : \alpha; 2 : \beta | g(1, 2) | 1 : \gamma; 2 : \delta \rangle &= \iint d\vec{r}_1 d\vec{r}_2 \sum_{\sigma_1\sigma_2} \sum_{\tau_1\tau_2} \varphi_\alpha^*(\vec{r}_1\sigma_1\tau_1) \varphi_\beta^*(\vec{r}_2\sigma_2\tau_2) \\ &\quad \times g(\vec{r}_1, \vec{r}_2) \\ &\quad \times \varphi_\gamma(\vec{r}_1\sigma_1\tau_1) \varphi_\delta(\vec{r}_2\sigma_2\tau_2). \end{aligned} \quad (1.46)$$

Example. An example of local, spin-independent, isospin-dependent two-body operator is given by the Coulomb interaction $V_c(1, 2)$ between two nucleons (of charge 0 for neutrons

and e for protons). The matrix elements of $V_c(1, 2)$ in position basis reads

$$\begin{aligned} V_c(\vec{r}_1, \sigma_1, \tau_1, \vec{r}_2, \sigma_2, \tau_2) &\equiv \langle 1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2 | V_c(1, 2) | 1 : \vec{r}_1' \sigma_1' \tau_1' ; 2 : \vec{r}_2' \sigma_2' \tau_2' \rangle \\ &= \frac{e^2}{\hbar^2} \frac{(\tau_1 + \frac{\hbar}{2})(\tau_2 + \frac{\hbar}{2})}{|\vec{r}_1 - \vec{r}_2|} \delta(\vec{r}_1 - \vec{r}_1') \delta_{\sigma_1 \sigma_1'} \delta_{\tau_1 \tau_1'} \\ &\quad \times \delta(\vec{r}_2 - \vec{r}_2') \delta_{\sigma_2 \sigma_2'} \delta_{\tau_2 \tau_2'}. \end{aligned} \quad (1.47)$$

In the arbitrary basis $\mathcal{B}_{2'}(1, 2) = \{|1 : \mu, 2 : \nu\rangle\}$, the matrix elements are then easily obtained by inserting Eq. (1.47) into Eq. (1.43)

$$\begin{aligned} \langle 1 : \alpha ; 2 : \beta | V_c(1, 2) | 1 : \gamma ; 2 : \delta \rangle &= \iint d\vec{r}_1 d\vec{r}_2 \sum_{\sigma_1 \sigma_2} \sum_{\tau_1 \tau_2} \varphi_\alpha^*(\vec{r}_1 \sigma_1 \tau_1) \varphi_\beta^*(\vec{r}_2 \sigma_2 \tau_2) \\ &\quad \times \frac{e^2}{\hbar^2} \frac{(\tau_1 + \frac{\hbar}{2})(\tau_2 + \frac{\hbar}{2})}{|\vec{r}_1 - \vec{r}_2|} \varphi_\gamma(\vec{r}_1 \sigma_1 \tau_1) \varphi_\delta(\vec{r}_2 \sigma_2 \tau_2). \end{aligned} \quad (1.48)$$

k -body operator

The above considerations are easily extended to the representation of a k -body operator on a basis \mathcal{B}_k of \mathcal{H}_k .

1.2.5 Identical particles

In the case of N identical particles⁷, not all states of \mathcal{H}_N are in fact physically allowed states. This feature leads to the necessity to add two postulates that apply differently to particles carrying an integer spin (γ, Z^0, H , etc), and those carrying an a half-integer spin (protons, neutrons, quarks, etc). Particles belonging to the first category are denoted as Bosons whereas those belonging to the second category are denoted as Fermions.

In order to formulate the additional postulates, it is necessary to introduce the notion of permutations of N elements as well as to define symmetrization and antisymmetrization operators.

Symmetric group

The permutations of N elements form the symmetric group whose basic properties are now listed.

1. The group contains $N!$ elements p called permutations of $(1, 2, \dots, N)$. The set of permutations is denoted as \mathcal{P} .
2. Any permutation $p \in \mathcal{P}$ can be decomposed as a product of elementary transpositions t_{ij} defined through

$$t_{ij}(i) = j \quad ; \quad t_{ij}(j) = i \quad ; \quad t_{ij}(k) = k \quad \text{if } k \neq i, j. \quad (1.49)$$

For example, the permutation⁸ $p: (1, 2, 3) \rightarrow (3, 1, 2)$ can be written as $p = t_{23}t_{13}$.

3. One associates a number $\pi_p = \pm 1$ to each permutation $p \in \mathcal{P}$, called the signature of the permutation, such that

$$\begin{aligned} \pi_{pp'} &= \pi_p \pi_{p'}, \\ \pi_{\text{Id}} &= +1, \\ \pi_{t_{ij}} &= -1. \end{aligned}$$

⁷By identical particles, we mean here particles carrying the same intrinsic quantum numbers, e.g. mass, charge, spin, isospin.

⁸This permutation is a cycle.

