Beyond Standard Model physics: Elements of statistical analysis

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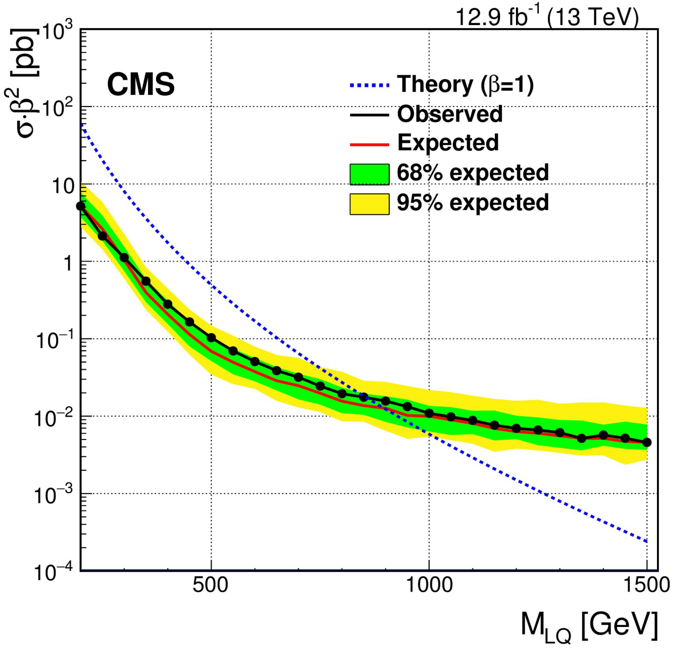
Introduction

or: why spending 3 hours to do maths ?

- Use of statistics in HEP is a very broad topic
  - There are complete courses on the topic
  - See bibliography / references
- Goals of today's lecture:
  - Teach / remind some basic notions
  - Focus on aspects used nowadays in the majority of BSM searches at the LHC
  - Understand the main plots often shown in searches or measurements

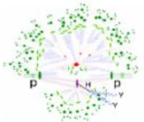


Outline

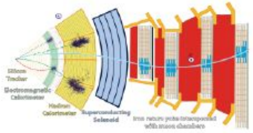
- Probabilities
- Parameter estimation
- Building a likelihood
- Hypothesis testing
  - Significances
  - Limits

Why probas / stats ?

Goals



forward modeling
generation
simulation

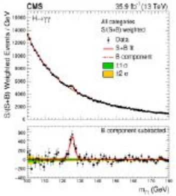

Silicon Tracker
Electromagnetic Calorimeter
Hadron Calorimeter
Superconducting Solenoid
Iron return yoke interspersed with muon chambers

PREDICTION

θ
parameters of interest

p(x, z | θ, ν)

x
observed data
simulated data

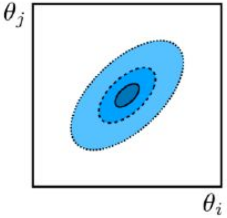


z
latent variables
Monte Carlo truth

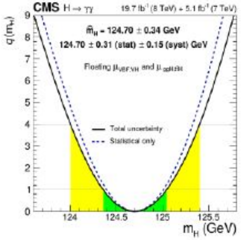
ν
nuisance parameters

s(x)
summary statistic

INFERENCE



inverse problem
measurement
parameter estimation



Key tasks in statistics

- Point estimation: what single “measured” value of a parameter to report ?
    - m_H = 125.09 GeV
- Interval estimation: what confidence interval to report ?
    - m_H = 125.09 ± 0.24 GeV
- Hypothesis testing
    - Tell aparts different models: model selection
        - Is Higgs 0^+ or 0^- ?
    - Test a specific value of a parameter vs any other value
    - Goodness of fit: test a specific model vs anything else
        - Is the data consistent with the SM expectation ?
- Decision making: what action should be taken based on the observed data ?
    - Usually based on more or less explicit conventions
        - Ex: The small difference between the measurement and theory is probably a fluctuation, more data are needed.

Bayesian vs Frequentist statistics

Two philosophies coexist !

- Bayesian:
  - Closer to everyday reasoning, where probability is interpreted as a degree of belief that something will happen, or that a parameter will have a given value.
- Frequentist:
  - Closer to scientific reasoning, where probability means the relative frequency of something happening. This makes it more objective, since it can be determined independently of the observer, but restricts its application to repeatable phenomena.
- So what ?
  - For practical matters, results tend to be very similar in the asymptotics regime
  - There exist nonetheless some important differences (coverage, goodness of fit…)

Bayesian vs Frequentist: take-home messages

> **“Bayesians address the questions everyone is interested in by using assumptions that no one believes. Frequentists use impeccable logic to deal with an issue that is of no interest to anyone.” (Louis Lyons)**

- Communities tend to lean towards one approach
  - Cosmology is mostly using Bayesian statistics (there is only 1 universe…)
  - HEP is more frequentist

- Will use frequentist approach in the following
  - By far the most common at the LHC
  - Bayesian treatment used for historical reasons in some new physics searches

Probability and random variables

Random Variables

Random variable: **a variable that represents the outcome of a random phenomenon.**

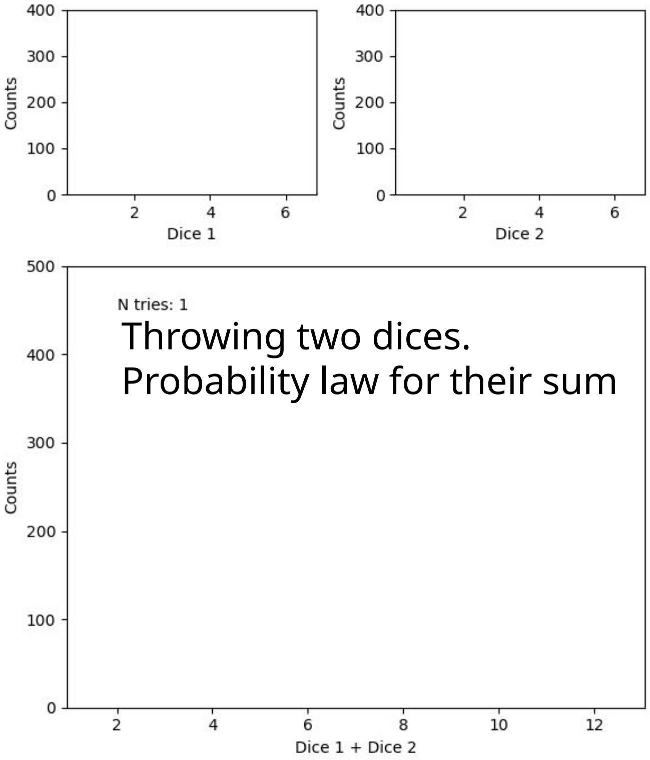
- Examples: tossing a coin, lifetime of a particle, throwing dices…

- Random variables are usually denoted with a capital letter (e.g. X)

- A function P is a probability function of X if (Kolmogorov axioms):
  - $P(x_i) \geq 0$ for all i
  - $P(x_i \text{ or } x_j) = P(x_i) + P(x_j)$
  - $\sum P(x_i) = 1$

- Frequentist probability: $P(A) = \lim_{N \to \infty} n/N$

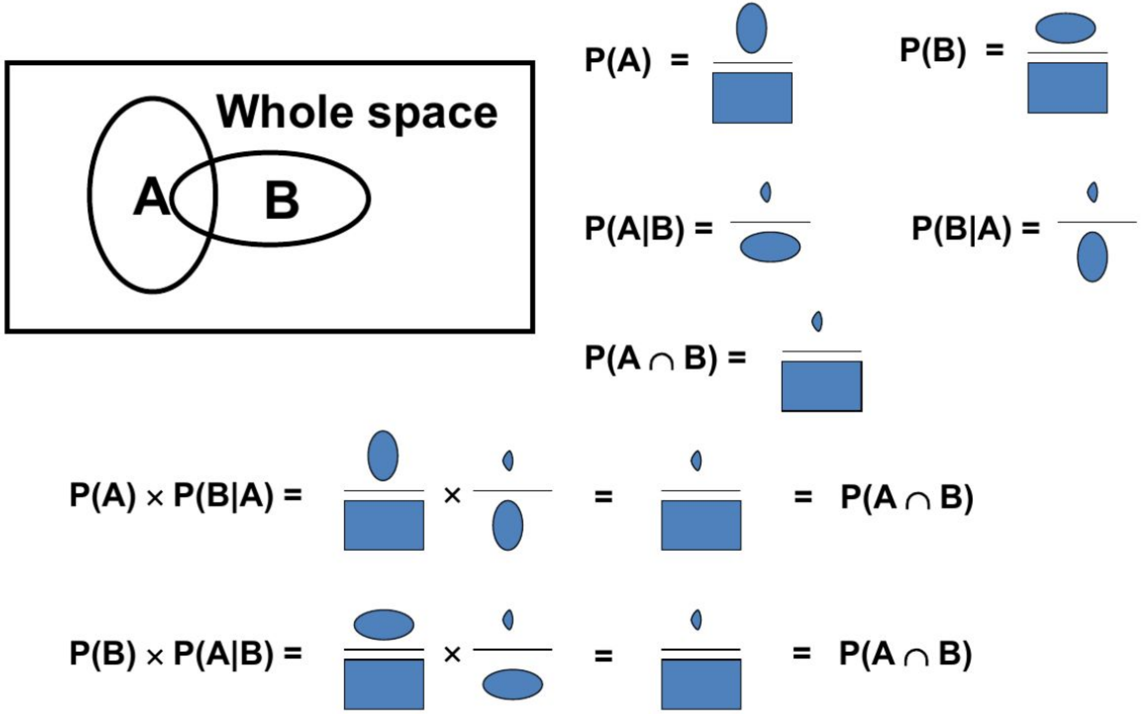
- **Probability distribution** of the random variable



Throwing two dices.
Probability law for their sum

Bayes' theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$



Useful example

A muon detection experiment measures:

- P(muon ID|muon), i.e., efficiency for tagging muons
- P(muon ID|not a muon), i.e., efficiency for background
- P(no muon ID|muon) = 1 – P(muon ID|muon)
- P(no muon ID|not a muon) = 1 – P(muon ID|not a muon)

Question: Given a selection of particles identified as muons, what fraction of them is muons? I.e., what is **P(muon|muon ID)** ?

Answer: Cannot be determined from the given information ! Need in addition: **P(muon)**, the true fraction of all particles that are muons.

Then Bayes' theorem inverts the conditionality:
P(muon|muon ID) = P(muon ID|muon)P(muon) / P(muon ID)

Useful example, contd.

- P(muon ID|muon) is the **efficiency** for tagging muons
- P(muon|muon ID) is the **purity** of a sample of particles identified as muons

⇒ helpful to keep in mind when one encounters cases where it is tempting or confusing to make the logical error of equating P(A|B) and P(B|A).

