From nuclei to stars
Theoretical course

NPAC 2019-2020

Mid-term exam 15/11/2019

1. Nuclei are usually modelled in terms of protons and neutrons. Which other degrees of freedom could be instead used to describe atomic nuclei, and what would be their limitations?

2. What is a hypernucleus? Why is it interesting to study hypernuclei?

3. Give an example of

   a. A basis of the one-nucleon Hilbert space $\mathcal{H}_1$;

   b. A basis of the two-nucleon Hilbert space $\mathcal{H}_2$;

   c. A basis of the two-nucleon Hilbert space $\mathcal{H}_2$ that has well-defined (anti)symmetry properties in coordinate, spin and isospin Hilbert subspaces separately.

4. Given $P_\sigma \equiv \vec{S}^2 - 1$, where $\vec{S} \equiv \frac{1}{2}\left[\vec{\sigma}_1 + \vec{\sigma}_2\right]$ is the total-spin operator for a pair of nucleons

   a. Prove that $P_\sigma = \dfrac{1 + \vec{\sigma}_1\vec{\sigma}_2}{2}$;

   b. Determine the eigenvalues of P_σ ;

   c. Show that P_σ is a spin-exchange operator.

5. On which experimental data the nucleon-nucleon potential is typically adjusted? Which data can be used to fit three-nucleon interactions?

6. Explain how we can conclude that in the nucleon-nucleon interaction

   a. There is a non-negligible spin-orbit component ;

   b. There is a non-negligible tensor component.

7. Prove the anticommutation relations between fermionic creation and annihilation operators a^+_μ and a_ν.

   Hint: Evaluate the action of the anticommutators on a generic Slater determinant $|\alpha\beta\ldots\rangle$.

MID-TERM EXAM - SOLUTIONS

1) One could directly use quarks (and gluons) as basic degrees of freedom. The difficulty resides in the non-perturbative character of QCD at low energy, which makes its solution in the energy regime relevant to atomic nuclei extremely challenging.

2) A hypernucleus is a nucleus in which one of the nucleons is substituted with a strange baryon. They are of interest because they could provide additional (complementary) information on nucleon-nucleon (more in general baryon-baryon) interactions and because of the possible presence of hyper-nuclear matter in neutron stars.

3a) $|\vec{r}\,\vec{\sigma}\,\tau\rangle$

3b) $|\vec{r}_1\,\vec{\sigma}_1\,\vec{\tau}_1\,;\,\vec{r}_2\,\vec{\sigma}_2\,\vec{\tau}_2\rangle$

3c) $|JMLS\rangle \otimes |T\,M_T\rangle$

4a) $$P_\sigma = \vec{S}^2 - 1 = \frac{1}{4}\Big(\underbrace{\vec{\sigma}_1^{\,2}}_{=3} + \underbrace{\vec{\sigma}_2^{\,2}}_{=3} + 2\,\vec{\sigma}_1\cdot\vec{\sigma}_2\Big) - 1$$

$$= \frac{1}{2} + \frac{1}{2}\,\vec{\sigma}_1\cdot\vec{\sigma}_2$$

4b) $P_\sigma = \vec{S}^2 - 1$ has eigenvalues $S(S+1) - 1 = \begin{cases} -1 & \text{if } S=0 \\ +1 & \text{if } S=1 \end{cases}$

$\Rightarrow P_\sigma^2\,|S\,M_S\rangle = |S\,M_S\rangle$

4c) Inverting

$$\begin{cases} |S=0 \ M_S=0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right) \\ |1\,1\rangle = |\uparrow\uparrow\rangle \\ |1\,0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\uparrow\downarrow\rangle\right) \\ |1\,-1\rangle = |\downarrow\downarrow\rangle \end{cases}$$

one has

$$\begin{cases} |\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}\left(|1\,0\rangle + |0\,0\rangle\right) \\ |\uparrow\uparrow\rangle = |1\,1\rangle \\ |\downarrow\downarrow\rangle = |1\,-1\rangle \\ |\downarrow\uparrow\rangle = \frac{1}{\sqrt{2}}\left(|1\,0\rangle - |0\,0\rangle\right) \end{cases}$$

Using $P_\sigma|S=0 \ M_S=0\rangle = -|00\rangle$ and $P_\sigma|S=1 \ M_S\rangle = |1 \ M_S\rangle$

one finds

$$P_\sigma|\uparrow\downarrow\rangle = \frac{1}{\sqrt{2}}\left(P_\sigma|1\,0\rangle + P_\sigma|0\,0\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|1\,0\rangle - |0\,0\rangle\right) = |\downarrow\uparrow\rangle$$

etc...

5) NN potential is typically adjusted on NN scattering data and properties of the deuteron.

NNN potential can be adjusted on properties of $A=3,4$ nuclei e.g. binding energy and charge radius of ^3H, ^3He, ^4He or n-d scattering data

6a) A non-negligible spin-orbit component can be deduced e.g. by looking at NN scattering phase shifts in the $L=1$ partial waves (P-waves).

6b) The presence of a strong tensor force can be inferred by comparing different P-wave phase shifts or by computing the magnetic moment of the deuteron

7) Consider the action of $\{a_\mu, a_\nu^\dagger\}$ on Slater determinant $|\alpha\beta\ldots\rangle$

Four cases

a) μ and ν are occupied in $|\alpha\beta\ldots\rangle$

b) μ and ν are unoccupied in $|\alpha\beta\ldots\rangle$

c) μ occupied and ν unoccupied in $|\alpha\beta\ldots\rangle$

d) μ unoccupied and ν occupied in $|\alpha\beta\ldots\rangle$

Now

a) $\{a_\mu, a_\nu^\dagger\}|\alpha\beta\ldots\rangle = (-1)^n a_\nu^\dagger |\alpha\beta\ldots\rangle$

$$= \begin{cases} 0 & \text{if } \nu \neq \mu \quad (\nu \text{ is already occupied}) \\ (-1)^{2n}|\alpha\beta\ldots\rangle & \text{if } \nu = \mu \end{cases}$$

$$= \delta_{\mu\nu}|\alpha\beta\ldots\rangle$$

b) $\{a_\mu, a_\nu^\dagger\}|\alpha\beta\ldots\rangle = a_\mu |\nu\alpha\beta\ldots\rangle$

$$= \begin{cases} 0 & \text{if } \nu \neq \mu \\ |\alpha\beta\ldots\rangle & \text{if } \nu = \mu \end{cases}$$

$= \delta_{\mu\nu} |\alpha\beta \ldots \rangle$

c) $\{a_\mu, a_\nu^+\} |\alpha\beta \ldots \rangle = a_\mu |\nu\alpha\beta \ldots \rangle + (-1)^n a_\nu^+ |\alpha\beta \ldots \rangle$

$= (-1)^{n+1} |\nu\alpha\beta \ldots \rangle + (-1) |\nu\alpha\beta \ldots \rangle$

$= 0$

d) $\{a_\mu, a_\nu^+\} |\alpha\beta \ldots \rangle = \underbrace{a_\mu a_\nu^+ |\alpha\beta \ldots \rangle}_{=0} + \underbrace{a_\nu^+ a_\mu |\alpha\beta \ldots \rangle}_{=0}$

$= 0$