Given the set of permutation \mathcal{P} , $N!$ operators P_p are defined on \mathcal{H}_N through their action on the basis states

$$P_p|1 : \alpha_1; \dots; N : \alpha_N\rangle \equiv |1 : \alpha_{p(1)}; \dots; N : \alpha_{p(N)}\rangle \quad (1.50a)$$

$$\equiv |p(1) : \alpha_1; \dots; p(N) : \alpha_N\rangle, \quad (1.50b)$$

where the permutation can equally act on particle or single-particle labels. The first case of particular interest concerns the identity whose associated operator reads as $P_{\text{Id}} = \mathbb{1}_N$. The second case of particular interest relates to operators associated with transpositions whose traditional short-hand notation is given by $P_{ij} \equiv P_{t_{ij}}$ and whose action reads as

$$P_{ij}|1 : \alpha_1; \dots; i : \alpha_i; \dots; j : \alpha_j; \dots\rangle = |1 : \alpha_1; \dots; i : \alpha_j; \dots; j : \alpha_i; \dots\rangle. \quad (1.51)$$

Permutation operators are unitary, verify $P_p P_{p'} = P_{pp'}$ but are a priori not hermitian. As for the particular case of transpositions, one has

$$P_{ij}^2 = 1, \quad (1.52a)$$

$$P_{ij}^\dagger = P_{ij}. \quad (1.52b)$$

Symmetrization and antisymmetrization operators

One introduces symmetrization and antisymmetrization operators acting on \mathcal{H}_N and respectively defined through

$$\mathcal{S} \equiv \frac{1}{N!} \sum_{p \in \mathcal{P}} P_p, \quad (1.53a)$$

$$\mathcal{A} \equiv \frac{1}{N!} \sum_{p \in \mathcal{P}} \pi_p P_p, \quad (1.53b)$$

whose main properties are⁹

1. $\mathcal{S}^\dagger = \mathcal{S}$ and $\mathcal{A}^\dagger = \mathcal{A}$,
2. $P_p \mathcal{S} = \mathcal{S}$ and $P_p \mathcal{A} = \pi_p \mathcal{A}$,
3. $\mathcal{S}^2 = \mathcal{S}$ and $\mathcal{A}^2 = \mathcal{A}$,
4. $\mathcal{S} \mathcal{A} = \mathcal{A} \mathcal{S} = 0$.

Examples. For $N = 2$, one finds that

$$\mathcal{S} \equiv \frac{1}{2}(\mathbb{1}_2 + P_{12}), \quad (1.54a)$$

$$\mathcal{A} \equiv \frac{1}{2}(\mathbb{1}_2 - P_{12}), \quad (1.54b)$$

whereas for $N = 3$ the expressions of \mathcal{S} and \mathcal{A} in terms of transposition operators are

$$\mathcal{S} \equiv \frac{1}{6}(\mathbb{1}_3 + P_{12} + P_{13} + P_{23} + P_{12}P_{13} + P_{12}P_{23}), \quad (1.55a)$$

$$\mathcal{A} \equiv \frac{1}{6}(\mathbb{1}_3 - P_{12} - P_{13} - P_{23} + P_{12}P_{13} + P_{12}P_{23}). \quad (1.55b)$$

⁹Although \mathcal{S} and \mathcal{A} are orthogonal projectors, it must not be concluded that their associated subspaces (onto which they project) are necessarily in direct sum. Indeed, except for $N = 2$, one does not have $\mathcal{S} + \mathcal{A} = \mathbb{1}_N$.

Additional postulates

6. The N -body Hilbert space \mathcal{H}_N must be reduced to the subspace of physically accessible states \mathcal{H}_N^B or \mathcal{H}_N^F , i.e. the subspace of totally symmetric or totally antisymmetric states under the exchange of any two particles, respectively. In the following, we limit ourselves to considering fermions. A basis \mathcal{B}_N^F of \mathcal{H}_N^F is made of normalized and fully antisymmetric product states (i.e. Slater determinants) of \mathcal{H}_N obtained from direct-product states through the action of the antisymmetrizer

$$|\alpha_1 \dots \alpha_N\rangle \equiv \sqrt{N!} \mathcal{A} |1 : \alpha_1 ; \dots ; N : \alpha_N\rangle \quad (1.56a)$$

$$= \frac{1}{\sqrt{N!}} \sum_{p \in \mathcal{P}} \pi_p |p(1) : \alpha_1 ; \dots ; p(N) : \alpha_N\rangle \quad (1.56b)$$

$$= \frac{1}{\sqrt{N!}} \begin{vmatrix} |1 : \alpha_1\rangle & \dots & |N : \alpha_1\rangle \\ \vdots & \ddots & \vdots \\ |1 : \alpha_N\rangle & \dots & |N : \alpha_N\rangle \end{vmatrix}, \quad (1.56c)$$

where the last line testifies that the antisymmetrized many-body state can be written under the form of a determinant, hence the denomination as a Slater determinant. The Slater determinant $|\Phi\rangle \equiv |\alpha_1 \dots \alpha_N\rangle$ is antisymmetric under the exchange of any pair of particles (or equivalently of any pair of single-particle states)

$$P_{ij} |\Phi\rangle = -|\Phi\rangle. \quad (1.57)$$

which is obvious given that the determinant is antisymmetric under the exchange of two columns or of two rows (corresponding respectively to an exchange of two particles or two single-particle states). As a result of the antisymmetrization, and by opposition to direct-product states (Eq. (1.26)), it is not possible anymore to state that particle 1 is occupying single-particle state $|\alpha_1\rangle$, that particle 2 is occupying single-particle state $|\alpha_2\rangle$ etc in state $|\Phi\rangle$. Therefore, there exist intrinsic correlations between the N particles, even though they are minimal for a Slater determinant as they are the fingerprint of the sole Pauli exclusion principle.