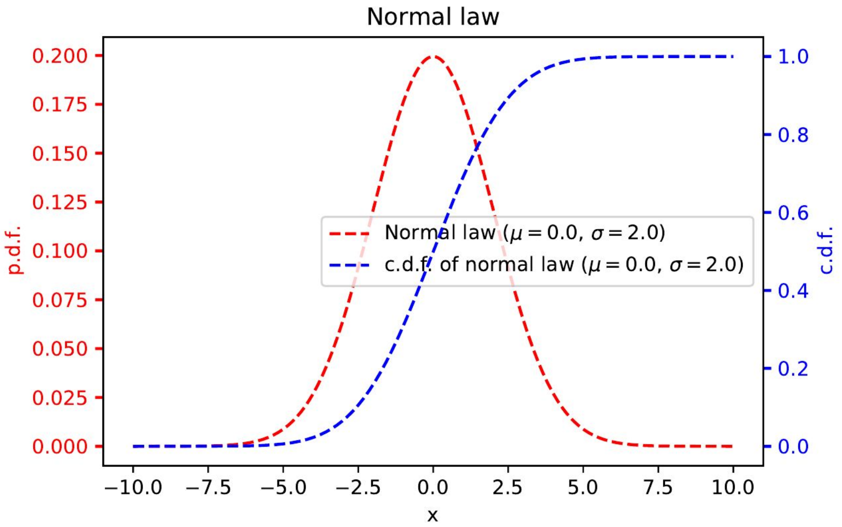
Probability density function

For continuous random variables, the probability density function f is defined by:

$f(x)\,dx = P(X \in [x, x + dx])$

It is related to the cumulative function:

- F so that $F(x_0) = P(x \leq x_0)$
  - $F(a) = 0$
  - $F(b) = 1$
- $f(x)\,dx = F(x + dx) - F(x)$

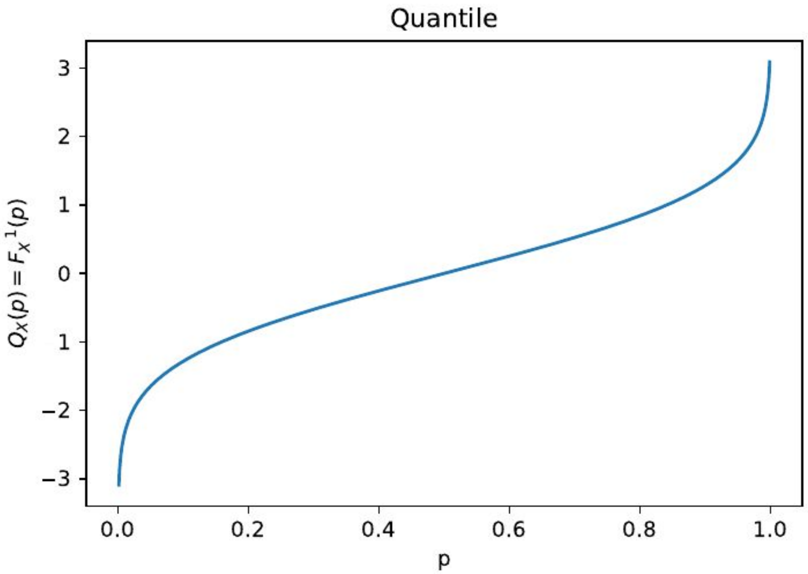


Quantiles

The **quantile x_α** is the value of the random variable x at which the cumulative distribution is equal to α. It is the inverse of the cumulative distribution function:

$$x_\alpha = F^{-1}(\alpha)$$

Quantiles of the Normal law:



Special case:

- **Median**: $x_{med} = x_{1/2}$

Moments

- **Mean μ**

$$\mu = E(X) = \int x f(x) dx$$

- **Variance V, standard deviation σ**

$$V = \sigma^2 = E\left((X-\mu)^2\right) = E(X^2) - \mu^2$$

- Higher moments ($E((X-\mu)^n)$):
  - **skew** (n=3): measures left-right asymmetry of the pdf
  - **kurtosis** (n=4): measures the size of the tails of the distribution (if positive, then larger tails than a Gaussian).

Multi-dimensional case

Example: 3 variables x, y, z, with a joint pdf f

A **marginal pdf** is defined as:

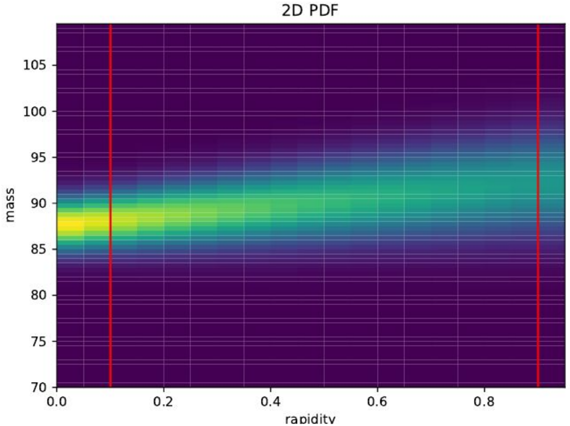
$$f_X(x) = \int dy'\, dz'\, f(x, y', z')$$

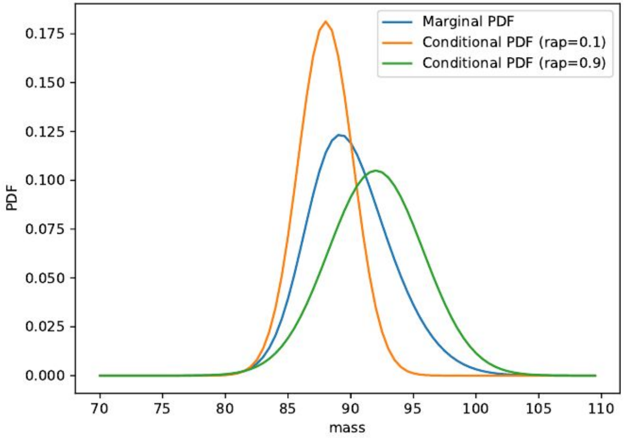
f_X is a **projection** of f. The other variables are **integrated**.

A **conditional pdf** is defined as:

$$f_C(x, y_0, z_0) = \frac{f(x,y_0,z_0)}{\int dx'\, f(x',y_0,z_0)}$$

f_C is a **slice** of f





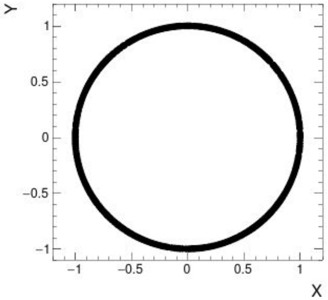
Independence and correlation

Two variables X and Y are **independent** iff $f(x,y) = f_X(x)f_Y(y)$

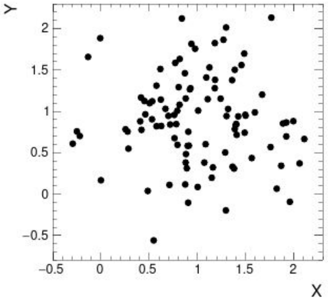
Correlation coefficient between two variables X and Y:

$$\rho(X,Y) = \frac{C(X,Y)}{\sigma_x \sigma_y}$$

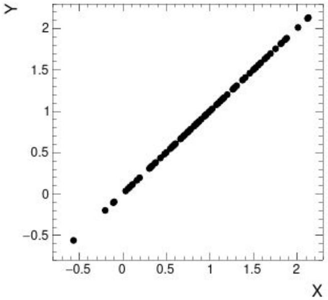
with $C(X, Y) = E((X - \mu_X)(Y - \mu_Y)) = E(XY) - E(X)\,E(Y)$



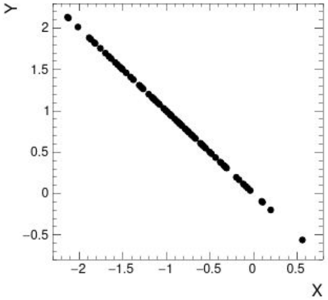
$\rho = 0,$
not independent



$\rho = 0,$
independent



$\rho = 1,$
complete correlation



$\rho = -1,$
complete
anti-correlation

Independent
⇒ ρ = 0

The opposite is **not true**

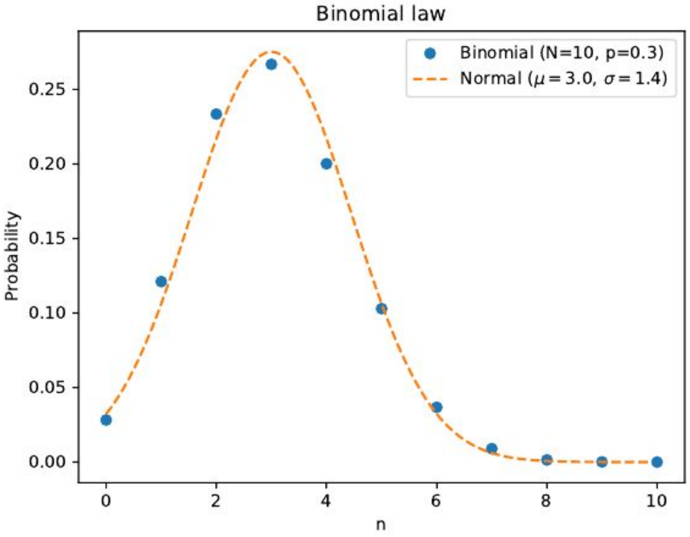
Binomial law

A random process of probability of success p is repeated N times.

The number of successes n follows a binomial distribution:

$$P(n) = B(n|N,p) = \binom{N}{n} p^n (1-p)^{N-n}$$

- Mean E(n) = pN
- Variance V(n) = Np(1-p)
- Example: out of 1000 collisions, how many will produce a W boson ?
- In the limit of small p and large N, with pN constant, the binomial law converges towards the **Poisson law**



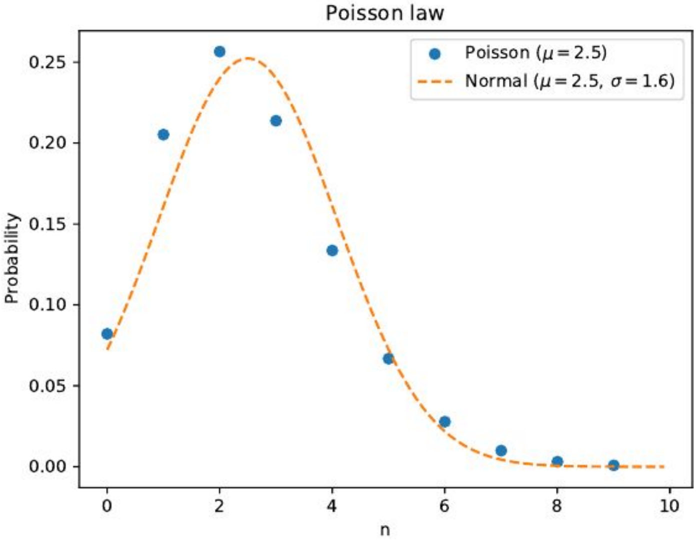
Poisson law

Typical case of random memoryless processes.

