3. NUCLEAR INTERACTIONS

Generalities

- Relevant degrees of freedom : structureless nucleons
- Non-relativistic framework (+ relativistic corrections)

Nucleon dynamics in nuclei is to a large extent non-relativistic

$$\frac{|\vec{p}|}{m} \approx \frac{200 \text{ MeV}}{939 \text{ MeV}} \Rightarrow \left(\frac{v}{c}\right)^2 < 0{,}1$$

- Interaction is istantaneous, potential is time independent
- Strong + weak + EM accounted for
- Here we consider the ab initio Hamiltonian (as opposed to effective)

AB INITIO	EFFECTIVE
in vacuum	in medium
nucleus independent	nucleus dependent
links nuclei to QCD	focuses on phenomenological description
$H\,\lvert\Psi^A\rangle = E^A\,\lvert\Psi^A\rangle$	$H^A_{eff}\,\lvert\Psi^A_{eff}\rangle = E^A\,\lvert\Psi^A_{eff}\rangle$

- General form

$$H = \sum_{i=1}^{A} \frac{p_i^2}{2M} + \frac{1}{2}\sum_{i\neq j=1}^{A} V(i,j) + \frac{1}{3!}\sum_{i\neq j\neq k=1}^{A} W(i,j,k) + \dots$$

which is the form of these operators?

how many do we need to include?

Symmetries and operator structure

- Most general form of two-nucleon interaction

$$V(1,2) = V(\vec{r}_1, \vec{p}_1, \vec{s}_1, \vec{t}_1, \vec{r}_2, \vec{p}_2, \vec{s}_2, \vec{t}_2)$$

- Nuclear interactions are invariant under
  - exchange of the two nucleons (permutation)
  - translations in space
  - translations in time
  - rotations
  - Galilean boost
  - parity
  - time reversal
  - ~ isospin

potential invariant under rotations in isospin space

$$V(p,p) = V(n,n) = V(n,p)$$

from spectra of "mirror" nuclei $^{A}_{B}X_{C}$ and $^{A}_{C}X_{B}$ (in the notation $^{A}_{Z}X_{N}$)

from nucleon-nucleon scattering cross sections

+ add isospin-symmetry-breaking corrections

$$V^{ISB}(p,p) \neq V^{ISB}(n,n) \neq V^{ISB}(n,p)$$

differences in masses, cross sections, scattering lenght, ...

The most general operator structure consistent with all symmetry constraints can be then derived.

Two examples

1) isospin operators

Potential must be a scalar with respect to isospin rotations

$\Rightarrow$ can depend only on scalar product

$$\Rightarrow V(\ldots, \vec{t}_1, \vec{t}_2) = F(\ldots, \vec{t}_1 \cdot \vec{t}_2)$$

$$= F_0(\ldots) + F_1(\ldots)\, \vec{t}_1 \cdot \vec{t}_2 + F_2(\ldots)\left(\vec{t}_1 \cdot \vec{t}_2\right)^2 + \ldots$$

One can prove that

$$\left(\vec{t}_1 \cdot \vec{t}_2\right)^2 |T M_T\rangle = \left(3 - 2\vec{t}_1 \cdot \vec{t}_2\right) |T M_T\rangle$$

$\rightarrow$ exercise

$$\Rightarrow F_0(\ldots) + F_1(\ldots)\, \vec{t}_1 \cdot \vec{t}_2$$

ii) linear dependence on spin operators

Since the potential is a scalar under spatial rotations, terms linear in $\vec{s}$ must appear under the form of a scalar product

$$V(\ldots \vec{s}_1, \vec{s}_2) = F(\ldots, \underbrace{\vec{a} \cdot \vec{s}_1 + \vec{b} \cdot \vec{s}_2}_{})$$

invariance under permutations $1 \leftrightarrow 2$ implies $\vec{a} = \vec{b}$

$$\Rightarrow \vec{a} \cdot (\vec{s}_1 + \vec{s}_2) = \vec{a} \cdot \vec{S}$$

possibilities : $\underbrace{\vec{r}, \vec{p}}$, $\vec{L}$, $\vec{S}$ — would be trivial (a constant)

wouldn't be invariant under parity

only nontrivial possibility

At the end, the most general operator structure reads

$$V(1,2) = V^0 + V^s \vec{s}_1 \cdot \vec{s}_2 + V^t \vec{t}_1 \cdot \vec{t}_2 + V^{st} (\vec{s}_1 \cdot \vec{s}_2)(\vec{t}_1 \cdot \vec{t}_2)$$

with

$$V^i = \sum_{k=1}^{5} f_k^i (\vec{r}^{\,2}, \vec{p}^{\,2}, \vec{L}^{\,2}) O_k \qquad (i = 0, s, t, st)$$