The many-body wave function also takes the form of a determinant of the N occupied single-particle wave functions

$$\begin{aligned} \langle 1 : \vec{r}_1 \sigma_1 \tau_1 ; \dots ; N : \vec{r}_N \sigma_N \tau_N | \alpha_1 \dots \alpha_N \rangle &\equiv \Phi_{\alpha_1 \dots \alpha_N}(\vec{r}_1 \sigma_1 \tau_1 ; \dots ; \vec{r}_N \sigma_N \tau_N) \quad (1.58) \\ &= \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{\alpha_1}(\vec{r}_1 \sigma_1 \tau_1) & \dots & \psi_{\alpha_1}(\vec{r}_N \sigma_N \tau_N) \\ \vdots & \ddots & \vdots \\ \psi_{\alpha_N}(\vec{r}_1 \sigma_1 \tau_1) & \dots & \psi_{\alpha_N}(\vec{r}_N \sigma_N \tau_N) \end{vmatrix} \end{aligned}$$

which can be compared to the one associated with the direct-product state in the absence of antisymmetrization that was given by

$$\langle 1 : \vec{r}_1 \sigma_1 \tau_1 ; \dots ; N : \vec{r}_N \sigma_N \tau_N | 1 : \alpha_1 ; \dots ; N : \alpha_N \rangle = \prod_{i=1}^N \psi_{\alpha_i}(\vec{r}_i \sigma_i \tau_i). \quad (1.59)$$

Example. For $N = 2$, the wave function associated with a Slater determinant is

$$\begin{aligned} \Phi_{\alpha_1 \alpha_2}(\vec{r}_1 \sigma_1 \tau_1 ; \vec{r}_2 \sigma_2 \tau_2) &\equiv \langle 1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2 | \alpha_1 \alpha_2 \rangle \\ &= \langle 1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2 | \sqrt{2!} \mathcal{A} | 1 : \alpha_1 ; 2 : \alpha_2 \rangle \\ &= \frac{1}{\sqrt{2}} \begin{vmatrix} \psi_{\alpha_1}(\vec{r}_1 \sigma_1 \tau_1) & \psi_{\alpha_1}(\vec{r}_2 \sigma_2 \tau_2) \\ \psi_{\alpha_2}(\vec{r}_1 \sigma_1 \tau_1) & \psi_{\alpha_2}(\vec{r}_2 \sigma_2 \tau_2) \end{vmatrix} \\ &= \frac{1}{\sqrt{2}} [\psi_{\alpha_1}(\vec{r}_1 \sigma_1 \tau_1) \psi_{\alpha_2}(\vec{r}_2 \sigma_2 \tau_2) \\ &\quad - \psi_{\alpha_2}(\vec{r}_1 \sigma_1 \tau_1) \psi_{\alpha_1}(\vec{r}_2 \sigma_2 \tau_2)], \quad (1.60) \end{aligned}$$

which should be compared to

$$\langle 1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2 | 1 : \alpha_1 ; 2 : \alpha_2 \rangle = \psi_{\alpha_1}(\vec{r}_1 \sigma_1 \tau_1) \psi_{\alpha_2}(\vec{r}_2 \sigma_2 \tau_2) . \quad (1.61)$$

The antisymmetric two-body wave function computed in Eq. (1.60) is indeed normalized

$$\begin{aligned} \langle \alpha_1 \alpha_2 | \alpha_1 \alpha_2 \rangle &= \sum_{\sigma_1 \sigma_2} \sum_{\tau_1 \tau_2} \iint d\vec{r}_1 d\vec{r}_2 |\Phi_{\alpha_1 \alpha_2}(\vec{r}_1 \sigma_1 \tau_1 ; \vec{r}_2 \sigma_2 \tau_2)|^2 \\ &= \frac{1}{2} \left[\sum_{\sigma_1 \tau_1} \int d\vec{r}_1 |\psi_{\alpha_1}(\vec{r}_1 \sigma_1 \tau_1)|^2 \sum_{\sigma_2 \tau_2} \int d\vec{r}_2 |\psi_{\alpha_2}(\vec{r}_2 \sigma_2 \tau_2)|^2 \right. \\ &\quad \left. + \sum_{\sigma_1 \tau_1} \int d\vec{r}_1 |\psi_{\alpha_2}(\vec{r}_1 \sigma_1 \tau_1)|^2 \sum_{\sigma_2 \tau_2} \int d\vec{r}_2 |\psi_{\alpha_1}(\vec{r}_2 \sigma_2 \tau_2)|^2 \right] \\ &= 1 , \end{aligned}$$

where the orthonormalization of one-body wave functions was used after having inserted the completeness on \mathcal{H}_2

$$\mathbb{1}_2 = \sum_{\sigma_1 \sigma_2} \sum_{\tau_1 \tau_2} \iint d\vec{r}_1 d\vec{r}_2 |1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2 \rangle \langle 1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2 |$$

in the original overlap.

7. Considering identical particles also reduces the space of allowed operators that must be symmetric under the exchange of any pair of particles, i.e.

$$\forall (i, j) \in \llbracket 1, N \rrbracket^2 \quad V(1, \dots, i, \dots, j, \dots, N) = V(1, \dots, j, \dots, i, \dots, N) . \quad (1.62)$$

Examples. The potential of two identical particles has to satisfy

$$V(1, 2) = V(2, 1) . \quad (1.63)$$

Similarly a three-body operator should satisfy

$$V = V(1, 2, 3) = V(1, 3, 2) = V(2, 1, 3) = V(2, 3, 1) = V(3, 2, 1) = V(3, 1, 2) . \quad (1.64)$$

Therefore any three-body operator V can be written as

$$\begin{aligned} V &\equiv V_{\overline{123}} \\ &= \frac{1}{6} (V(1, 2, 3) + V(2, 1, 3) + V(1, 3, 2) + V(3, 1, 2) + V(3, 2, 1) + V(2, 3, 1)) \\ &= \frac{1}{3} \left(\frac{V(1, 2, 3) + V(2, 1, 3)}{2} + \frac{V(1, 3, 2) + V(3, 1, 2)}{2} + \frac{V(3, 2, 1) + V(2, 3, 1)}{2} \right) \\ &= \frac{1}{3} (V_{\overline{123}} + V_{\overline{132}} + V_{\overline{231}}) , \end{aligned} \quad (1.65)$$

where $\overline{123}$ means that the potential is symmetric under any exchange of particles 1, 2 and 3, whereas $\overline{12}$ means that it is symmetrized under the exchange of 1 and 2 only. Two of the three terms can be rewritten in terms of the third one and of permutation operators for two particles such that

$$V = \frac{1}{3} (V_{\overline{123}} + P_{23} P_{12} V_{\overline{123}} P_{12} P_{23} + P_{13} P_{12} V_{\overline{123}} P_{12} P_{13}) . \quad (1.66)$$

Equation (1.66) demonstrates that a three-body operator is fully specified through the sole given of $V_{\overline{123}}$.