The probability to observe n events in a given interval follows a Poisson distribution:

$$P_\mu(n) = \frac{\mu^n e^{-\mu}}{n!}$$

- Mean E(n) = μ
- Variance V(n) = μ
- $P_{\mu 1} + P_{\mu 2} = P_{\mu 1+\mu 2}$
- Example: how many Higgs bosons are produced for a luminosity L = 140 fb^{-1} ?
- In the limit of large μ, the Poisson distribution converges towards a **Gaussian distribution**

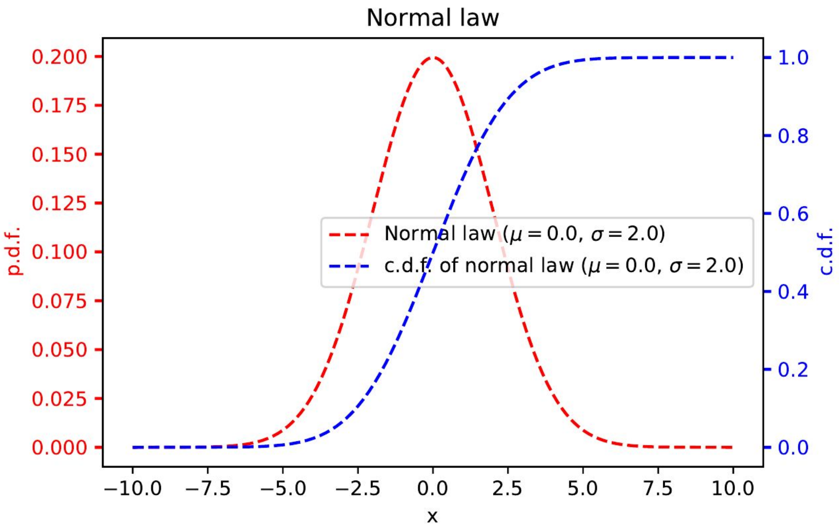


Gaussian distribution

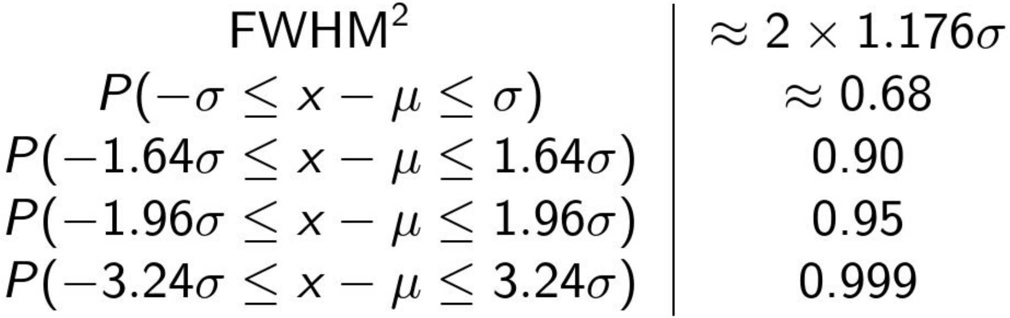
A continuous random variable x follows a Gaussian distribution of parameters μ and σ:

$$f_{\mu,\sigma}(x) = G(x|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Mean E(x) = μ
- Variance V(x) = σ^2
- Special case μ=0, σ=1 is called the **Normal law**
- Gaussian distributions play a very special role in statistics because of the **Central Limit Theorem**

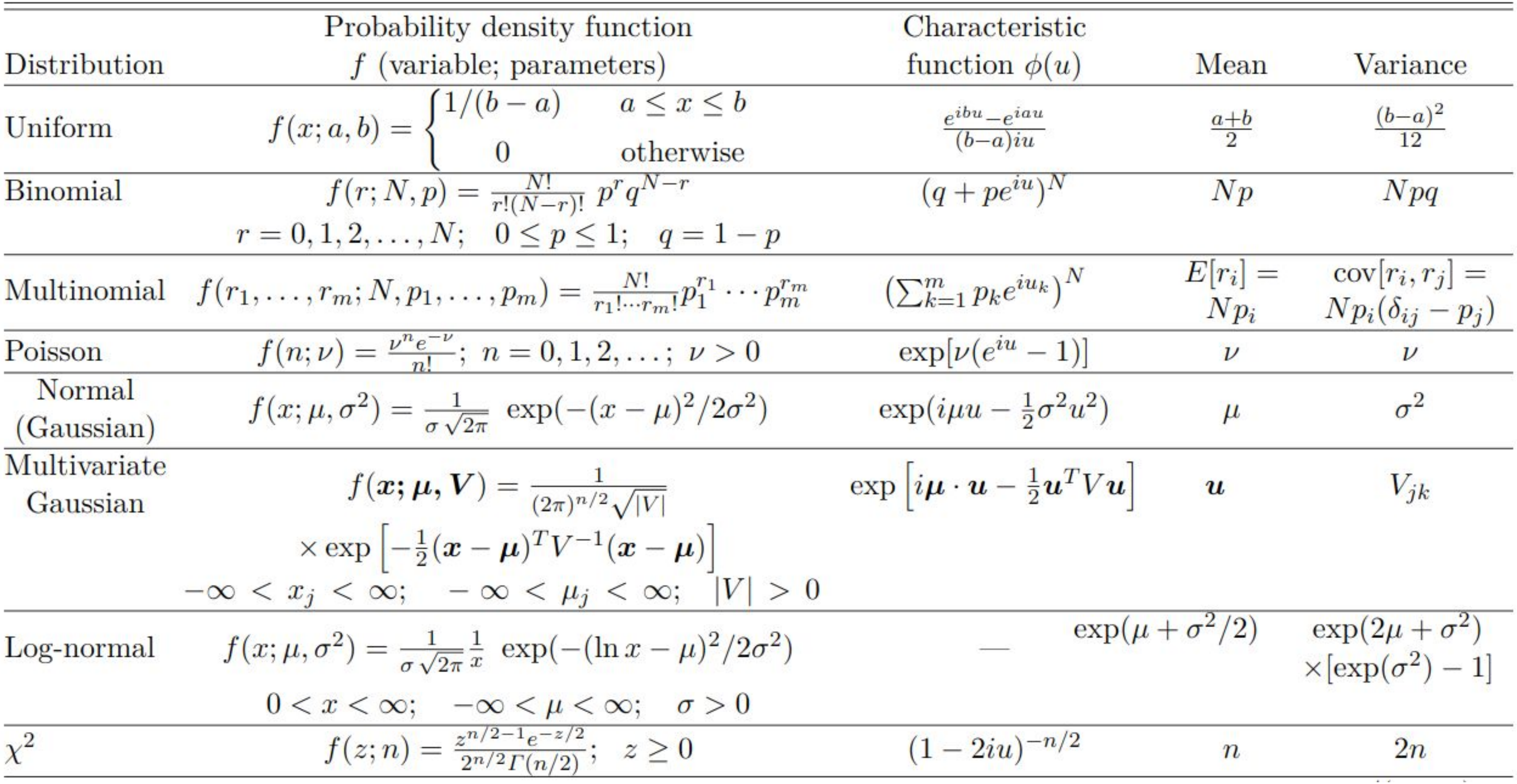


Normal law: useful numbers



$$\begin{array}{c|c}
\mathrm{FWHM}^2 & \approx 2 \times 1.176\sigma \\
P(-\sigma \leq x - \mu \leq \sigma) & \approx 0.68 \\
P(-1.64\sigma \leq x - \mu \leq 1.64\sigma) & 0.90 \\
P(-1.96\sigma \leq x - \mu \leq 1.96\sigma) & 0.95 \\
P(-3.24\sigma \leq x - \mu \leq 3.24\sigma) & 0.999
\end{array}$$

Summary of common and useful distributions



Distribution	Probability density function f (variable; parameters)	Characteristic function $\phi(u)$	Mean	Variance
Uniform	$f(x;a,b) = \begin{cases} 1/(b-a) & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$	$\frac{e^{ibu}-e^{iau}}{(b-a)iu}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Binomial	$f(r;N,p) = \frac{N!}{r!(N-r)!}\, p^r q^{N-r}$ $r = 0,1,2,\ldots,N; \quad 0 \le p \le 1; \quad q = 1-p$	$(q + pe^{iu})^N$	Np	Npq
Multinomial	$f(r_1,\ldots,r_m;N,p_1,\ldots,p_m) = \frac{N!}{r_1!\cdots r_m!} p_1^{r_1}\cdots p_m^{r_m}$	$\left(\sum_{k=1}^{m} p_k e^{iu_k}\right)^N$	$E[r_i] =$ Np_i	$\mathrm{cov}[r_i,r_j] =$ $Np_i(\delta_{ij} - p_j)$
Poisson	$f(n;\nu) = \frac{\nu^n e^{-\nu}}{n!}; \; n = 0,1,2,\ldots; \; \nu > 0$	$\exp[\nu(e^{iu}-1)]$	ν	ν
Normal (Gaussian)	$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-(x-\mu)^2/2\sigma^2)$	$\exp(i\mu u - \frac{1}{2}\sigma^2 u^2)$	μ	σ^2
Multivariate Gaussian	$f(\boldsymbol{x};\boldsymbol{\mu},\boldsymbol{V}) = \frac{1}{(2\pi)^{n/2}\sqrt{\vert V\vert}}$ $\times \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T V^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$ $-\infty < x_j < \infty; \quad -\infty < \mu_j < \infty; \quad \vert V\vert > 0$	$\exp\left[i\boldsymbol{\mu}\cdot\boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^T V\boldsymbol{u}\right]$	$\boldsymbol{u}$	V_{jk}
Log-normal	$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\frac{1}{x} \exp(-(\ln x-\mu)^2/2\sigma^2)$ $0 < x < \infty; \quad -\infty < \mu < \infty; \quad \sigma > 0$	—	$\exp(\mu + \sigma^2/2)$	$\exp(2\mu + \sigma^2)$ $\times[\exp(\sigma^2) - 1]$
χ^2	$f(z;n) = \frac{z^{n/2-1}e^{-z/2}}{2^{n/2}\Gamma(n/2)}; \quad z \ge 0$	$(1-2iu)^{-n/2}$	n	$2n$

Parameter estimation

Parameter estimation ?

- Suppose we have a model, represented by a pdf f(x|θ)
  - x is a **random variable**
  - θ represents **parameters** that affect the shape of the pdf

- Now, let us collect a sample of observed data $x=(x_1, x_2,..., x_N)$

- We want to say something of the parameters θ using the data

- An **estimator** is a function of the data (a.k.a a **statistic**), that is used to **estimate the value** of a parameter:
  - $t_N(x)$
  - $t_N(x) \rightarrow \theta$?

Estimator properties

Not all estimators are born equal

X is a random variable of pdf $f(x|\theta_0)$, with θ_0 unknown. An estimator t_N of θ_0 can be:

- **unbiased** (**accuracy**): if the bias $b_N = E(t_N) - \theta_0 = 0$.
- **convergent** (or consistent): mathematical convergence towards the true value for large enough N
- **efficient** (**precision**): if the variance of the estimator $V(t_N)$ converges towards a minimum variance bound
- **optimal**: if t_N minimises the Mean Square Error (MSE):
  $MSE(t_N) = V(t_N) + b_N^2$
- **robust**: if it does not depend on a hypothesis on the pdf

Usual method to build estimators

- Moments method
  - aka the sample mean !
- Maximum likelihood method
  - today's focus
- Least squares method
  - still useful in many occasions

Likelihood function

A random variable x follows a pdf $f(x|\theta)$ where θ represents parameter(s).