and

$$O_k = \begin{cases} \mathbb{1} & \\ \vec{L} \cdot \vec{S} & \text{spin-orbit} \\ S_{12}^r \equiv 3(\vec{s}_1 \cdot \bar{r})(\vec{s}_2 \cdot \bar{r}) - \vec{s}_1 \cdot \vec{s}_2 & \text{tensor} \\ S_{12}^p \equiv 3(\vec{s}_1 \cdot \bar{p})(\vec{s}_2 \cdot \bar{p}) - \vec{s}_1 \cdot \vec{s}_2 & \text{tensor} \\ Q_{12} \equiv \frac{1}{2}\left[(\vec{s}_1 \cdot \vec{L})(\vec{s}_2 \cdot \vec{L}) + (\vec{s}_2 \cdot \vec{L})(\vec{s}_1 \cdot \vec{L})\right] & \text{quadratic spin-orbit} \end{cases}$$

where $\bar{x} \equiv \frac{\vec{x}}{|\vec{x}|}$

- Tensors are defined with an additional $\vec{s}_1 \cdot \vec{s}_2$ such that the average over the angles of $\vec{r}$ and $\vec{p}$ gives zero
- Q_{12} is called quadratic spin-orbit because it can be reexpressed as a function of $(\vec{L} \cdot \vec{S})^2$, $\vec{L} \cdot \vec{S}$ and $\vec{L}^{\,2}$

Experimental constraints: NN scattering

- Different types of nucleon-nucleon (NN) scattering can be studied
  - np scattering is the easiest: neutron beams on hydrogen targets

- pp scattering: technically easy, but EM interaction needs to be substracted (beyond Coulomb, radiative corrections, ...)

- nn scattering: only indirect information (no n targets)
  - nd scattering (then substract np component)
  - two neutrons in final state, e.g. $n + d \to n + n + p$
  - comparison between different reactions

$$^3He + {}^3He \longrightarrow \alpha + p + p$$
$$^3He + {}^3H \longrightarrow \alpha + p + n$$
$$^3H + {}^3H \longrightarrow \alpha + n + n$$

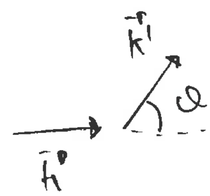
For energies beyond the pion production threshold ($\sim 2\, m_\pi$ in lab frame) an inelastic component appears in the potential.

Cross section and partial-wave expansion

Relevant quantities

usually independent of φ

- differential cross section $\dfrac{d\sigma}{d\Omega}(E, \Theta)$
- total cross section $\sigma(E)$



$\to$ elastic scattering $\Rightarrow |\vec{k}| = |\vec{k}'|$ and $E = \dfrac{k^2}{2\mu}$ with $\mu = \dfrac{M}{2}$ reduced mass

Wave function at $r \to \infty$

$$\Psi_k(\vec{r}) \xrightarrow[r \to \infty]{} \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming}} + \underbrace{f(k, \theta)\, \frac{e^{ikr}}{r}}_{\text{outgoing}}$$

$f(k,\theta)$ is the scattering amplitude $\rightarrow$ $\boxed{\dfrac{d\sigma(k,\theta)}{d\Omega} = |f(k,\theta)|^2}$ (6)

By expanding $\Psi_{\vec{k}}(\vec{r})$ in partial waves one obtains

$$\Psi_k(\vec{r}) \xrightarrow[r\to\infty]{} \sum_{l=0}^{\infty} (2l+1)\, P_l(\cos\theta)\, \frac{\overbrace{(-1)^{l+1} e^{-ikr}}^{\text{incoming}} + \overbrace{S_l(k)\, e^{ikr}}^{\text{outgoing}}}{2ikr}$$

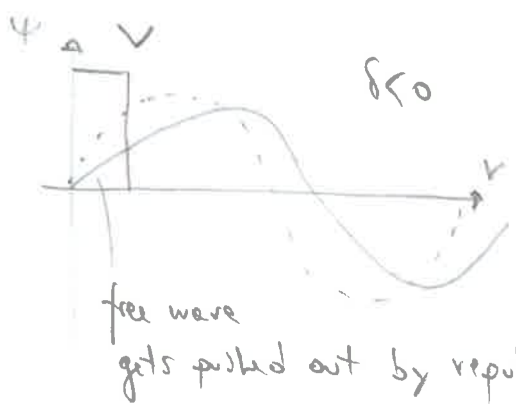
$S_l(k) = 1 + 2ik\, f_l(k)$ is the scattering matrix, usually parametrised as