1.3 One-nucleon states

1.3.1 Eigenbasis of position

Let us recall direct-product states $|\vec{r}\sigma\tau\rangle \equiv |\vec{r}\rangle \otimes |\sigma\rangle \otimes |\tau\rangle$ defined such that¹⁰

$$\vec{r}|\vec{r}\sigma\tau\rangle = \vec{r}|\vec{r}\sigma\tau\rangle , \quad (1.67)$$

$$\vec{s}^2 |\vec{r}\sigma\tau\rangle = \frac{1}{2} \left(\frac{1}{2} + 1 \right) |\vec{r}\sigma\tau\rangle , \quad s_z |\vec{r}\sigma\tau\rangle = \sigma |\vec{r}\sigma\tau\rangle , \quad (1.68)$$

$$\vec{\tau}^2 |\vec{r}\sigma\tau\rangle = \frac{1}{2} \left(\frac{1}{2} + 1 \right) |\vec{r}\sigma\tau\rangle , \quad \tau_z |\vec{r}\sigma\tau\rangle = \tau |\vec{r}\sigma\tau\rangle , \quad (1.69)$$

constitute a complete continuous orthonormal basis of $\mathcal{H}_1 = \mathcal{H}_{1,\vec{r}} \otimes \mathcal{H}_{1,\vec{s}} \otimes \mathcal{H}_{1,\vec{\tau}}$ with $\vec{r} \in \mathbb{R}^3$ and $\sigma, \tau \in \{+1/2, -1/2\}$ ¹¹.

The orthogonality and completeness relations can be written as

$$\langle \vec{r}\sigma\tau | \vec{r}'\sigma'\tau' \rangle = \delta(\vec{r} - \vec{r}') \delta_{\sigma\sigma'} \delta_{\tau\tau'} , \quad \int d\vec{r} \sum_{\sigma} \sum_{\tau} |\vec{r}\sigma\tau\rangle \langle \vec{r}\sigma\tau| = \mathbb{1}_1 , \quad (1.70)$$

where $\mathbb{1}_1$ is the unity operator on \mathcal{H}_1 .

1.3.2 Eigenbasis of linear momentum

Direct-product states $|\vec{p}\sigma\tau\rangle \equiv |\vec{p}\rangle \otimes |\sigma\rangle \otimes |\tau\rangle$ made out of eigenstates of the momentum operator $\vec{p} = -i\hbar\vec{\nabla}$, s_z and τ_z also constitute a basis of \mathcal{H}_1 . The relationship between such states and the previous basis is given by

$$\langle \vec{r}\sigma\tau | \vec{p}\sigma'\tau' \rangle = \frac{e^{i\vec{r}\cdot\vec{p}}}{(2\pi)^{3/2}} \delta_{\sigma\sigma'} \delta_{\tau\tau'} . \quad (1.71)$$

1.3.3 Eigenbasis of orbital angular momentum

The orbital angular-momentum operator is defined as $\vec{l} \equiv \vec{r} \times \vec{p}$. It is hermitian since \vec{r} and \vec{p} are hermitian. The orbital angular-momentum components follow the commutation relations

$$[l_i, l_j] = i\hbar \sum_k \varepsilon_{ijk} l_k , \quad (1.72)$$

which derive from those fulfilled by \vec{r} and \vec{p} . One can easily show that l^2 commute with each components l_i of \vec{l} . One can thus introduce the eigenstates $|lm_l\rangle$ of l^2 and l_z such that

$$\vec{l}^2 |lm_l\rangle = \hbar^2 l(l+1) |lm_l\rangle , \quad l_z |lm_l\rangle = \hbar m_l |lm_l\rangle , \quad (1.73)$$

where $l \in \mathbb{N}$ and $m_l \in \mathbb{Z}$ such that $|m_l| \leq l$. In spherical coordinates, $|\vec{r}\rangle \equiv |r\theta\varphi\rangle$ with $r = \|\vec{r}\|$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$, it is possible to check that l acts only on angular coordinates (θ, φ) . The spherical harmonics

$$Y_l^{m_l}(\theta, \varphi) \equiv \langle \theta\varphi | lm_l \rangle , \quad (1.74)$$

are the wave functions associated with $|lm_l\rangle$ in this space and one has

$$|lm_l\rangle = \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi |\theta\varphi\rangle Y_l^{m_l}(\theta, \varphi) .$$

¹⁰For some of the operators, e.g. \vec{r} , we denote the operator and the associated quantum numbers in the same way. This should not be too confusing.

¹¹Throughout the lecture series, the quantum number τ is sometimes referred to as a letter, i.e. n for neutrons and p for protons, or as a number, i.e. $+1/2$ for protons and $-1/2$ for neutrons.