N independent observations of x are obtained: $x_1, \ldots, x_N$

The joint pdf of the N observations is then:

$$P(\mathbf{X}|\theta) = \prod f(x_i|\theta)$$

The likelihood function is this pdf, evaluated with **fixed data X** and regarded **as a function of the parameters θ** only:

$$L(\theta) = P(\mathbf{X}|\theta)$$

Notes:

- $L(\theta)$ is **not** a pdf for θ. The area under L is **meaningless**
- It is not even normalised to unity. The **absolute value of the likelihood is also meaningless**

Maximum likelihood estimators

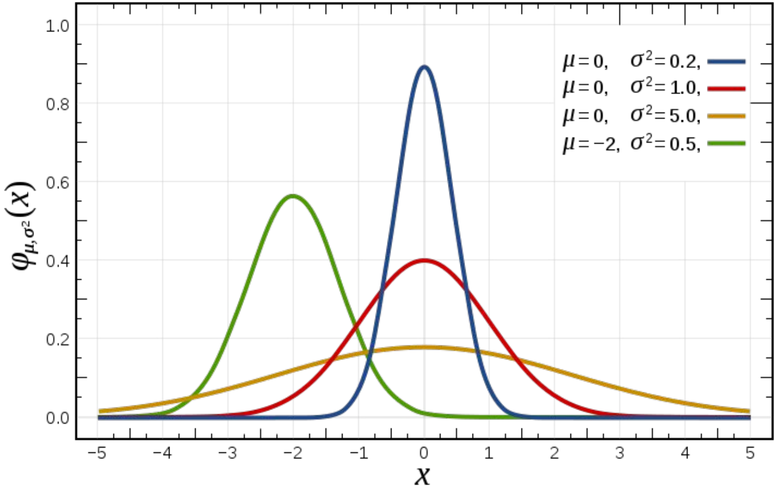
If the hypothesized θ is close to the true value, then there is a high probability to get data like the observed one.

The maximum likelihood (ML) estimator(s) are defined as the parameter value(s) for which the likelihood is maximum

- In practice, we usually minimize **-ln L(θ)** or **-2ln L(θ)**
- ML estimators are **not** guaranteed to be always unbiased, neither optimal
- **In practice they are very good**: asymptotically unbiased, with a MLE distribution asymptotically Gaussian
- ML estimators are not robust: **the shape of the pdf must be known**

Example: estimation of Gaussian parameters

- Random process following a Gaussian law of unknown mean and variance:
  - Example: Invariant mass distribution of $Z \to e^+e^-$
  - Parameters: θ ↦ μ mean, σ standard error
  - Observables: x_i
  - PDF: f ↦ $G(x \mid \mu, \sigma) = 1/\sqrt{(2\pi\sigma^2)}\exp(-(x-\mu)^2 / 2\sigma^2)$



Parameter estimation

- Likelihood function to maximize:

$$\mathcal{L}(x_i|\mu,\sigma) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

- In practice, we minimize the negative log-likelihood:

$$\boxed{NLL = -\log\mathcal{L}(x_i|\mu,\sigma) = \frac{N}{2}\log(2\pi\sigma^2) + \sum_{i=1}^{N}\frac{(x_i-\mu)^2}{2\sigma^2}}$$

- which yields:

$$\left.\begin{array}{l} \dfrac{\partial -\log\mathcal{L}(x_i|\mu,\sigma)}{\partial\mu} = 0 \\[2ex] \dfrac{\partial -\log\mathcal{L}(x_i|\mu,\sigma)}{\partial\sigma} = 0 \end{array}\right\}$$

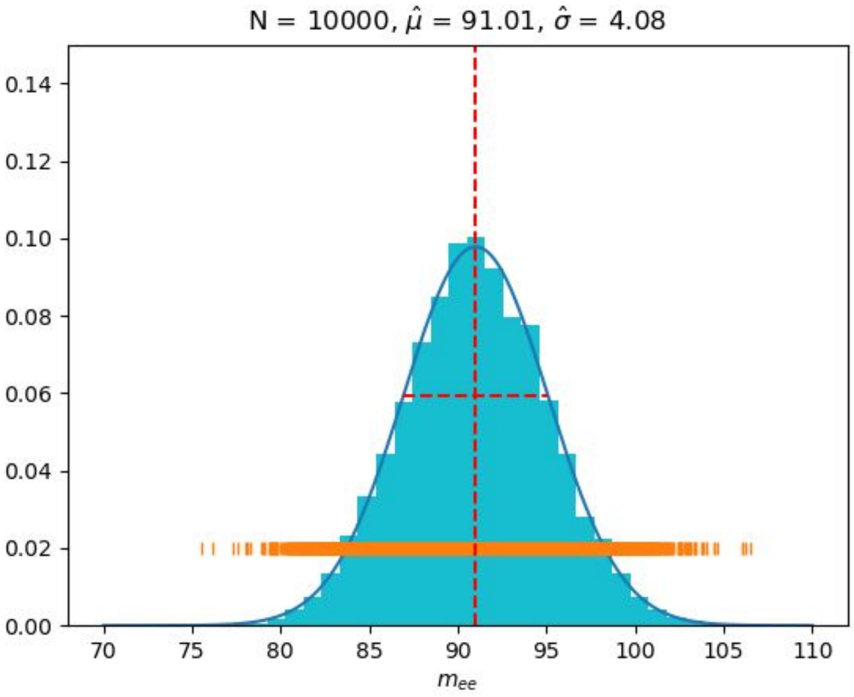
$$\boxed{\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i}$$

Sample mean !

$$\boxed{\hat{\sigma} = \sqrt{\frac{1}{\boxed{N}}\sum_{i=1}^{N}(x_i-\hat{\mu})^2}}$$

Biased estimator !
(but asymptotically unbiased)

Live example



- When adding data
  - μ̂ converges to μ=91
  - σ̂ converges to σ=4
  - Uncertainty in the estimate decreases as well
- Maximisation of likelihood function
  - Analytical calculation here
  - Usually relying on numerical minimisers: `Minuit`

Coverage probability and confidence level

Beyond parameter estimation: uncertainty in the parameter

- Frequentists report **confidence intervals**, which will contain the true value of the parameter θ a certain fraction of the time (called the **confidence level**).

- **Frequentist Principle** (Neyman): Construct statements such that a fraction $f \geq 1 - \alpha$ of them are true over an ensemble of statements.
  - f is called the coverage probability
  - $1 - \alpha$ is called the confidence level
  - An ensemble of statements that obeys the FP is said to cover

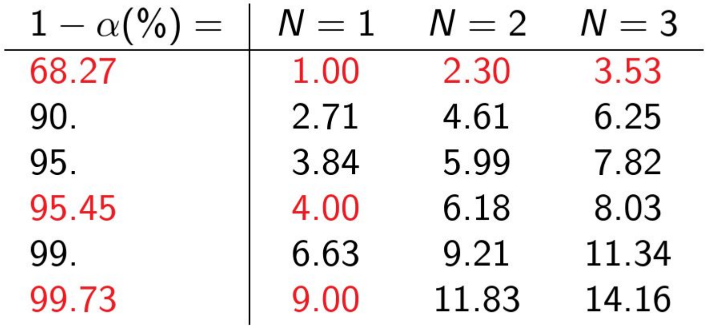
- Application to confidence intervals: if we report a confidence interval I and we repeat the experiment N times, then a fraction f of the intervals I will contain the true value of the parameter

Confidence intervals for ML estimators

- Finding procedures that give correct coverage (i.e neither undercoverage nor overcoverage) is in general not trivial

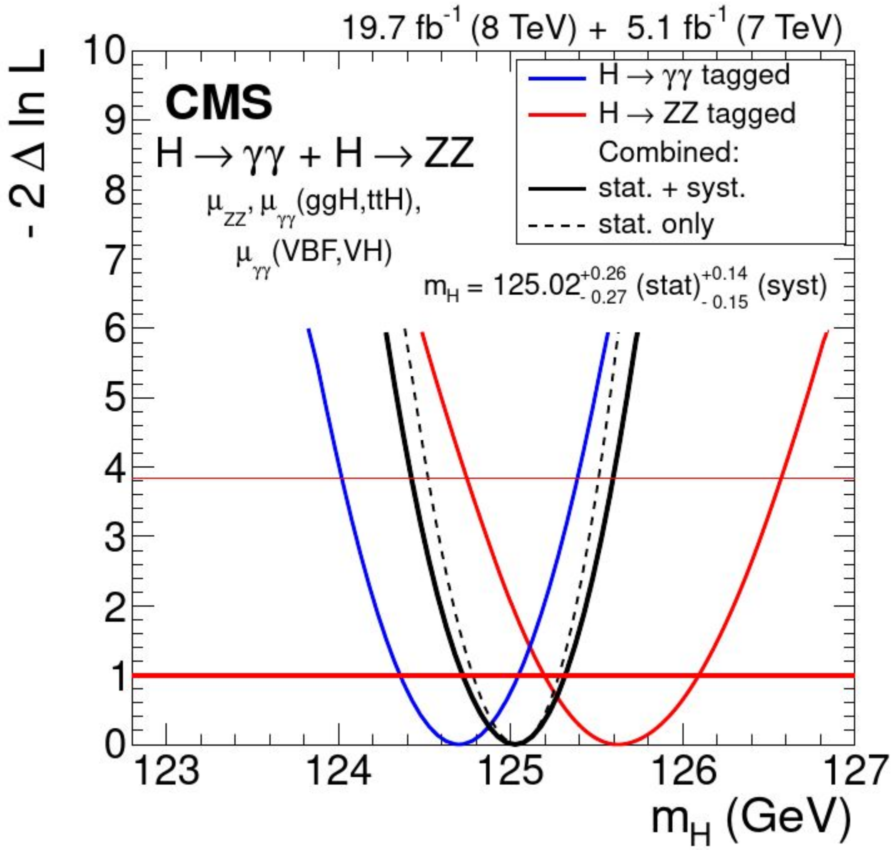
- Asymptotic properties of log-likelihoods to the rescue:
  - **Wald's approximation**: the likelihood shape is asymptotically gaussian around its maximum
  - **Wilk's theorem**: -2 ln L(θ) asymptotically follows a χ^2 law with d degrees of freedom, where d is the dimensionality of θ
  - **Consequence**: Confidence intervals can be obtained from the inverse quantiles of a χ^2 distribution with d degrees of freedom: the so-called **likelihood intervals**

Values of $\Delta\chi^2$ or $2\Delta \ln L$ corresponding to a coverage probability 1−α in the large data sample limit, for joint estimation of N parameters.