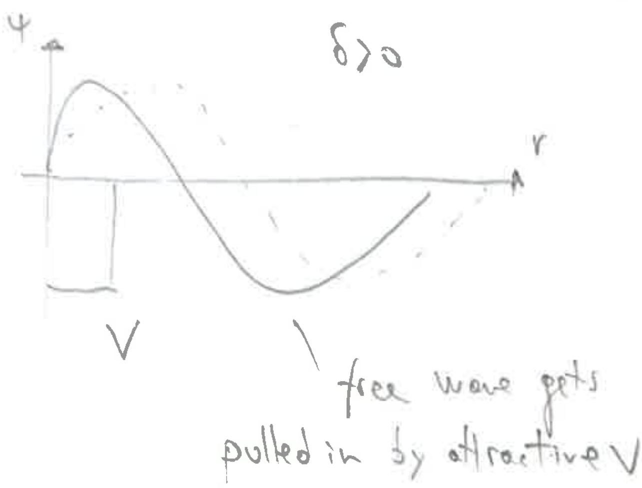
$$S_l(k) = e^{2i\delta_l(k)} \qquad \text{(S-matrix is unitary)}$$

Then

$$\boxed{f_l(k) = \frac{S_l(k)-1}{2ik} = \frac{e^{i\delta_l(k)}}{k} \sin\delta_l(k) = \frac{1}{k\cot\delta_l(k) - ik}}$$

Outside the range of the interaction $(r\to\infty)$ the potential leads to a change in the phase of the outgoing wave





$$\Rightarrow \sigma_{tot}(E) = \int d\Omega\, \frac{d\sigma(k,\theta)}{d\Omega} = \int d\Omega\, |f(k,\theta)|^2 = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1)\, \sin^2\delta_l(k)$$

Low-energy Scattering

- Schwinger showed that $k^{2l+1} \cot \delta_l(k)$ has an analytic expansion

$$l=0 \quad \rightarrow \quad k \cot \delta_0(k) = -\frac{1}{a} + \frac{1}{2} r_e k^2 - P^2 r_e^3 k^4 + \mathcal{O}(k^6)$$

$$l=2 \quad \rightarrow \quad k^3 \cot \delta_1(k) = -\frac{3}{a_p^3} + \mathcal{O}(k^2)$$

Effective range expansion

In the limit $k \to 0$ only $l=0$ scattering is present and

$$f_0(k) \to -a \quad \Rightarrow \quad \frac{d\sigma}{d\Omega} = a^2 \quad \Rightarrow \quad \sigma_{tot} = 4\pi a^2$$

in this limit a is the radius of an absorption disc equivalent to $\sigma_{tot}(0)$

→ effective range represents first correction ~ range of the interaction

- Scattering length → useful to expand $\cot \delta_0(k)$ ⇒ get $\delta_0(k) = -ka$

$a > 0$ bound state (see inset)
- a small → strong attraction
- a large → weak attraction

$a < 0$ no bound state
- a small → repulsion
- a large → weak repulsion / weak attraction

$$f_0(k) \xrightarrow[k \to 0]{} \frac{1}{-\frac{1}{a_0} - ik}$$

pole at $k = \frac{i}{a}$

$$\Rightarrow e^{ikr} \to e^{-\frac{r}{a}}$$

• Case of nucleon-nucleon scattering

→ At low energies, only $L=0$ ⇒ two possible channels:
spin singlet $^1S_0$ and spin triplet $^3S_1$
(s) (t)

$a_{s\,(np)} \simeq -23.7$ fm $\qquad r_{s\,(np)} \simeq 2.73$ fm

$a_{t\,(np)} \simeq +5.4$ fm $\qquad r_{t\,(np)} \simeq 1.75$ fm

$a_{s\,(pp)} \simeq -17.1$ fm $\qquad r_{s\,(pp)} \simeq 2.84$ fm

(after correcting for Coulomb)

$a_{s\,(nn)} \simeq -16.6$ fm $\qquad r_{s\,(nn)} \simeq 2.66$ fm

→ Charge symmetry realised to a very good extent

→ Charge independence more approximate

→ Bound state in $^3S_1$ channel (n-p) ⇒ deuteron

→ Scattering length large also in $^1S_0$ channel
⇒ almost-bound state (resonance) of two neutrons
(two protons excluded because of Coulomb) di-neutron

Phase shifts and high-energy scattering

Phase shifts are used to parametrise scattering data based on partial wave analysis

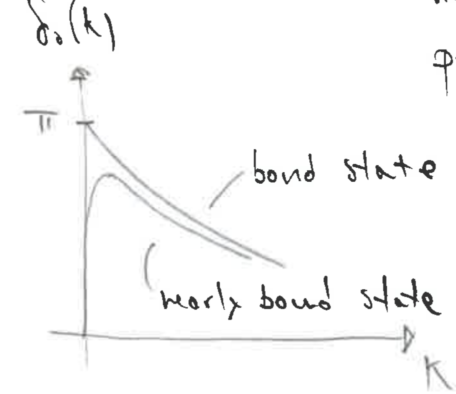
→ see e.g. Nijmegen partial wave analysis nn-online.org/NN