1.3.4 Eigenbasis of total angular momentum

The total angular momentum operator of a nucleon is $\vec{j} \equiv \vec{l} + \vec{s}$. The components of \vec{j} obey to the same commutation relationships than \vec{l} and \vec{s} . One can define eigenstates $|(l1/2)jm\rangle$ common to \vec{l}^2 , \vec{s}^2 , \vec{j}^2 and j_z , because these four operators commute with each other, and write

$$\vec{l}^2 |(l1/2)jm\rangle = \hbar^2 l(l+1) |(l1/2)jm\rangle , \quad (1.75)$$

$$\vec{s}^2 |(l1/2)jm\rangle = \frac{3}{4} \hbar^2 |(l1/2)jm\rangle , \quad (1.76)$$

$$\vec{j}^2 |(l1/2)jm\rangle = \hbar^2 j(j+1) |(l1/2)jm\rangle, \quad j_z |(l1/2)jm\rangle = \hbar m |(l1/2)jm\rangle , \quad (1.77)$$

where $2j \in \mathbb{N}^*$ and $2m \in \mathbb{Z}$ such that $|m| \leq j$. Such eigenstates can be expressed by a linear combination of the direct-product states $|lm_l 1/2\sigma\rangle \equiv |lm_l\rangle \otimes |1/2\sigma\rangle$

$$|(l1/2)jm\rangle = \sum_{m_l \sigma} |lm_l 1/2\sigma\rangle \langle l1/2m_l \sigma | jm\rangle , \quad (1.78)$$

where $\langle l1/2m_l \sigma | jm\rangle$ are the so-called Clebsch-Gordan coefficients. Equation (1.78) is important because it shows how to add angular momentum operators \vec{l} and \vec{s} . The operator \vec{j} acts on both spatial and spin coordinates such that the wave function associated with $|(l1/2)jm\rangle$ is a spin-angular spherical harmonics that reads in spherical and spin coordinates as

$$\langle \theta \varphi \sigma' | (l1/2)jm\rangle \equiv \mathcal{Y}_{(l1/2)jm}(\theta, \varphi, \sigma') = \sum_{m_l \sigma} \langle l1/2m_l \sigma | jm\rangle Y_l^{m_l}(\theta, \varphi) \chi_{1/2}^{\sigma}(\sigma') , \quad (1.79)$$

where $\chi_{1/2}^{\sigma}(\sigma') \equiv \langle 1/2\sigma' | 1/2\sigma\rangle = \delta_{\sigma'\sigma}$ is the wave function of the state of spin σ . The most general wave function of a nucleon of isospin τ can be expanded using the complete set of spin-angular spherical harmonics according to

$$\psi_{\tau'}(r, \theta, \varphi, \sigma, \tau) = \sum_{ljm} f_{ljm\tau'}(r) \mathcal{Y}_{(l1/2)jm}(\theta, \varphi, \sigma) \chi_{1/2}^{\tau'}(\tau) , \quad (1.80)$$

where $\chi_{1/2}^{\tau'}(\tau) \equiv \langle 1/2\tau | 1/2\tau'\rangle = \delta_{\tau\tau'}$ is the wave function of the state of isospin τ' .

1.4 Two-nucleon states

1.4.1 Eigenbases of position and linear momentum

The two-nucleon Hilbert space \mathcal{H}_2 is the tensor product $\mathcal{H}_2 \equiv \mathcal{H}_1(1) \otimes \mathcal{H}_1(2)$ of two one-nucleon Hilbert spaces. Starting from the direct-product bases of \mathcal{H}_1 introduced in Secs. 1.3.1 and 1.3.2, one obtains direct-product bases of \mathcal{H}_2

$$|1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2\rangle \equiv |1 : \vec{r}_1 \sigma_1 \tau_1\rangle \otimes |2 : \vec{r}_2 \sigma_2 \tau_2\rangle , \quad (1.81)$$

and similarly for $|1 : \vec{p}_1 \sigma_1 \tau_1 ; 2 : \vec{p}_2 \sigma_2 \tau_2\rangle$.

Completeness relation

The completeness relation on the two-body Hilbert space \mathcal{H}_2

$$\mathbb{1}_2 = \int \int d\vec{r}_1 d\vec{r}_2 \sum_{\sigma_1 \sigma_2} \sum_{\tau_1 \tau_2} |1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2\rangle \langle 1 : \vec{r}_1 \sigma_1 \tau_1 ; 2 : \vec{r}_2 \sigma_2 \tau_2| . \quad (1.82)$$

where $\mathbb{1}_2$ is the unity operator on \mathcal{H}_2 .

Antisymmetrized basis

If one aims at working with fermions, the Hilbert space of physically accessible states one is really dealing with is the subspace of \mathcal{H}_2 of antisymmetric states under the exchange of particles 1 and 2. Bases of such a subspace are obtained by applying the antisymmetrization operator $\mathcal{A}_{12} = \mathbb{I} - P_{12}$, where P_{12} is the two-body exchange operator, onto the direct-product states

$$\begin{aligned} |\vec{r}_1\sigma_1\tau_1; \vec{r}_2\sigma_2\tau_2\rangle &= \sqrt{2}\mathcal{A}_{12}|1: \vec{r}_1\sigma_1\tau_1; 2: \vec{r}_2\sigma_2\tau_2\rangle \\ &= \frac{1}{\sqrt{2}} \left[|1: \vec{r}_1\sigma_1\tau_1; 2: \vec{r}_2\sigma_2\tau_2\rangle - |1: \vec{r}_2\sigma_2\tau_2; 2: \vec{r}_1\sigma_1\tau_1\rangle \right], \end{aligned} \quad (1.83)$$

and similarly for $|\vec{p}_1\sigma_1\tau_1; \vec{p}_2\sigma_2\tau_2\rangle$. A crucial point is that the states defined through Eq. 1.83 are globally antisymmetric under the exchange of space, spin and isospin coordinates but do not have specific properties under the exchange of only spatial, spin or isospin coordinates. It is sometimes preferred to build antisymmetric two-body states out of the tensor product of two-body spatial, spin and isospin states that each has specific properties, i.e. symmetric or antisymmetric, under the exchange of the two particles. Such a building of antisymmetric two-body states is different from what has been done above and is discussed in the remaining of the present notes.