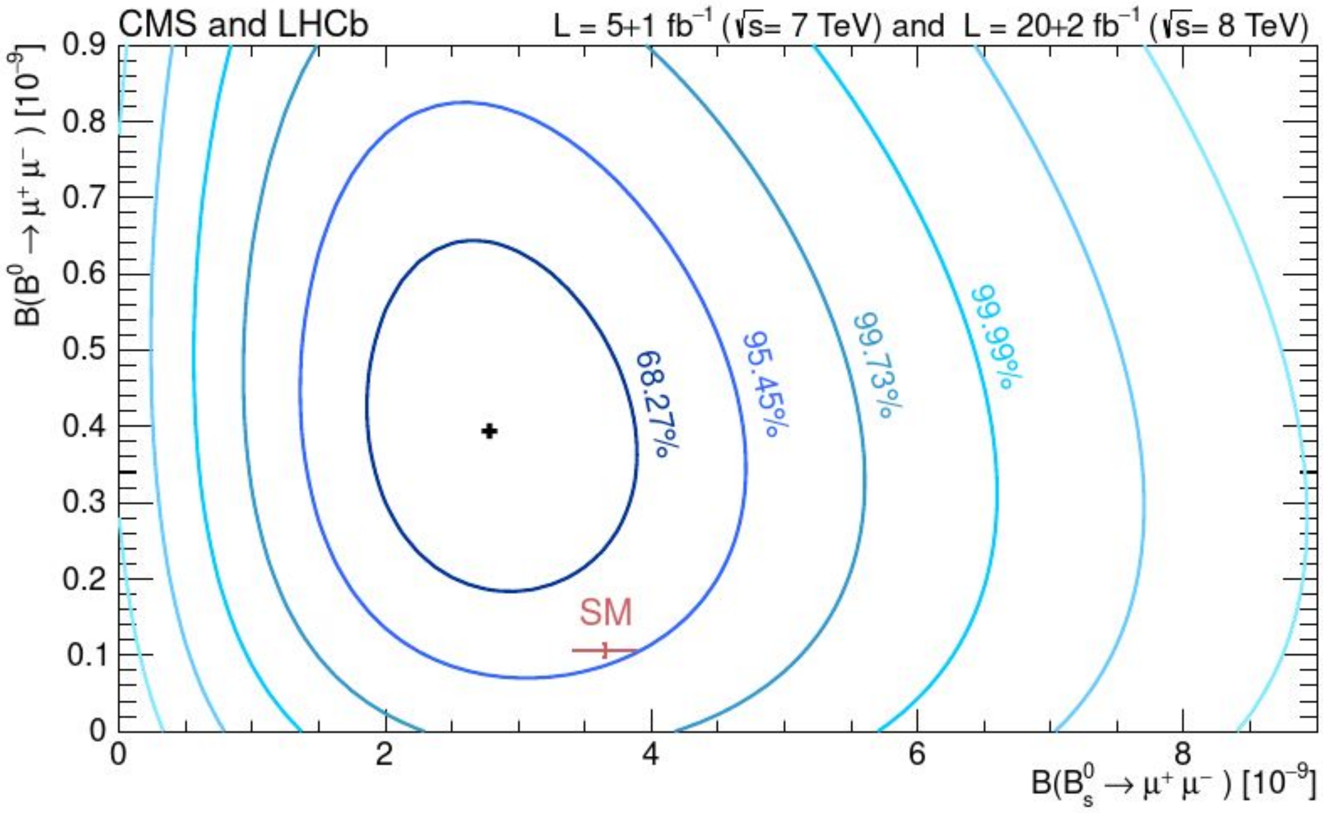


$1-\alpha(\%)=$	$N=1$	$N=2$	$N=3$
68.27	1.00	2.30	3.53
90.	2.71	4.61	6.25
95.	3.84	5.99	7.82
95.45	4.00	6.18	8.03
99.	6.63	9.21	11.34
99.73	9.00	11.83	14.16

Example: Higgs mass measurement



Example in 2D: measurement of $B_s \rightarrow \mu\mu$ and $B_0 \rightarrow \mu\mu$



Contours: $\Delta \ln(\mathcal{L}) = 2.30,\ 6.18,\ 11.83,\ \ldots$

Building a likelihood

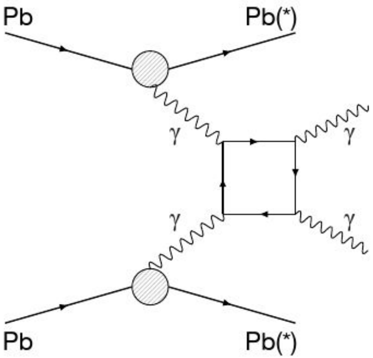
Likelihoods in collider experiments

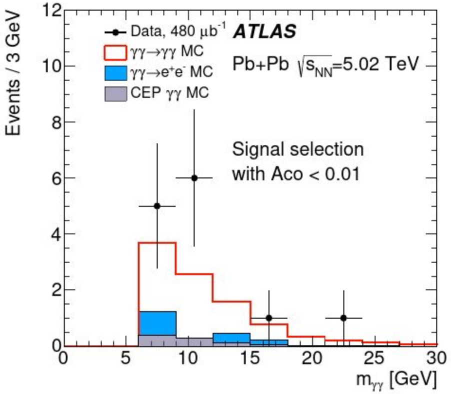
- Observables are **numbers of events**
  - After selections, in categories, bins…
  - Due to the nature of collisions (independent), they obey **Poisson laws**

- Simplest case: **number counting experiment**
  - D observed events
  - s expected signal events (parameter of interest)
  - b expected background events (known)
  - $$p(D|s,b) = \frac{(s+b)^D e^{-(s+b)}}{D!}$$
  - What is the MLE of s ?
    - L(s) = p(D|s,b)
    - dL/ds = 0 ⇒ s = D-b

Example: light-by-light scattering

- D = 13
- b = 2.6 ± 0.7
- s = 10.4





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Extension: multiple analysis regions, multiple bins

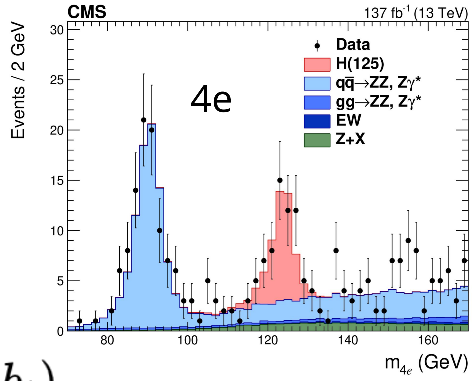
In practice, almost all analyses have more than one observable

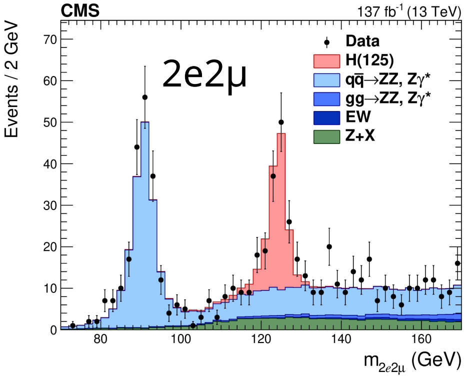
- **Signal strength μ**: often used as the main **parameter of interest**
  - $\mu = \sigma/\sigma_{SM}$
- Likelihood is a product of Poisson:

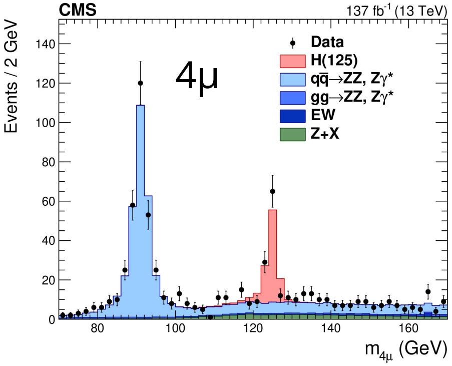
$$L(n_1,\ldots,n_{Nbins}|\mu) = \prod_{i=1}^{Nbins} P(n_i|\mu s_i + b_i)$$

expected numbers of events in bin i

- Special cases:
  - N=1: counting experiment
  - N=∞: unbinned analysis (s_i and b_i become pdf values)
- In practice, binned pdfs are often used
  - Software to do it easily: **HistFactory**







Dealing with uncertainties: nuisance parameters

- In a realistic model, the expectations for s_i and b_i are uncertain
  - Affected by systematic uncertainties

- This uncertainty can be added to our likelihood model as new parameters affecting the shape of the pdfs
  - $b_i \rightarrow b_i(\theta)$, $s_i \rightarrow s_i(\theta)$
  - They are new parameters of the likelihood: $L(\mu) \rightarrow L(\mu,\theta)$
  - But they are of no interest for our measurement: **nuisance parameters (NP)**

- Often we do have additional knowledge on these parameters
  - Ex: Background estimation performed in a dedicated control region (with some uncertainty)
  - Ex: Luminosity calibrated in a dedicated measurement (with some uncertainty)
  - This knowledge should be incorporated into the likelihood
  - Factorizable: $p(n_i|\mu) \rightarrow p(n_i,y_j|\mu,\theta) = \mathrm{Poiss}(n_i|\mu,\theta) \times \mathrm{pdf}(y_j|\theta)$
  - For the likelihood: $L(\mu,\theta) = L_{meas}(\mu,\theta) \times C(\theta)$

auxiliary measurements

data bins **constraint terms**

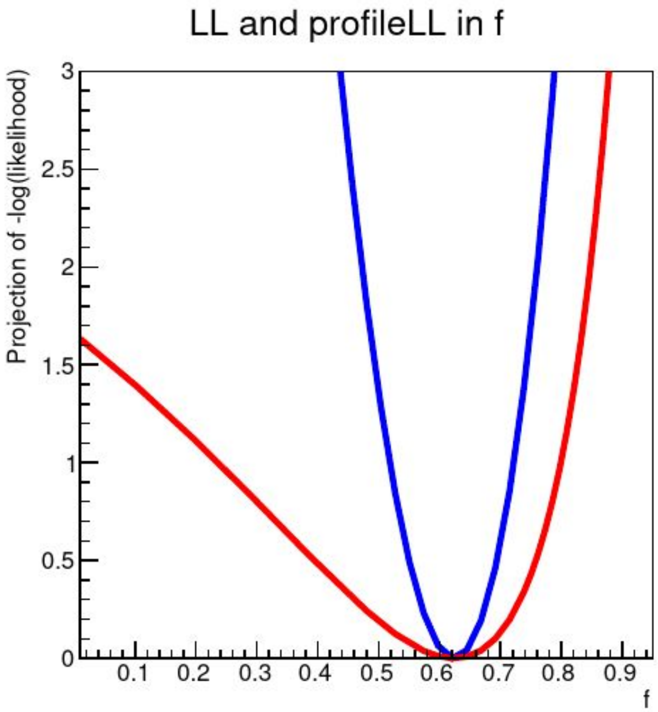
Constraint terms are most often Normal (Gaussian), but other distributions sometimes used (log-normal)