Discussion of $T=1$ and $T=0$ phase shifts (see figures)

- Presence of a bound state in $^3S_1$ channel (consistent with scattering length) and a nearly-bound state in $^1S_0$

  Levinson's theorem $\delta_l(k=0) = n_l \pi$

  n_l: number of bound states in partial wave l



- S-waves become repulsive at high momentum → short distances

  ⇒ short-range repulsion of NN interaction

- Consider ^3P waves

  i) Central component ~ average of $^3P_0$, $^3P_1$ and $^3P_2$ is small at small energies

  ⇒ there must be something else ⇒ spin-orbit

  ii) Consider spin-orbit operator

$$\vec{L}\cdot\vec{S} = \frac{1}{2}\left(\vec{J}^{\,2} - \vec{L}^{\,2} - \vec{S}^{\,2}\right)$$

$$= \frac{1}{2}\left[J(J+1) - L(L+1) - S(S+1)\right]$$

$$\underset{\text{for } ^3P}{=} \frac{1}{2} J(J+1) - 2$$

$$= \begin{cases} -2 & \text{for } ^3P_0 \Rightarrow \text{repulsive} \quad ?? \Rightarrow \text{there must be something else} \\ -1 & \text{for } ^3P_1 \Rightarrow \text{repulsive} \longleftrightarrow \delta < 0 \\ +1 & \text{for } ^3P_2 \Rightarrow \text{attractive} \longleftrightarrow \delta > 0 \end{cases}$$

$V_{LS}\,\vec{L}\cdot\vec{S}$ where $V_{LS} < 0$ → tensor term

Experimental constraints: the deuteron

The deuteron is the only bound system of two nucleons.

Main properties

- Binding energy $E = 2.2245$ MeV
- Total angular momentum $J = 1$
- Magnetic dipole moment $\mu = 0.8574\,\mu_N$ with $\mu_N = \frac{e}{2m_p}$
- Electric quadrupole moment $Q = 0.2859\ e\,fm^2$
- R.m.s. radius $r = 1.97$ fm

Proton-neutron system can exist in $T=0$ or $T=1$. Given $J=1$, there are four possibilities $^3S_1$, $^1P_1$, $^3P_1$, $^3D_1$

$$\quad + \qquad - \qquad - \qquad + \quad \longleftarrow \text{parity} = (-1)^L$$

Since parity has to be conserved, it can be at most a combination of the two states of the same parity.

Quadropole moment $=0$ for a spherical nucleus

$\Rightarrow$ cannot be pure $L=0$ state

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If it were a superposition of the P states, it would mix $T=0$ and $T=1$. Not impossible, but not very probable given that we expect isospin symmetry (specially since there is no Coulomb).

Further hints come from the magnetic moment.

$$\mu = \langle \Psi_{J\,M=J} | \mathcal{M}_z | \Psi_{J\,M=J} \rangle$$

for a nucleus in the state $|\Psi_{JM}\rangle$.

$\mathcal{M}_z$ is the third component of the magnetic moment operator

$$\vec{\mathcal{M}} = \mu_N \left[\sum_{i=1}^{Z} \left(\vec{l}_i + g_p \vec{s}_i \right) + \sum_{i=1}^{N} g_n \vec{s}_i \right]$$

proton orbital angular momenta

nucleon spin g-factors

$$g_p = 5.585 \qquad g_n = -3.826$$

For the deuteron one has

$$\mu = \mu_N \langle \Psi_{11} | \left(\vec{l}_p + g_p \vec{s}_p + g_n \vec{s}_n \right) | \Psi_{11} \rangle$$

and one finds

- for a superposition $|\Psi_{11}\rangle = \cos\omega_0 |{}^3S_1\rangle + \sin\omega_0 |{}^3D_1\rangle$

$$\mu = \mu_N \left(0.880 - 0.570 \sin^2\omega_0 \right)$$

- for a superposition $|\Psi_{11}\rangle = \cos\omega_1 |{}^1P_1\rangle + \sin\omega_1 |{}^3P_1\rangle$

$$\mu = \mu_N \left(0.500 + 0.190 \sin^2\omega_1 \right)$$

$\Rightarrow$ it has to be ${}^3S_1 + {}^3D_1$ (cf. experimental value $\mu = 0.8574\,\mu_N$

The only term in the nuclear interaction mixing different L components is the tensor

only operator that does not commute with $\vec{L}^2$