Centre of mass decoupling

Assuming that both nucleons have the same mass, one introduces relative and center of mass coordinates through

$$\vec{R} \equiv \frac{\vec{r}_1 + \vec{r}_2}{2}, \quad \vec{r} \equiv \vec{r}_1 - \vec{r}_2, \quad \vec{P} \equiv \vec{p}_1 + \vec{p}_2, \quad \vec{p} \equiv \frac{\vec{p}_1 - \vec{p}_2}{2}, \quad (1.84)$$

such that, in the center-of-mass frame, the spatial part of the direct-product state can be written as $|1: \vec{r}_1; 2: \vec{r}_2\rangle \equiv |\vec{R}\vec{r}\rangle$. Considering that the nuclear interaction only depends on the relative position vector \vec{r} and rewriting the kinetic energy operator as

$$T \equiv \frac{\vec{p}_1^2}{2m} + \frac{\vec{p}_2^2}{2m} = \frac{\vec{P}^2}{4m} + \frac{\vec{p}^2}{m} = \frac{\vec{P}^2}{4m} + \frac{\vec{p}^2}{2\mu}, \quad (1.85)$$

where $\mu = m/2$ denotes the reduced mass of the two-body system, one recovers the standard result that the center of mass motion decouples and that the relative motion of the two-nucleon system reduces to the motion of a fictitious one-body system of mass μ . Being only interested in the relative motion of the nucleons, we can thus omit the motion associated with the coordinate \vec{R} as is assumed in the following.

1.4.2 Eigenbasis of orbital angular momentum

The total orbital angular-momentum of the two-body system is defined as $\vec{L}_{tot} \equiv \vec{l}_1 + \vec{l}_2$ with $\vec{l}_1 = \vec{r}_1 \times \vec{p}_1$, $\vec{l}_2 = \vec{r}_2 \times \vec{p}_2$. Using Eq. (1.84), one finds that $\vec{L}_{tot} \equiv \vec{L}_{com} + \vec{L}$ where

$$\vec{L}_{com} \equiv \vec{R} \times \vec{P}, \quad \vec{L} \equiv \vec{r} \times \vec{p}. \quad (1.86)$$

\vec{L}_{com} is the center-of-mass orbital angular-momentum whereas \vec{L} is the relative orbital angular-momentum of the two nucleons. In the center-of-mass frame, \vec{L}_{com} is null such that the orbital angular-momentum reduces to \vec{L} . The operator \vec{L} possesses all properties of an angular momentum, e.g. standard commutation relations, and leads to the introduction of the basis $|LM_L\rangle$ that happens to be convenient to describe the relative motion one is interested in. The corresponding wave functions are

$$Y_L^{ML}(\theta, \varphi) \equiv \langle \theta\varphi | LM_L \rangle, \quad (1.87)$$

where angles θ and φ provide the orientation of the relative position vector \vec{r} . When the two nucleons are exchanged, $\vec{r} = \vec{r}_1 - \vec{r}_2$ is changed into $-\vec{r}$, which is equivalent to changing θ and φ into $\pi - \theta$ and $\varphi + \pi$. As $Y_L^{M_L}(\pi - \theta, \varphi + \pi) = (-1)^L Y_L^{M_L}(\theta, \varphi)$, the exchange of the two particles introduces a phase equal to $(-1)^L$. Consequently, one sees that a state $|LM_L\rangle$ with L even is symmetric under the exchange of the two particles whereas a state with an odd L is antisymmetric under such an exchange. Such a property makes the basis $|LM_L\rangle$ of $\mathcal{H}_{2,\vec{r}}$ very suited to the construction of fully antisymmetrized states down the road.

1.4.3 Eigenbasis of spin

The two-nucleon spin operator is $\vec{S} = \vec{s}_1 + \vec{s}_2$. Eigenstates $|SM_S\rangle$ of \vec{S}^2 and S_z are expressed in terms of the direct-product states $|1 : \sigma_1 ; 2 : \sigma_2\rangle$ as

$$|0 \ 0\rangle = \frac{|1 : \uparrow \ 2 : \downarrow\rangle - |1 : \downarrow \ 2 : \uparrow\rangle}{\sqrt{2}} , \quad (1.88)$$

$$|1 \ 1\rangle = |1 : \uparrow \ 2 : \uparrow\rangle , \quad (1.89)$$

$$|1 \ 0\rangle = \frac{|1 : \uparrow \ 2 : \downarrow\rangle + |1 : \downarrow \ 2 : \uparrow\rangle}{\sqrt{2}} , \quad (1.90)$$

$$|1 \ -1\rangle = |1 : \downarrow \ 2 : \downarrow\rangle , \quad (1.91)$$

whose eigenvalues $\hbar^2 S(S+1)$ and M_S of \vec{S}^2 and of S_z , respectively, take the values $S = 0$ or 1 and $|M_S| \leq S$. As is customary, "spin-up" and "spin down" arrows have been used to denote $\sigma = +1/2$ and $\sigma = -1/2$, respectively. The $S = 0$, or spin-singlet, state is antisymmetric under the exchange of particles 1 and 2 while $S = 1$, or spin-triplet, states are symmetric. One can define the spin-exchange operator P_σ through

$$P_\sigma = \vec{S}^2 - 1 \equiv \frac{1 + \vec{s}_1 \cdot \vec{s}_2}{2} , \quad (1.92)$$

such that $P_\sigma|1 : \sigma_1 ; 2 : \sigma_2\rangle = |1 : \sigma_2 ; 2 : \sigma_1\rangle$ and $P_\sigma|SM_S\rangle = (-1)^{1-S}|SM_S\rangle$, i.e. $P_\sigma|00\rangle = -|00\rangle$ and $P_\sigma|1M_S\rangle = +|1M_S\rangle$.