Profile likelihood

- Nuisance parameters are not of interest for our measurement
- **Profile likelihood**:

$$L_p(\mu) = L(\mu, \hat{\hat{\theta}})$$

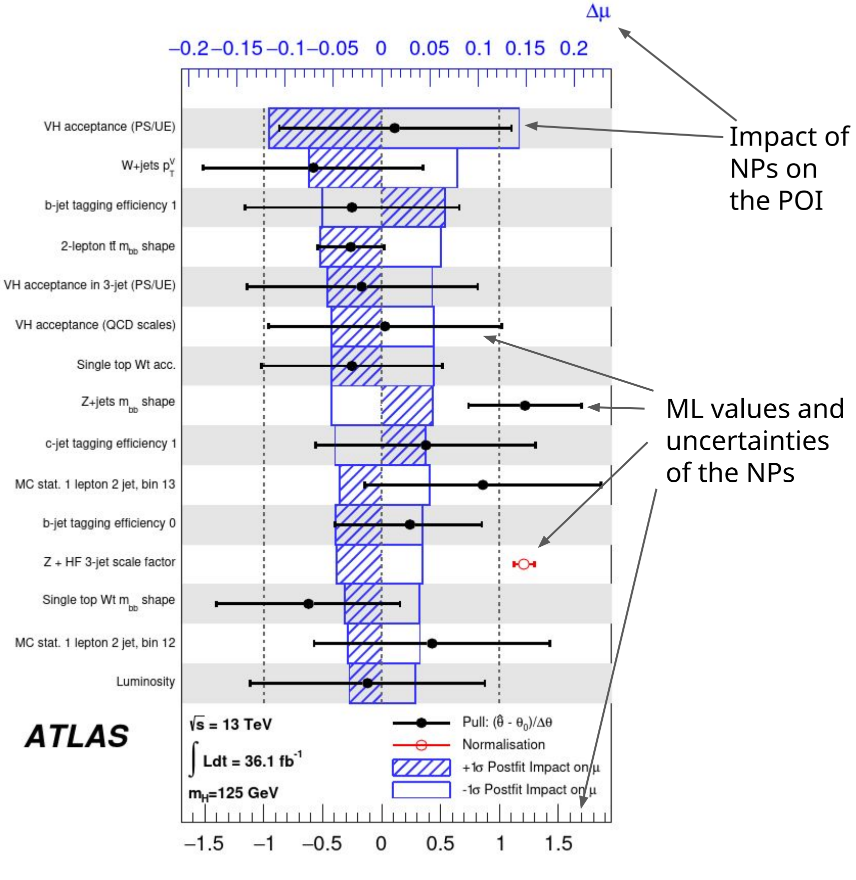
  - $\hat{\hat{\theta}}$ are the values of the parameters θ that maximise the likelihood for a given value μ of the parameters of interest
  - L_p(μ) is a function of the POI only: for each μ, new values of $\hat{\hat{\theta}}$ are obtained.
  - It reduces the dimensionality of L to that of μ (usually 1-d)



Working with profile likelihoods I

Profile likelihoods are complex objects !

- Working with >200 NPs is commonplace
- ML estimators make sense and have good properties, **assuming the pdf** (i.e the model) **is correct** !
- Great care should be taken to ensure this is the case
  - Goodness-of-fit tests
  - Do the ML fitted values for the NPs make sense (**pulls**) ?
  - Do the ML uncertainties for the NPs make sense (**constraints**) ?
- Keep track of how NPs affect the estimated POI (**impact**)



Impact of NPs on the POI

ML values and uncertainties of the NPs

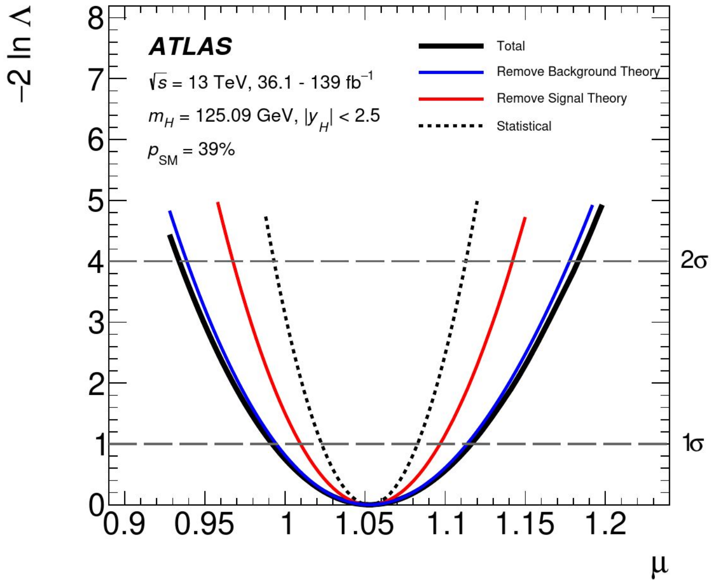
Working with profile likelihoods II

Error decomposition:

- **Total uncertainty** comes from the profile likelihood scan: $L_p(\mu) = L(\mu, \hat{\hat{\theta}})$
- **Statistical** comes from a scan where all NPs are set to their best fit value $\hat{\theta} = \hat{\hat{\theta}}(\hat{\mu})$:

$$L(\mu, \hat{\theta}) = L(\mu, \hat{\hat{\theta}}(\hat{\mu}))$$

- Other curves are intermediate cases where some NP are profiled while others are set to their best fit value in the scan
  - Allows to estimate the fraction of the total uncertainty coming from some NPs

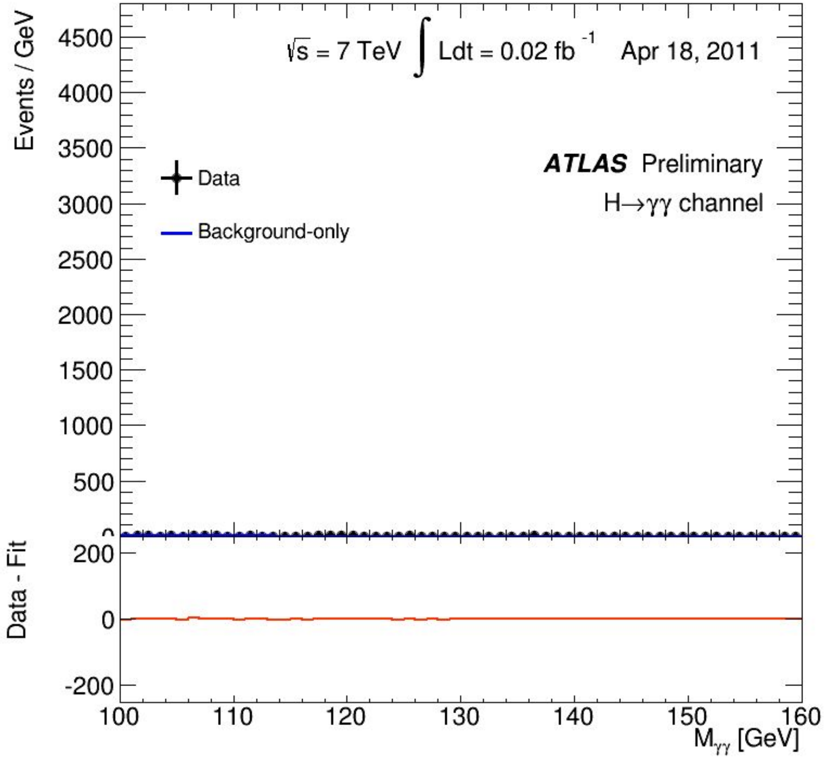


Hypothesis testing

Motivation: July 2012

- Analysis ready: take data and wait

- We “see” a bump at 125 GeV:
  - Is that a discovery ?
  - How do we quantify it ?



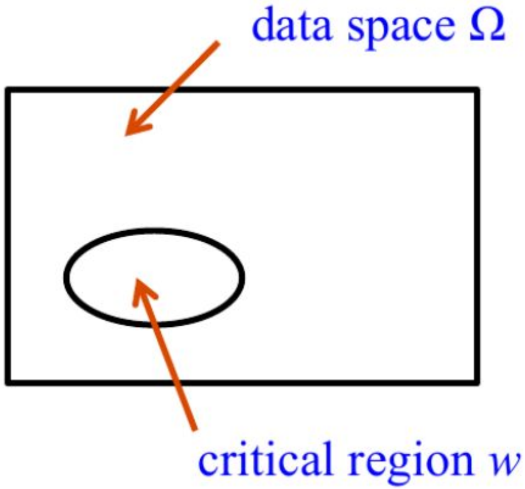
Hypothesis testing

Types of hypotheses

- **Simple hypothesis**: fully specified, including parameter values
  - eg: H_0 = Higgs is 0^+ vs H_1 = Higgs is 0^-
- **Composite hypothesis**: ensemble of simple hypotheses
- **Nested hypotheses**: most common case for searches
  - $\mu = 0$ (background-only) vs $\mu>0$ (new physics signal !)
  - $\mu = 1$ (SM expectation) vs $\mu \neq 1$ (SM is broken !)

Two ingredients for a hypothesis test

- A **test statistic** t(x)
- A **critical region** w such the hypothesis H_0 is false (with a given probability) if t in w
  - The alternative hypothesis is named H_1

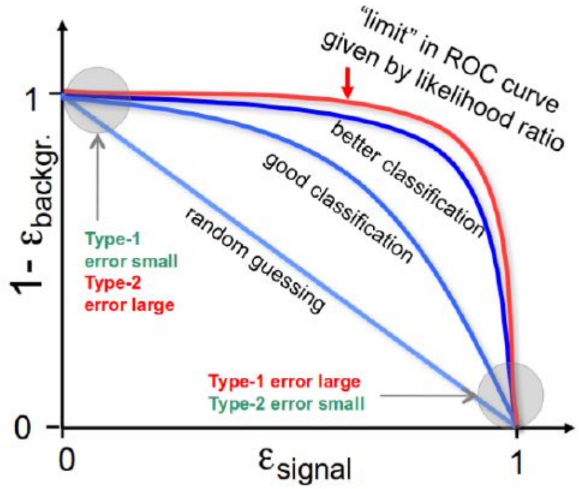


Power and size of a test

- **Size of the test** (**level of significance**): $\alpha = P(t \in w | H_0 \text{ true})$
    - Also called error of the first kind (Type I error)
    - “false discovery claim”: probability of rejecting H_0 when it is true
- **Power of the test**: $1 - \beta$ with $\beta = P(t \notin w \mid H_1)$
    - Also called error of the second kind (Type II error)
    - Probability of not claiming a discovery when there is one

	$P(t \notin w)$	$P(t \in w)$
H_0 true	$1 - \alpha$	α
H_1 true	β	$1 - \beta$

There is a **tradeoff** between Type I and Type II errors



Example: muon ID experiment

A muon detection experiment measures:

- P(muon ID|muon), i.e., efficiency for tagging muons
- P(muon ID|not a muon), i.e., efficiency for background
- P(no muon ID|muon) = 1 – P(muon ID|muon)
- P(no muon ID|not a muon) = 1 – P(muon ID|not a muon)

Hypotheses:

- H_0: not a muon
- H_1: muon

Then:

- Size of the test α = P(muon ID | not a muon)
- Power of the test β = P(no muon ID | muon)

Neyman-Pearson lemma

In the comparison of two simple hypotheses H0 and H1, the optimal discriminator is the **likelihood ratio** (LR):

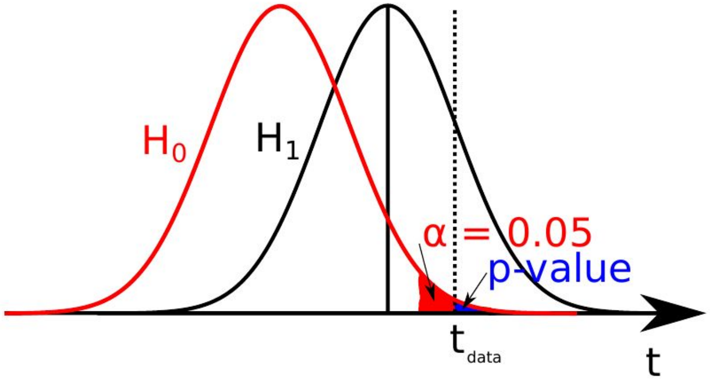
$$t(x) = \frac{L(x|H_1)}{L(x|H_0)}$$

Notes:

- **Optimal**: minimizes Type II error for a given Type I level of significance
- Valid for any **monotonic function of t**
  - Ex: q(x) = -2 ln t(x)
  - Ex: In a counting experiment, number of events
- Strictly valid for simple hypotheses only.
  - However, in practice, **works extremely well for our nested hypotheses**

Procedure for hypothesis testing

- Specify the null and the alternate hypotheses
  - Ex: H_0 = SM background, H_1 = BSM signal
- Build a test statistic t(x) using e.g Neyman-Pearson lemma
- Specify the significance of the test (what we accept as a false discovery rate)
  - Ex: 2.9 10^{-7} (5σ) for discovery
  - Ex: 0.05 for exclusion
- See where the measurement is t_{obs}
- Depending on whether t_{obs} is in or out of the critical region: decide on H_0



p-value and significance

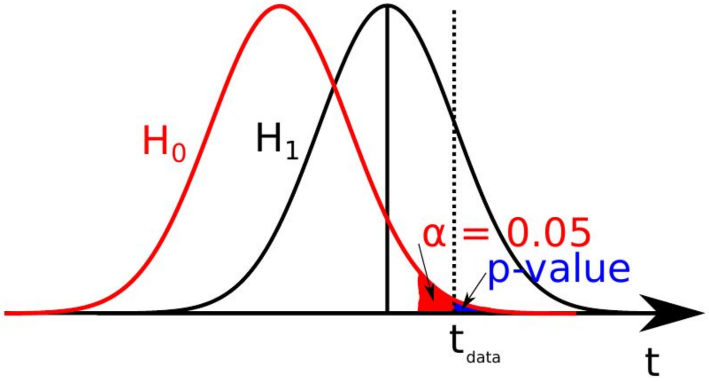
- p-value: $p_0 = p(t \geq t_{obs} \mid H_0)$
  - Significance level of the test α: chosen prior to look at the data
  - p-value: interesting quantity to compute when looking at the data

- Interpretation:
  - probability for the test statistic t to be larger than the observed one t_{obs}, under the null hypothesis H_0
  - **NOT** "the probability that H_0 is true"

- "Significance" in number of sigmas:
  - translation of the p-value using the integral in one tail of a Gaussian

$$p_0 = \int_Z^\infty G(x|0,1)\mathrm{d}x = 1 - \Phi(Z)$$

  - **Convention**: **3σ is evidence, 5σ is discovery**



z-value (σ)	p-value
1.0	0.159
2.0	0.0228
3.0	0.00135
5.0	2.87×10^{-7}

Profile likelihood ratio and asymptotics formulae

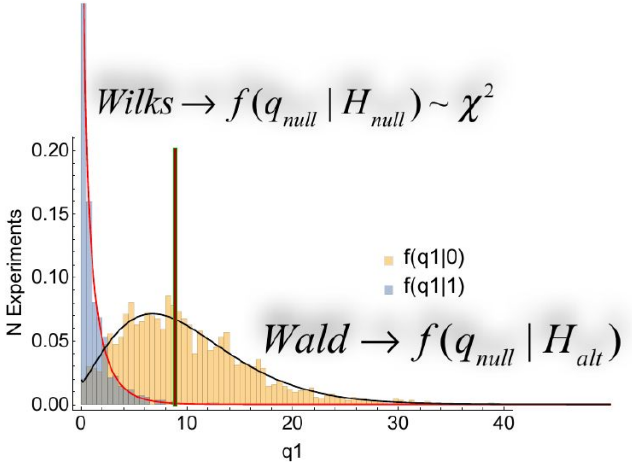
- At the LHC, to deal with systematics, the basis of test statistics used for hypothesis testing is the **Profile Likelihood Ratio** (PLR):

$$\lambda(\mu) = \frac{L(\mu,\hat{\hat{\theta}})}{L(\hat{\mu},\hat{\theta})}$$

- Then the test statistic for discovery is:

$$q_0 = \begin{cases} -2\ln\lambda(0) & \hat{\mu} \geq 0 \\ 0 & \hat{\mu} < 0 \end{cases}$$

- **Asymptotics properties** of the PLR make it easy to work with:
  - Wald's approximation, Wilks' theorem
  - Cowan, Cranmer, Gross and Vitells, EPJC 71 (2011) 1554
  - Median expected properties from the **Asimov dataset**
    - No need for CPU intensive toys !
  - $Z = \sqrt{-q_0} = \sqrt{-2\ln\lambda(0)}$



p-value in counting experiments

Very useful formula for e.g optimization studies

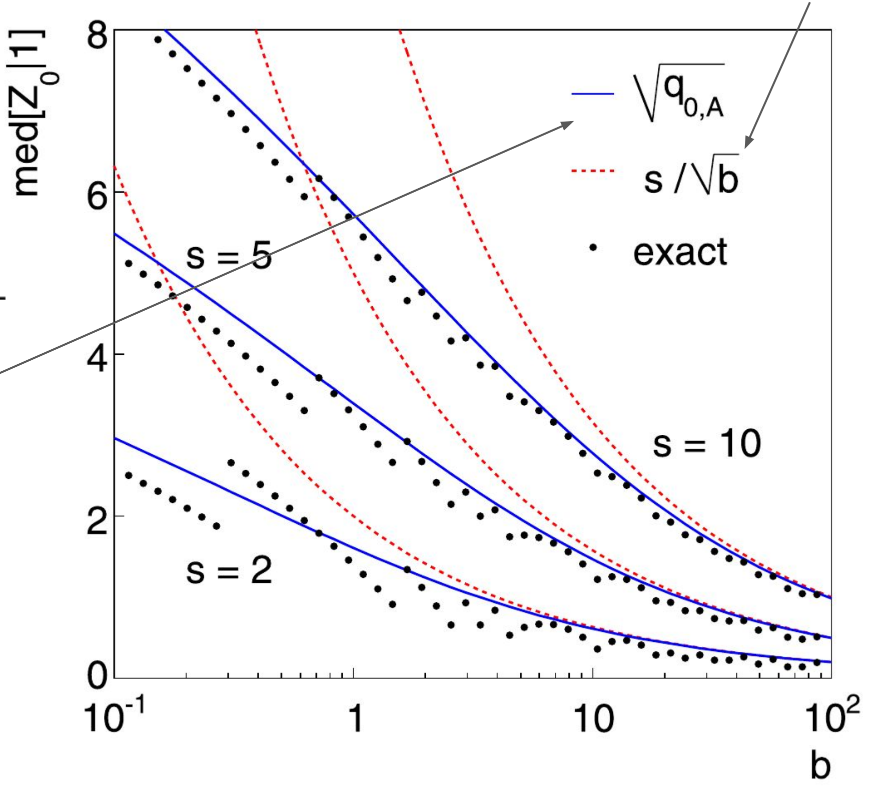
- n observed events, b background
  - n = µ.s + b

$$L(\mu) = \frac{(\mu s+b)^n e^{-(\mu s+b)}}{n!}$$

$$Z = \sqrt{-2\ln\frac{L(0)}{L(\hat{\mu})}} = \sqrt{2\left(n\ln\left(1+\frac{\hat{\mu}s}{b}\right)-\hat{\mu}s\right)}$$

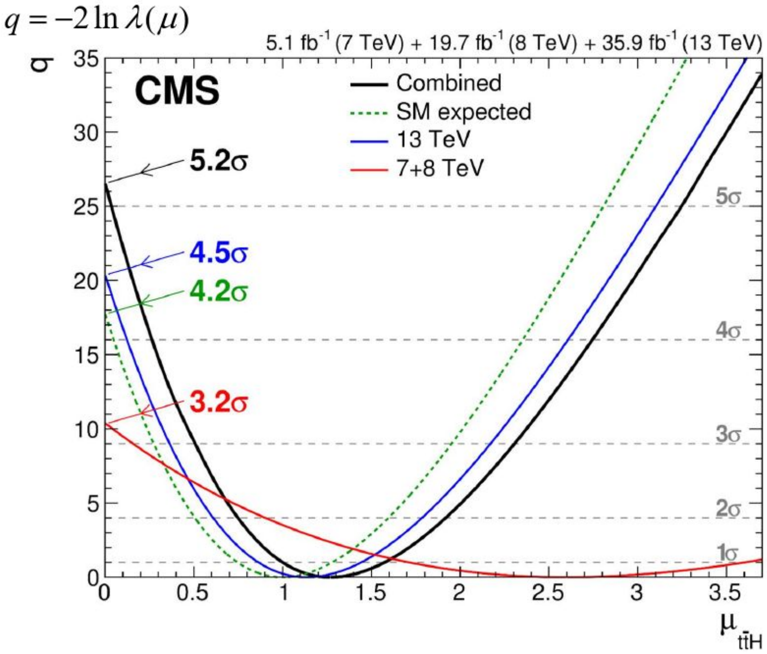
$$\boxed{Z = \sqrt{2\left(n\ln\left(\frac{n}{b}\right)+b-n\right)}}$$

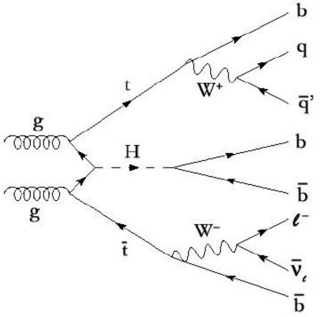
Gaussian approximation



Example: ttH observation

p-value (transformed as a significance) can be directly read on the y axis: $q_0 = q(\mu=0)$