1.4.4 Eigenbasis of isospin

The two-nucleon isospin operator is $\vec{T} = \vec{\tau}_1 + \vec{\tau}_2$. Noting $\hbar^2 T(T+1)$ and M_T the eigenvalues of \vec{T}^2 and of T_z , respectively, one has $T = 0$ or 1 and $|M_T| \leq T$. Eigenstates $|TM_T\rangle$ of \vec{T}^2 and T_z are expressed in terms of the direct-product states $|1 : \tau_1 ; 2 : \tau_2\rangle$ as

$$|0 \ 0\rangle = \frac{|1 : n \ 2 : p\rangle - |1 : p \ 2 : n\rangle}{\sqrt{2}} , \quad (1.93)$$

$$|1 \ 1\rangle = |1 : n \ 2 : n\rangle , \quad (1.94)$$

$$|1 \ 0\rangle = \frac{|1 : n \ 2 : p\rangle + |1 : p \ 2 : n\rangle}{\sqrt{2}} , \quad (1.95)$$

$$|1 \ -1\rangle = |1 : p \ 2 : p\rangle , \quad (1.96)$$

where n and p have been used to denote τ equal to $+1/2$ and $-1/2$, respectively. The $T = 0$, or isospin-singlet, state is antisymmetric under the exchange of particles 1 and 2, while the $T = 1$, or isospin-triplet, states are symmetric. One can define the isospin-exchange operator P_τ through

$$P_\tau = \vec{T}^2 - 1 \equiv \frac{1 + \vec{\tau}_1 \cdot \vec{\tau}_2}{2} , \quad (1.97)$$

such that $P_\tau|1 : \tau_1 ; 2 : \tau_2\rangle = |1 : \tau_2 ; 2 : \tau_1\rangle$ and $P_\tau|TM_T\rangle = (-1)^{1-T}|T \ M_T\rangle$, i.e. $P_\tau|00\rangle = -|00\rangle$ and $P_\tau|1M_T\rangle = +|1M_T\rangle$.

1.4.5 Eigenbasis of total angular momentum

In the center-of-mass frame, one can couple the relative orbital angular-momentum \vec{L} and the total spin \vec{S} to form the total angular-momentum $\vec{J} = \vec{L} + \vec{S}$. Doing so, one introduces the basis of $\mathcal{H}_{2,\vec{r}} \otimes \mathcal{H}_{2,\vec{s}}$

$$|(LS)JM\rangle = \sum_{M_L M_S} \langle L S M_L M_S | JM \rangle |L M_L\rangle |S M_S\rangle , \quad (1.98)$$

whose wave function is a spin-angular spherical harmonics $\mathcal{Y}_{(L S)JM}(\theta, \varphi, \sigma)$ and which are eigenstates of \vec{J}^2 and J_z with eigenvalues $\hbar^2 J(J+1)$ and $\hbar M$, respectively. Thanks to the angular momentum coupling rules, one has $|L - S| \leq J \leq L + S$ and $M = M_L + M_S$. Ordering the two-nucleon states according to J , one has $|J - S| \leq L \leq J + S$, such that

$$\left\{ \begin{array}{l} \text{for } J = 0 \quad L = S \\ \text{for } J > 0 \quad L = J - S, J, J + S \end{array} \right\} \left\{ \begin{array}{l} \text{if } S = 0 : \quad L = 0 \\ \text{if } S = 1 : \quad L = 1 \\ \text{if } S = 0 : \quad L = J \\ \text{if } S = 1 : \quad L = J - 1, J, J + 1 \end{array} \right. . \quad (1.99)$$

Consequently, $J=0$ gives rise to a spin-singlet state $S=L=0$ and to a spin-triplet state $S=L=1$, whereas for $J>0$ there exists a spin-singlet state $S=0, L=J$ and three spin-triplet states $S=1, L=J-1, J, J+1$.

Spectroscopic notation

To characterise such a spatial-spin content of the two-body state, one usually uses the spectroscopic notation $^{(2S+1)}[L]_J$ where $[L]$ denotes one of the letters S, P, D, F, \dots corresponding to $L = 0, L = 1, L = 2, L = 3, \dots$ partial waves, respectively. The two-nucleon states obtained for the first few J values are thus denoted as

$$\begin{aligned} J = 0 : & \quad {}^1S_0, {}^3P_0 \\ J = 1 : & \quad {}^1P_1, {}^3S_1, {}^3P_1, {}^3D_1 \\ J = 2 : & \quad {}^1D_2, {}^3P_2, {}^3D_2, {}^3F_2 \\ J = 3 : & \quad {}^1F_3, {}^3D_3, {}^3F_3, {}^3G_3 \end{aligned} . \quad (1.100)$$

1.4.6 Spin-isospin channels

Finally, one obtains a complete basis of \mathcal{H}_2 through the tensor product of the basis $|(LS)JM\rangle$ of $\mathcal{H}_{2,\vec{r}} \otimes \mathcal{H}_{2,\vec{s}}$ and of the basis $|TM_T\rangle$ of $\mathcal{H}_{2,\vec{\tau}}$. As already stated, two-fermion states must be antisymmetric under the exchange of the two fermions. Summarizing the above discussion, we see that such an exchange brings a phase $(-1)^{S-1}$ for the spin part, $(-1)^{T-1}$ for the isospin part and $(-1)^L$ for the spatial part. To fulfill the overall antisymmetry, one must thus have $(-1)^{L+S-1+T-1} = -1$, i.e. $L + S + T$ must be odd. In the listing of Eq. (1.100), the two $J = 0$ states have an even value of $L + S$, and thus have to be isospin-triplet ($T = 1$) states. For $J = 1$ states, the first, second and fourth states have $L + S$ odd, and thus can only be isospin-singlet ($T = 0$) states. The third state has $L + S$ even, and thus must be an isospin-triplet $T = 1$ state. And so on and so forth.