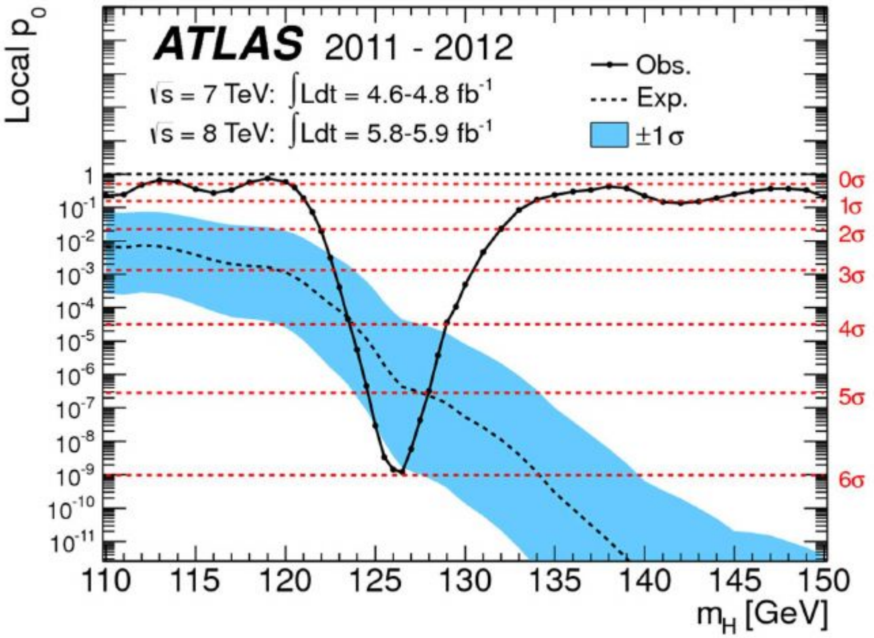


Example: Higgs boson discovery

- Each Higgs mass hypothesis is scanned independently

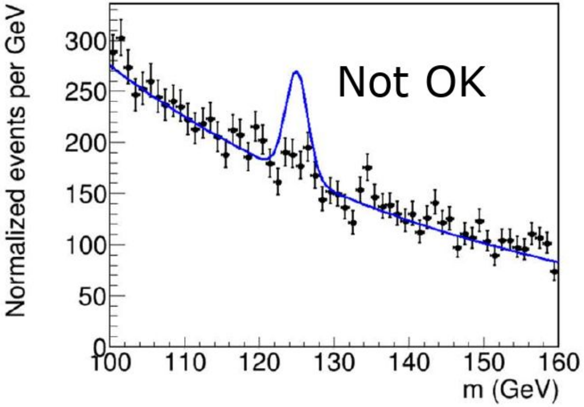
- For each mass:
  - **Observed**: p-value observed in data
  - **Expected**: median of the p-value expected in the presence of the SM Higgs boson
  - Blue band: interval containing 68% of the p-values under SM Higgs hypothesis

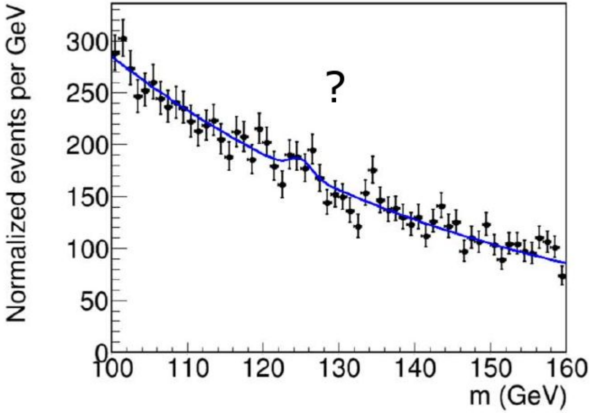
- “Local” p_0
  - Many mass points scanned
  - **Look-elsewhere effect**: global p_0 to correct for number of trials



Exclusion limits

- Similar procedure to discovery case, but hypotheses are inverted:
  - H_0: signal + background hypothesis
  - H_1: background-only hypothesis
- Goal: disprove H0 by estimating the probability of downward fluctuation of s+b
- Size of test less stringent: α = 0.05
  - 95% CL limits
- Upper limit: minimal signal strength for which H_0 can be excluded at 95% CL





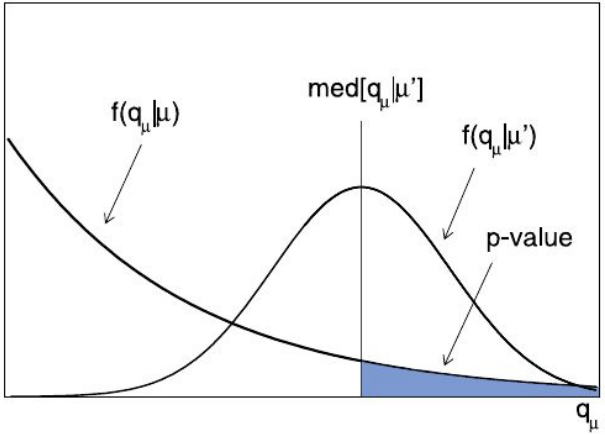
Test statistics for exclusion

- Still using PLR-based test statistic:

$$q_\mu = \begin{cases} -2\ln\lambda(\mu) & \hat{\mu} \leq \mu \\ 0 & \hat{\mu} > \mu \end{cases}$$

  - NB: one does not regard an upwards fluctuation of the data as representing incompatibility with the hypothesized μ

$$p_\mu = \int_{q_{\mu,\mathrm{obs}}}^{\infty} f(q_\mu|\mu)\, dq_\mu$$



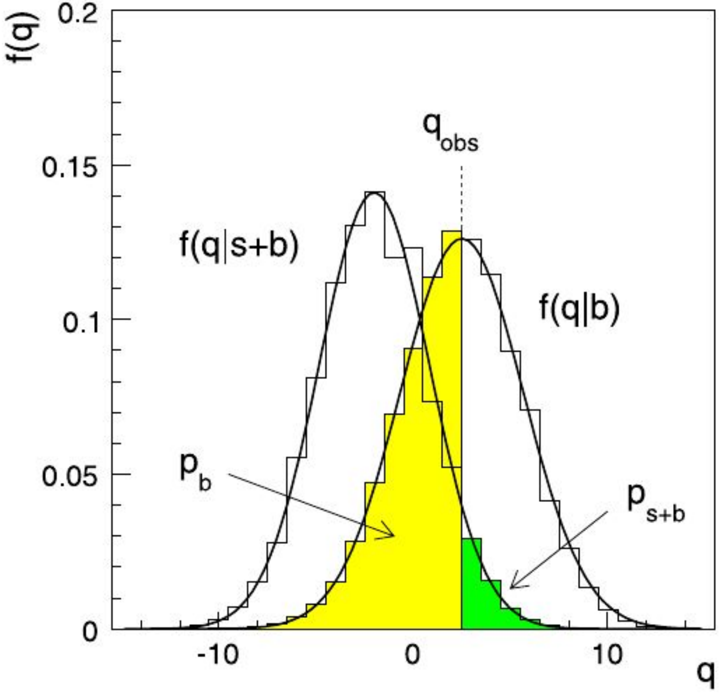
- In this case as well asymptotics formulae exist for the different distributions
  - Fairly quick computation of limits

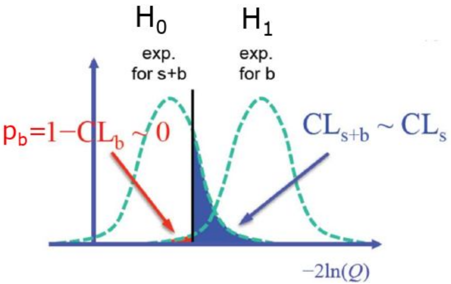
CL_s correction

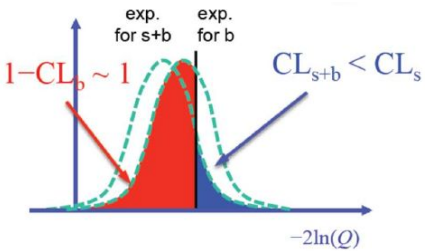
- Potential problem when setting limits: spurious exclusion when there is a downward fluctuation of the data even wrt the background-only hypothesis

- Definition: $\mathrm{CL}_s = \frac{p_\mu}{1-p_b} = \frac{\mathrm{CL}_{s+b}}{\mathrm{CL}_b}$

  - If the two distributions are well separated, small change wrt CL_{s+b}
  - If distributions are close, prevents spurious exclusion



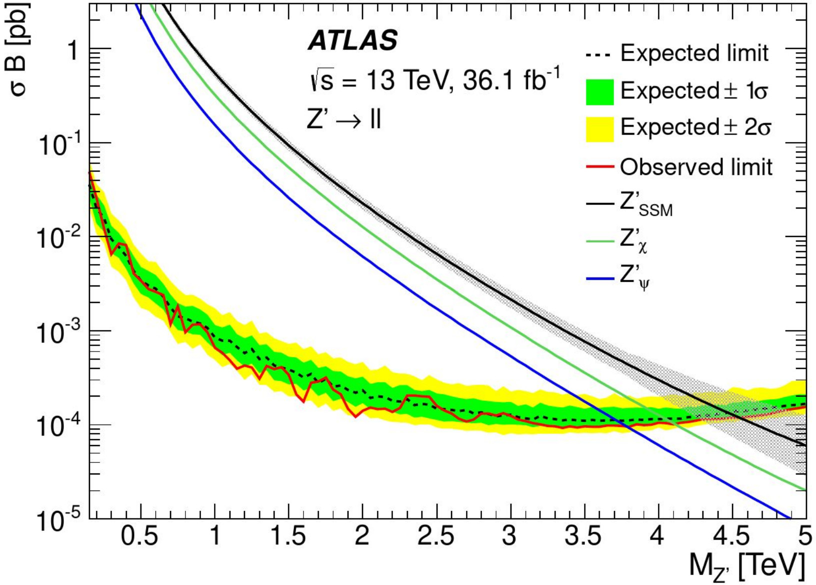




Example: search for high mass dilepton resonances

- For each hypothetized $M_{Z'}$ value, compute:
  - **Expected limit**: **median** value of upper limit under bkg-only hypo.
  - **Expected ±1σ**: interval containing 68% of the upper limit values under bkg-only
  - **Expected ±2σ**: interval containing 95% of the upper limit values under bkg-only
  - **Observed limit**: upper limit obtained using data actually observed
  - Theoretical curves: often superimposed. Crossing point gives lower limit on the Z′ mass for the given model

- All expected limits can be obtained with asymptotics formulae
  - At very high masses, very low number of events: good practice to cross-check limit with toys



Conclusions

Some knowledge of statistics is necessary to perform and understand BSM searches at the LHC

- Statistical analyses rely on likelihood functions:
  - Parameters of interest we want to measure (cross-section, mass…)
  - Other parameters of the model are called nuisance parameters

- Parameter estimation uses maximum likelihood values as estimators
  - Asymptotic properties of the likelihoods allows to set easily confidence intervals

- Hypothesis testing is used to claim discovery or to set limits
  - Use Profile likelihood ratio-based test statistics
  - Null and alternative hypotheses have to be set appropriately
  - Significance of the test: 0.05 for exclusion, 5σ for discovery, etc…
  - Asymptotic formulas allow to compute limits and significances without the need for massive amount of toy data.

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