It is customary to start classifying complete two-nucleon states according to S and T quantum numbers. This gives birth to the four so-called spin-isospin channels

$$(S = 0, T = 0) \quad , \quad (S = 0, T = 1) \quad , \quad (S = 1, T = 0) \quad , \quad (S = 1, T = 1) \quad , \quad (1.101)$$

which are spin-singlet/isospin-singlet, spin-singlet/isospin-triplet, spin-triplet/isospin-singlet and spin-triplet/isospin-triplet channels, respectively. Channels $(S = 0, T = 0)$ and $(S = 1, T = 1)$ must correspond to an odd L and are denoted as singlet-odd and triplet-odd channels, respectively. Similarly, channels $(S = 0, T = 1)$ and $(S = 1, T = 0)$ must

correspond to an even L and are denoted as singlet-even and triplet-even, taking the spin as a reference. Using such a convention, two-nucleon states (Eq. (1.100)) are distributed within the four channels according to

S	T	channel	neutron/proton states	$J = 0$	$J = 1$	$J = 2$	$J = 3$	$J = 4$
0	1	singlet even	nn, pp, (np+pn)/ $\sqrt{2}$	1S_0		1D_2		1G_4
1	0	triplet even	(np-pn)/ $\sqrt{2}$		3S_1 , 3D_1	3D_2	3D_3 , 3G_3	3G_4
0	0	singlet odd	(np-pn)/ $\sqrt{2}$		1P_1		1F_3	
1	1	triplet odd	nn, pp, (np+pn)/ $\sqrt{2}$	3P_0	3P_1	3P_2 , 3F_2	3F_3	3F_4 , 3H_4

Parts of the array are empty, e.g. neither the singlet-even state with an odd J nor the singlet-odd state with an even J are allowed.

The (S, T) channel classification is useful because the nuclear interaction does not only commute with \vec{S}^2 but also commutes, to a good approximation, with \vec{T}^2 . The latter represents the charge independence of the nuclear force. Consequently, the nuclear interaction cannot mix two-nucleon states that belong to different channels. However, the nuclear interaction can be different in each channel such that there are four decoupled parts of the nuclear interaction v_{12}^{ST} ; one for each channel. Experimental nucleon-nucleon scattering experiments have confirmed such a feature.

It is often said that the charge independence of nuclear forces implies that the nuclear interaction between two neutrons is the same as between two protons or between a neutron and a proton. Such a statement is wrong. In fact, the interaction between two neutrons is the same than the interaction between two protons or between one neutron and one proton only if they occupy the same orbital and spin states, that is to say if they are in the same channel. It is what is shown in the array. The identity between the proton-proton and neutron-neutron forces in the $T = 1$ channel denotes the charge symmetry of the nuclear force. However, the singlet-even channel interaction is not equal to the triplet-odd channel interaction.

The interaction in even channels is attractive, while it is repulsive in odd channels. Moreover, the interaction in the triplet even channel is more attractive than in the singlet even channel. It is the proton-neutron interaction in the triplet even channel that binds the deuteron, whereas there is no bound di-neutron or di-protons. In heavier nuclei, even channels play an important role, e.g. the proton-neutron triplet even interaction plays an essential role in binding nuclei while proton-proton and neutron-neutron singlet-even interactions are responsible of pairing correlations and superfluid properties of nuclei.

Finally, let us note that, in the center-of-mass frame, the general form of the two-nucleon wave-function is similar to the one-body wave function Eq. (1.80) with mass $\mu = m/2$

$$\psi(r, \theta, \varphi, M_S, M_{T'}) = \sum_{LSJM} \sum_{TM_T} f_{LSTM_T}^{JM}(r) \mathcal{Y}_{(LS)JM}(\theta, \varphi, M_S) \chi_T^{M_T}(M_{T'}) , \quad (1.102)$$

where $\chi_T^{M_T}(M_{T'}) \equiv \langle TM_{T'} | TM_T \rangle = \delta_{M_{T'}, M_T}$. Thanks to nuclear interaction symmetries, the wave function has to be a eigenstate of \vec{J}^2 , J_z , \vec{S}^2 , \vec{T}^2 , T_z and of the parity Π . In this case, Eq. (1.102) becomes

$$\psi_{STM_T}^{JM\Pi}(r, \theta, \varphi, M_S, M_T) = \sum_{L: (-)^L = \Pi} f_{LSTM_T}^J(r) \mathcal{Y}_{(LS)JM}(\theta, \varphi, M_S) \chi_T^{M_T}(M_T) . \quad (1.103)$$

Bibliography

General

- K. S. Krane, *Introductory Nuclear Physics*, Wiley & Sons, 1988
- S. G. Nilsson, I. Ragnarsson, *Shapes and Shells in Nuclear Structure*, Cambridge University Press, 1995
- M. A. Preston, R. K. Bhaduri, *Structure of the Nucleus*, Addison-Wesley, 1975

Many-body problem

- P. Ring, P. Schuck, *The nuclear many-body problem*, Springer-Verlag, 1980
- J.-P. Blaizot, G. Ripka, *Quantum theory of finite systems*, MIT Press, 1986
- P. Nozières, *Theory of interacting Fermi systems*, Westview Press, 1964
- I. Shavitt, R. J. Bartlett, *Many-body methods in chemistry and physics*, Cambridge University Press, 2009
- A. L. Fetter, J. D. Walecka, *Quantum Theory of Many-particle Systems*, McGraw-Hill, 1971

